Vol. B15 (1984) ACTA PHYSICA POLONICA No 10

BOUNDED ORBITS FOR CLASSICAL MOTION OF COLORED TEST PARTICLES IN THE PRASAD-SOMMERFIELD MONOPOLE FIELD

By L. Gy. Fehér

Bolyai Institute, H-6720 Szeged, Hungary

(*Received February 21, 1984*)

A reinvestigation is given for classical motion of colored test particles in the Prasad-Sommerfield monopole field, a problem that was treated already by J. Schechter soon after the discovery of the monopole. The equation used by us can be obtained from the Wong equation if one regards the Higgs field as a component of a five dimensional translation invariant Yang-Mills potential and reinterpret the motion in the fifth direction appropriately. Its nonrelativistic limit is treated for the static, SO(3) symmetrical monopole of unit charge with SU(2) as gauge group. The total angular momentum is conserved, though the particle's mass is variable. For large distances the space motion is not simply that of an electric pole in a Dirac monopole field, as Schechter has stated neglecting the coupling to the Higgs field. The asymptotical solution at large distances proves that beyond unbounded scattering solutions there exist bounded orbits too, caused by long range forces which arise from the zero mass Higgs field. The bounded motion takes place on a closed, periodical trajectory between two meridian circles of a cone, whose axis is the total angular momentum vector.

PACS numbers: 11.10.-z, 03.50.-z, 12.10.-g

1. *Introduction*

Soon after the discovery of the first monopole solution [1] of the SU(2) Yang-Mills-Higgs (Y-M-H) field theory J. Schechter has investigated [2] the motion of "colored" (Y-M) test particles in the monopole field. He worked in the Prasad-Sommerfield limit, where simple analytical formulas [3] are known. If $A_0 = 0$ then the t'Hooft electromagnetic tensor is the same as for an Abelian magnetic monopole. Since the motion of the electric poles is known [4], there are only unbounded (scattering) solutions. For description of motion of the colored particles Schechter has used the Wong equation [5], in which their color is coupled only to the Y-M field. The nonrelativistic real space motion at large distances was stated to be the same as for electric poles in a Dirac monopole field. This is inconsistent with the physical expectation, that long range forces arise from the Higgs field — which is of zero mass in this limit — and produce bounded orbits. We remind the reader that these forces can compensate [6] the repulsive magnetic ones between distant monopoles of equal charge. The cause of this confusion is the neglection of the coupling to the Higgs field in Schechter's work. We derive an equation of motion containing that coupling too. This gives in fact forces of the same asymptotic form as the Y–M field strength. Its solution in the nonrelativistic, large distance limit proves the existence of bounded orbits. A system of units will be used, where the gauge coupling constant, the velocity of light and the constant in the monopole solution are equal to unity.

2. *Equation of motion for colored test particles*

The Lagrangian in the Prasad-Sommerfield limit (with the usual definitions) is: $(\alpha, \mu, \nu = 0, \ldots, 3)$

$$\mathcal{L} = -\tfrac{1}{2}\langle F_{\mu\nu}, F^{\mu\nu}\rangle_{\mathscr{Y}} - \langle D_\alpha\phi, D^\alpha\phi\rangle_{\mathscr{Y}}. \tag{2.1}$$

Let $[,]$; $\langle,\rangle_{\mathscr{Y}}$ denote the Lie bracket and the Cartan–Killing form on the Lie algebra $\mathscr{Y}$ of the gauge group G. This field system can be regarded [7] as a pure Y–M field over a five dimensional flat space-time M^5, for which the corresponding connection is invariant with respect to translations of the fifth coordinate x^4. The five dimensional motion in this pure Y–M field is governed by the following Wong equations:

$$\frac{d^2x^4}{ds^2} = -\langle D_\alpha\phi, Q(s)\rangle_{\mathscr{Y}} \frac{dx^\alpha}{ds},$$

$$\frac{dQ}{ds} = \left[Q(s), A_\mu \frac{dx^\mu}{ds} + \phi \frac{dx^4}{ds}\right],$$

$$\frac{d^2x^\mu}{ds^2} = \langle F^\mu{}_\alpha, Q(s)\rangle_{\mathscr{Y}} \frac{dx^\alpha}{ds} + \langle D^\mu\phi, Q(s)\rangle_{\mathscr{Y}} \frac{dx^4}{ds}. \tag{2.2}$$

