LAGRANGE TRIANGLE OF DIRAC PARTICLES (A MODEL FOR THE THREE-QUARK PROBLEM)

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An approximate model is constructed for a system of three Dirac particles of equal masses interacting mutually through a universal static two-body potential. The introduced approximation corresponds to characteristic features of the Lagrange exact solutions to the classical three-body problem, where three distances between three particles are kept equal all the time, although they are, in general, varying. Wave equations are found for the internal motion in the cases of spin 1/2 and spin 3/2. Radial equations are derived.

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1. Introduction

Theorists, sometimes, construct simplified models of "real problems" in order to make their theory computable. The problem of three Dirac particles, as it appears for instance in the quark theory of baryons, is so involved that its simplified calculable models seem to be wanted. In this paper we propose such a model for a highly symmetrical configuration of three Dirac particles. It is inspired by the Lagrange exact solutions to the classical three-body problem.

In fact, since Lagrange's time there are known special solutions of the classical (non-relativistic) three-body problem with Newton gravitational attraction, where three mutually interacting particles of arbitrary masses form throughout the motion an equilateral triangle, in general, of varying orientation and size [1]. Then, in the centre-of-mass frame, the particles describe three coplanar conics, all with the same eccentricity and one common focus located at the centre of mass. If the conics are ellipses, the motion is periodic.

To some extent these solutions remind of the case of three mutually non-interacting particles attracted by a fixed gravitational source, but now in the Lagrange case their

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motion is correlated in space and time by the equilateral-triangle condition $r_{12} = r_{23} = r_{31}$ with $r_{ij} = |\vec{r}_i - \vec{r}_j|$. In the case of equal masses $m_1 = m_2 = m_3 (\equiv m)$, this condition, if supplemented by the dynamically necessary requirement of coplanarity of motion, can be characterized in momentum space by the equations

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0, \quad \vec{p}_1^{\,2} = \vec{p}_2^{\,2} = \vec{p}_3^{\,2} \tag{1}$$

or equivalently

$$\vec{P} = 0, \quad \vec{p} \cdot \vec{\pi} = 0, \quad \vec{p}^{\,2} = \tfrac{4}{3}\vec{\pi}^{\,2}, \tag{2}$$

where the centre-of-mass frame is used. Here,

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3, \quad \vec{p} = \tfrac{1}{3}(2\vec{p}_3 - \vec{p}_1 - \vec{p}_2), \quad \vec{\pi} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) \tag{3}$$

are the canonical momenta conjugate with the coordinates

$$\vec{R} = \tfrac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3), \quad \vec{r} = \vec{r}_3 - \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2), \quad \vec{\varrho} = \vec{r}_1 - \vec{r}_2, \tag{4}$$

respectively. In terms of these coordinates, the equilateral-triangle condition is: $\vec{r} \cdot \vec{\varrho} = 0$ and $r = \frac{\sqrt{3}}{2}\varrho$ with $r = |\vec{r}|$ and $\varrho = |\vec{\varrho}|$. Note that the total orbital angular momentum and kinetic energy are

$$L = \sum_i r_i \times p_i = \vec{R} \times \vec{P} + \vec{r} \times \vec{p} + \vec{\varrho} \times \vec{\pi} \tag{5}$$

and

$$T = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^{\,2}}{2\kappa} + \frac{\vec{\pi}^{\,2}}{2\mu}, \tag{6}$$

where

$$M = m_1 + m_2 + m_3 = 3m, \quad \kappa = \frac{(m_1 + m_2)m_3}{M} = \tfrac{2}{3}m,$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \tfrac{1}{2}m. \tag{7}$$

In the case of Eq. (1) or (2)

$$\vec{R} \times \vec{P} = 0, \quad \vec{r} \times \vec{p} = \vec{\varrho} \times \vec{\pi}. \tag{8}$$

It is not difficult to see that in the case of equal masses the equilateral-triangle solutions exist also for any static attraction or repulsion described by the potential

$$V = V(r_{12}) + V(r_{23}) + V(r_{31}) \tag{9}$$

built additively from a universal two-body interaction $V(r_{ij})$. In fact, if in this case we put $r_{12} = r_{23} = r_{31}$ (and consequently $r = \frac{\sqrt{3}}{2}\varrho$) in the canonical equation of motion derived from the hamiltonian $H = T+V$, we get three consistent equations

$$M\ddot{\vec{R}} = 0,$$

$$\tfrac{1}{3}m\ddot{\vec{r}} = -\frac{V'\left(\frac{2}{\sqrt{3}}r\right)}{\frac{2}{\sqrt{3}}r}\vec{r},$$