Here $Q \in \mathscr{Y}$ is the color charge (or isospin for SU(2)) of the test particle and s is an affine parameter on the path in M^5. We have used the $A_4 \to \phi$ correspondence. It is natural to assume, that the projections of the orbits from M^5 to the four space-time M^4 give the motion in the Y–M–H system (2.1). Its causality is guaranteed if we regard only causal curves in M^5, that is

$$g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} + \left(\frac{dx^4}{ds}\right)^2 = -\kappa^2 \quad (g_{00} = -1). \tag{2.3}$$

In the proper time τ of the projection curve the equations (2.2) obtain their final "physical" form:

$$m^2(\tau) = \kappa^2 + q^2(\tau), \tag{2.4a}$$

$$\frac{dq}{d\tau} = -\langle D_\alpha\phi, Q(\tau)\rangle_{\mathscr{Y}} \frac{dx^\alpha}{d\tau}, \tag{2.4b}$$

$$\frac{dQ}{d\tau} = \left[Q(\tau), A_\mu \frac{dx^\mu}{d\tau} + \frac{q}{m}\phi\right], \tag{2.4c}$$

$$m\frac{d^2x^\mu}{d\tau^2} = \langle F^\mu{}_\beta, Q\rangle_{\mathscr{Y}} \frac{dx^\beta}{d\tau} + \frac{q}{m}\langle D_\alpha\phi, Q\rangle_{\mathscr{Y}} \left\{g^{\mu\alpha} + \frac{dx^\alpha}{d\tau}\frac{dx^\mu}{d\tau}\right\}. \tag{2.4d}$$

We have used (2.2–3) and the definitions:

$$q(\tau) = \frac{dX^4}{ds}, \qquad m(\tau) = \frac{d\tau}{ds} \qquad (s = s(\tau)). \tag{2.5}$$

A new gradient force has appeared from the Higgs field beyond the usual Lorentz force in (2.4d). As it is well known, the origin of ϕ in (2.1) as a component of a higher dimensional Y–M potential is a very special case of geometrization [8, 9] of general Higgs fields with symmetry breaking potential. So we think, that (2.4) is also a special case of an equation of particle motion obtainable from geodesics [9] of a Kaluza-Klein theory, for which the dimensional reduction gives Y–M–H and other fields, e.g. gravity, Brans-Dicke scalar. For the initial value problem of (2.4) $q(\tau_0)$, $Q(\tau_0)$, $x^\mu(\tau_0)$, $\frac{dx^\mu}{d\tau}(\tau_0)\left(\left(q_{\alpha\beta}\frac{dx^\alpha}{d\tau}(\tau_0)\frac{dx_\beta}{d\tau}(\tau_0) = -1\right)\right.$ are to be given. In (2.4d) $m(\tau)$ appears as the mass of the moving particle. This is consistent with its role in the angulr momentum conservation as it turns out soon. $m(\tau)$ varies because of the presence of the scalar field. If $G = SU(2)$ then one can identify $\mathscr{Y}$ with operations $[,]$, $\langle,\rangle_{\mathscr{Y}}$ and the real three space endowed with the vector and scalar product. This is convenient when we are going to deal with the Prasad-Sommerfield monopole field [3]:

$$A_i^a = \frac{1}{r^2}\varepsilon_{aij}x_j[1-K(r)], \qquad K(r) = \frac{r}{\sinh r},$$

$$A_0^a = 0 \qquad (a, b, \ldots, i, j, \ldots = 1, 2, 3), \qquad (r = \langle\vec{r}, \vec{r}\rangle^{1/2}),$$

$$\phi^a = \frac{1}{r^2}x_a H(r), \qquad H(r) = r\coth r - 1. \tag{2.6}$$

3. *Nonrelativistic, asymptotical motion in the monopole field*

We consider only nonrelativistic motions here, that is in (2.4) substitute τ by $t = x^0$ and drop out the equation for x^0. Following Schechter, it is convenient to express the particle's isospin in the $\vec{r}$, $\vec{w} = \vec{r}\times\vec{v}$ $\left(\vec{v} = \frac{d\vec{r}}{dt}\right)$, $\vec{z} = \vec{r}\times\vec{w}$ orthogonal moving frame along the particle's trajectory:

$$\vec{Q} = \alpha\hat{\vec{r}} + \beta\hat{\vec{w}} + \gamma\hat{\vec{z}}, \qquad (\alpha^2+\beta^2+\gamma^2 = Q^2). \tag{3.1}$$

The hatted symbols denote the corresponding unit vectors, Q^2 = const. Let prime and dot refer in the following to derivation with respect to r and t. Using (2.4), (2.6), (3.1) our equations are:

$$m^2(t) = \kappa^2 + q^2(t), \tag{3.2a}$$

$$\dot{q} = -\frac{\alpha}{r^3}\langle\vec{r},\vec{v}\rangle\,[rH'-H] + HK\gamma\frac{|\vec{w}|}{r^3}, \tag{3.2b}$$

$$\dot{\vec{Q}} = \frac{K-1}{r^2}\,\vec{Q}\times\vec{w} + \frac{q}{m}\,\frac{H}{r^2}\,\vec{Q}\times\vec{r}, \tag{3.2c}$$

$$\begin{aligned} m\dot{\vec{v}} = {} & \frac{\vec{r}}{r^3}\left\{\frac{\beta|\vec{w}|}{r}\,[rK'+2(1-K)] + \frac{\alpha q}{m}\,[rH'-H(1+K)]\right\} \\ & + \frac{\vec{v}}{r^2}\,\frac{q}{m}\left\{\frac{\alpha\langle\vec{r},\vec{v}\rangle}{r}\,[rH'-H(1+K)] + \langle\vec{Q},\vec{v}\rangle HK\right\} \\ & + \frac{\vec{v}\times\vec{r}}{r^3}\,\{\alpha(1-K)^2\} + \frac{\vec{Q}}{r^2}\left\{\frac{q}{m}\,HK\right\} + \frac{\vec{Q}\times\vec{v}}{r^2}\,\{2(1-K)\} \\ & + \frac{\vec{r}\times\vec{Q}}{r^4}\,\{\langle\vec{r},\vec{v}\rangle\,[rK'+2(1-K)]\}. \end{aligned} \tag{3.2d}$$

Since the monopole (2.6) is static and SO(3) invariant, the sum of the particle's orbital angular momentum $\vec{l} = m\vec{r}\times\vec{v}$ and the angular momentum of the total field of the particle--monopole system is conserved [10]. This total angular momentum

$$\vec{N} = \vec{l} + K\vec{Q} + \frac{\alpha}{r}(1-K)\vec{r} \tag{3.3}$$

is in fact a constant of motion for (3.2). For simplicity, we investigate in the following the motion at large distances from the monopole. That is we neglect the exponentially decreasing functions of r. (3.2b–d) give rise to:

$$\dot{q} = \alpha\frac{d}{dt}\left(\frac{1}{r}\right), \quad \dot{\alpha} = 0, \tag{3.4}$$

$$\dot{\beta} = \frac{\gamma\alpha}{m}\,\frac{1}{r^2} + \frac{q\gamma}{m}\,\frac{H}{r}, \tag{3.5a}$$

$$\dot{\gamma} = -\frac{\beta\alpha}{m}\,\frac{1}{r^2} - \frac{q\beta}{m}\,\frac{H}{r}, \tag{3.5b}$$

$$m\dot{\vec{v}} = \frac{\alpha}{r^3}\left\{\vec{r}\times\vec{v} + \frac{q}{m}\,[\vec{r} + \langle\vec{r},\vec{v}\rangle\vec{v}]\right\}. \tag{3.6}$$

In Schechter's paper m is constant and only the first term appears in (3.6). Our extra forces are produced by the particle-Higgs coupling. From (3.4) we see that $\alpha = \alpha_0$ and

$$q = q(r(t)) = \frac{\alpha}{r} - \frac{\alpha}{r_0} + q_0 \tag{3.7}$$

and $m(t)$ is given by (3.2a). Here the zero as index refers to the initial data. The asymptotical real space motion (3.6) is decoupled from the isospin motion. The reader can easily verify that $l = |\vec{l}|$,

$$\varepsilon = m^2(v^2+1), \quad \vec{j} = \vec{l} + \alpha\hat{\vec{r}} \tag{3.8}$$

are constants of the "asymptotical motion". The solutions of (3.4–6) can be regarded as approximate ones to (3.2). We will see the physical effect of the test particle-Higgs field coupling by giving the solution for (3.6). We assume $j = |\vec{j}| \neq 0$; otherwise uniform motion (or rest) on a radial straight line takes place. The motion takes place on one of the halves of a rotation cone with axis $\vec{j}$, because of the constancy of $\vec{j}$ and $\langle \hat{\vec{j}}, \hat{\vec{r}} \rangle = \frac{\alpha}{j}$. This is why it is natural to introduce spherical coördinates (r, ϑ, φ), taking the polar axis parallel to $\vec{j}$. In these coordinates (3.6) and the constant of motion ε give rise to

$$\ddot{r} - \frac{l^2}{m^2 r^3} - \frac{\alpha}{r^2}\frac{q}{m^2}[1+(\dot{r})^2] = 0, \tag{3.9a}$$

$$\vartheta = \vartheta_0, \quad \cos\vartheta_0 = \frac{\alpha}{j}, \tag{3.9b}$$

$$\dot{\varphi} = \frac{j}{mr^2}, \tag{3.9c}$$

$$\varepsilon = m^2\left[1+(\dot{r})^2 + \frac{l^2}{m^2 r^2}\right]. \tag{3.9d}$$

In the case of the pure radial motion $l = 0$ (3.9c) does not appear of course. Using its first integral ε, (3.9a) can be regarded as a one dimensional potential problem for a point particle of unit mass and ε plays the role of the total energy in this auxiliary problem. Hence, the motion is described by (3.9c) and the following first order, separable differential equations obtained for $r(t)$ and for the path $r(\varphi)$ from (3.9):

$$\left(\frac{dr}{dt}\right)^2 = \frac{T(r)}{m^2 r^2} = \frac{T(r)}{P(r)}, \tag{3.10a}$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{1}{j^2} r^2 T(r). \tag{3.10b}$$

$P(r)$ and $T(r)$ are generally second order polynomials of r because of (3.7), (3.2a). $P(r)$ is positive and the motion is trivially restricted to the domain, where $T(r) \geqslant 0$

$$T(r) = (\varepsilon - m^2)r^2 - l^2. \tag{3.11}$$

We exclude the very special case, when the coefficient η of r^2 in $T(r)$

$$\eta = \frac{\varepsilon}{m_\infty^2} - 1, \quad m_\infty^2 = \kappa^2 + \left(q_0 - \frac{\alpha}{r_0}\right)^2 \tag{3.12}$$

vanishes. It is easy to see, that T can be rewritten as follows:

$$T(r) = m_\infty^2 \eta (r - r_1)(r + r_2 \operatorname{sign} \eta). \tag{3.13}$$

Here r_1, $(\operatorname{sign}\eta) r_2$ are the roots of T, $0 < r_2 < r_1$, and the detailed form of the roots is:

$$\{r_1, r_2 \operatorname{sign} \eta\} = \frac{1}{\eta m_\infty^2}\left\{\alpha\left(q_0 - \frac{\alpha}{r_0}\right)^2 \pm \left[\left(\alpha q_0 + \frac{l^2}{r_0}\right)^2 + (\dot{r}_0 m_0 j)^2\right]^{1/2}\right\}. \tag{3.14}$$

If $\eta < 0$ then the test particle moves between two meridian circles of the cone $r_2 \leqslant r(t) \leqslant r_1$. This is surely the case when $\alpha = \langle\hat{\vec{\phi}}, \vec{Q}\rangle$ is large enough. This means that the Higgs coupling produces bounded orbits. If $\eta > 0$ (for example when v_0^2 is large enough) then $r(t) \geqslant r_1$, the motion is unbounded. Note that in the special case of $\eta = 0$, that is when $T(r)$ is a first order polynomial of r, an analysis of the same type leads trivially to bounded or unbounded motion depending on the root and the shape of $T(r)$. From (3.9c) we see that φ increases monotonically with the time and from (3.10), (3.13) we get:

$$t - t_0 = \frac{1}{m_\infty |\eta|^{1/2}} \int_{r_0}^{r} \operatorname{sign} \dot{\varrho} \left[\frac{P(\varrho)}{\operatorname{sign}\eta(\varrho - r_1)(\varrho + r_2 \operatorname{sign}\eta)}\right]^{1/2} d\varrho, \tag{3.15a}$$

$$\varphi - \varphi_0 = \frac{j}{m_\infty |\eta|^{1/2}} \int_{r_0}^{r} \frac{\operatorname{sign}\dot{\varrho}}{\varrho[\operatorname{sign}\eta(\varrho - r_1)(\varrho + r_2 \operatorname{sign}\eta)]^{1/2}} d\varrho. \tag{3.15b}$$

The sign in (3.15) changes when the particle reaches a turning point of its orbit at r_1 or r_2. From (3.15b) one monotonical piece of the path $r(\varphi)$ is described by:

$$\varphi - \varphi_0 = \operatorname{sign} \dot{r}_0 \left\{\arcsin \frac{2r_1 r_2 + (r_1 - r_2) r_0}{r_0(r_1 + r_2)} - \arcsin \frac{2r_1 r_2 + (r_1 - r_2) r}{r(r_1 + r_2)}\right\}. \tag{3.16}$$

For example if $\eta < 0$ and the particle starts at $r_0 = r_2$ then after a finite time and $\Delta\varphi = \dfrac{\pi}{2}$ it gets to r_1 and the motion takes place on a periodical, closed trajectory between r_2 and r_1. If $\eta > 0$, $\dot{r}_0 > 0$ then the particle escapes to infinity, during infinite time (3.15a) of course, and scattering solution appears in the case of $\dot{r}_0 < 0$. For example if at $t_0 = -\infty$

$r_0 = \infty$ then the particle moves on a symmetrical scattering trajectory between r_1 and infinity, and the scattering angle $\Delta\varphi$ (witn $r(t = 0) = r_1$) is:

$$\Delta\varphi = \varphi(t = \infty) - \varphi(t = -\infty) = \pi - \arcsin \frac{r_1 - r_2}{r_1 + r_2}. \tag{3.17}$$

In tne case $l = 0$ tne purely radial motion is of course similarly eitner oscillating or unbounded, depending on the initial data. Knowing $r(t)$ given by (3.15a), $\varphi(t)$ and the asymptotic isospin motion can be calculated from (3.9c) and (3.5) respectively. This calculation would not, however, add new details to the effect of the test particle-Higgs field coupling we were mainly interested in.

I would like to thank Z. Horváth, P. T. Nagy and L. Palla for helpful comments and pieces of advice.

REFERENCES

[1] G. 't Hooft, *Nucl. Phys.* **B79**, 276 (1974); A. M. Polyakov, *JETP Lett.* **20**, 194 (1974).
[2] J. Schechter, *Phys. Rev.* **D14**, 524 (1976).
[3] M. K. Prasad, C. M. Sommerfield, *Phys. Rev. Lett.* **35**, 760 (1976).
[4] I. R. Lapidus, J. L. Pietenpol, *Am. J. Phys.* **28**, 17 (1960); G. Nadeau, *Am. J. Phys.* **28**, 566 (1960).
[5] S. K. Wong, *Nuovo Cimento* **A65**, 689 (1970); N. K. Nielsen, *Nucl. Phys.* **B167**, 249 (1980); C. A. Orzalesi, M. Pauri, *Physica* **A114**, 138 (1982); T. Fukuyama, *Gen. Relativ. Gravitation* **14**, 729 (1982).
[6] N. S. Manton, *Nucl. Phys.* **B126**, 525 (1977).
[7] A. Jaffe, C. Taubes, *Vortices and Monopoles*, Birkhauser, Boston 1980, p. 46.
[8] P. Forgács, N. S. Manton, *Commun. Math. Phys.* **72**, 15 (1980); Gu Chaohao, *Phys. Rep.* **80**, 251 (1981); R. Kerner, *J. Math. Phys.* **24**, 356 (1983).
[9] R. Kerner, TH. 3669 — CERN (1983).
[10] P. Hasenfratz, G. 't Hooft, *Phys. Rev. Lett.* **36**, 1119 (1976).