$$\tfrac{1}{3}m\ddot{\vec{\varrho}} = -\frac{V'(\varrho)}{\varrho}\vec{\varrho}, \tag{10}$$

the second and third of which being identical in form. This identity and the covariance of Eqs. (10) under spatial rotations allow to satisfy the condition $r = \frac{\sqrt{3}}{2}\varrho$ and $\vec{r}\cdot\vec{\varrho} = 0$ by the ansatz

$$\vec{\varrho} = \frac{2}{\sqrt{3}}\hat{O}_{\pi/2}\vec{r}, \quad \hat{O}_{\pi/2} = \begin{pmatrix} 0, & -1, & 0 \\ 1, & 0, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \tag{11}$$

where the z-axis is chosen perpendicular to the plane of motion: $\vec{r} = (x, y, 0)$ and $\vec{\varrho} = (\xi, \eta, 0)$. Since in Eqs. (10) the coordinates $\vec{R}$, $\vec{r}$ and $\vec{\varrho}$ are separated, the three-body problem is reduced in this special case to three one-body problems.

Note that the argument presented above does not work in the case of different masses, except for the Newton gravitational attraction, where the two-body interaction $V_{ij}(r_{ij})$ is proportional to $m_i m_j$.

In the classical mechanics we can select solutions (as e.g. the equilateral-triangle solutions) by imposing initial conditions on canonical variables. Such a possibility no longer exists in the quantum mechanics, if we want to look for energy levels. However, the high spatial symmetry of the classical equilateral-triangle solutions suggests that Eqs. (1) or (2), when applied to the quantum-mechanical hamiltonian, may provide a useful *approximate model* for a three-body quantum system, as e.g. the nucleon or Δ baryon, that possesses a high spatial symmetry.

In the present paper we exploit this idea to construct an approximate model for a system of three Dirac particles of equal masses interacting mutually through a universal static two-body potential. The model may be called *the Lagrange triangle of Dirac particles* and hopefully it may be useful for three-quark configurations of high spatial symmetry as those in nonstrange baryons (in particular, in the nucleon and Δ baryon). Such an application is planned as a subject of another paper.

2. The model

To begin with consider a system of three Dirac particles interacting mutually through a universal static vector potential $V(r_{ij})$. Then the wave equation is

$$[\sum_i (\vec{\alpha}_i \cdot \vec{p}_i + \beta_i m_i) + \sum_{i<j} V(r_{ij})]\psi = E\psi, \tag{12}$$

where $\vec{p}_i = -i\partial/\partial\vec{r}_i$ and the usual anticommutation and commutation relations hold for the Dirac matrices. In order to construct our model we put $m_1 = m_2 = m_3\,(\equiv m)$ and eliminate six coordinates of the nine $\vec{r}_1, \vec{r}_2, \vec{r}_3$ by means of the condition (1) which implies that

$$\vec{p}_1 = \hat{O}_{2\pi/3}\vec{p}, \quad \vec{p}_2 = \hat{O}^{-1}_{2\pi/3}\vec{p}, \quad \vec{p}_3 = \vec{p}, \tag{13}$$

where

$$\hat{O}_{2\pi/3} = \frac{1}{2}\begin{pmatrix} -1, & -\sqrt{3}, & 0 \\ \sqrt{3}, & -1, & 0 \\ 0, & 0, & 2 \end{pmatrix}, \quad \hat{O}^{-1}_{2\pi/3} = \hat{O}^{\mathrm{T}}_{2\pi/3}. \tag{14}$$

Here, three particles $i = 1, 2, 3$ are labelled anti-clock-wise in the plane of motion with the z-axis chosen (for the sake of this argument) perpendicular to this plane: $\vec{p} = (p_x, p_y, 0)$ Then

$$\vec{\alpha}_1 \cdot \vec{p}_1 + \vec{\alpha}_2 \cdot \vec{p}_2 + \vec{\alpha}_3 \cdot \vec{p}_3 = (\vec{\alpha}_1' + \vec{\alpha}_2' + \vec{\alpha}_3) \cdot \vec{p}, \tag{15}$$

where $\vec{\alpha}_1' = \hat{O}^{-1}_{2\pi/3}\vec{\alpha}_1$ and $\vec{\alpha}_2' = \hat{O}_{2\pi/3}\vec{\alpha}_2$. Since $\vec{\alpha}_1'$ and $\vec{\alpha}_2'$ satisfy the same anticommutation and commutation relations as the original $\vec{\alpha}_1$ and $\vec{\alpha}_2$ (and are Hermitian and space-independent like the latter), we can drop their prime label. So we can write the following wave equation for our Lagrange triangle of Dirac particles:

$$\left[(\vec{\alpha}_1 + \vec{\alpha}_2 + \vec{\alpha}_3) \cdot \vec{p} + (\beta_1 + \beta_2 + \beta_3)m + 3V\left(\frac{2}{\sqrt{3}}r\right)\right]\psi(\vec{r}) = E\vec{\psi}(r) \tag{16}$$

with $\vec{p} = -i\partial/\partial\vec{r}$. Here, as usual, $\vec{\alpha}_i = \gamma_i^5\vec{\sigma}_i$ with $\{\gamma_i^5, \beta_i\} = 0$ and $[\gamma_i^5, \vec{\sigma}_i] = 0 = [\beta_i, \vec{\sigma}_i]$.

If mass m is not negligible, the configuration of three Dirac particles in Eq. (16) depends much on the eigenvalues of matrices β_i (equal to ± 1). So, in order to provide high spatial symmetry of this configuration we will impose on our Lagrange triangle of Dirac particles the additional condition

$$\beta_1 = \beta_2 = \beta_3(\equiv \beta) \tag{17}$$

and, consequently,

$$\gamma_1^5 = \gamma_2^5 = \gamma_3^5(\equiv \gamma^5). \tag{18}$$

Then, the wave equation takes the form

$$\left[\gamma^5(\vec{\sigma}_1 + \vec{\sigma}_2 + \vec{\sigma}_3) \cdot \vec{p} + 3\beta m + 3V\left(\frac{2}{\sqrt{3}}r\right)\right]\psi(\vec{r}) = E\psi(\vec{r}) \tag{19}$$

which we will assume as the *definition* of our model. Of course, this equation is for internal motion (in the centre-of-mass frame).

We should like to stress that it is hard to justify the simplifying assumption (17) a priori. In its case the wave function ψ contains only the "large-large-large" and "small-small-small" components which introduce no difference between three particles as far as their masses are concerned. (Of course, in the case of identical particles these components do not modify the non-relativistic discussion of the overall symmetry of ψ (including space, spin and internal degrees of freedom)). However, the assumption (17) is not simply equivalent to neglecting in the wave equation (16) the mixed components "small-large-large" and "small-small-large" (and their permutations). It is so, because in the case of $\beta_1 = \beta_2 = \beta_3$ and $\gamma_1^5 = \gamma_2^5 = \gamma_3^5$ there are in Eq. (16) (or now in Eq. (19)) direct transitions "large-large-large" $\leftrightarrow$ "small-small-small", which are absent in the case of different β_i, when they can occur only indirectly in three steps via mixed components. So, we can see that our new "large-large-large" and "small-small-small' components describe in a way the effect of *all* previous components.

In the Dirac representation we can write

$$\beta = \begin{pmatrix} \mathbf{1}, & 0 \\ 0, & -\mathbf{1} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0, & \mathbf{1} \\ \mathbf{1}, & 0 \end{pmatrix}, \quad \sigma_i = \begin{pmatrix} \vec{\sigma}_i^{\mathrm{P}}, & 0 \\ 0, & \vec{\sigma}_i^{\mathrm{P}} \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{20}$$

where $\vec{\sigma}_1^{\mathrm{P}} = \vec{\sigma}^{\mathrm{P}} \otimes \mathbf{1}^{\mathrm{P}} \otimes \mathbf{1}^{\mathrm{P}}$, $\vec{\sigma}_2^{\mathrm{P}} = \mathbf{1}^{\mathrm{P}} \otimes \vec{\sigma}^{\mathrm{P}} \otimes \mathbf{1}^{\mathrm{P}}$, $\vec{\sigma}_3^{\mathrm{P}} = \mathbf{1}^{\mathrm{P}} \otimes \mathbf{1}^{\mathrm{P}} \otimes \vec{\sigma}^{\mathrm{P}}$ and $\mathbf{1} = \mathbf{1}^{\mathrm{P}} \otimes \mathbf{1}^{\mathrm{P}} \otimes \mathbf{1}^{\mathrm{P}}$ with $\vec{\sigma}^{\mathrm{P}}$ and $\mathbf{1}^{\mathrm{P}}$ being the usual 2×2 Pauli matrices. Then, the wave equation (19) can be represented in the form

$$(E \mp 3m - 3V)\psi_\pm = 2\vec{S}^{\mathrm{P}} \cdot \vec{p}\psi_\mp, \tag{21}$$

where

$$\vec{S} = \tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2 + \vec{\sigma}_3) = \begin{pmatrix} \vec{S}^{\mathrm{P}}, & 0 \\ 0, & \vec{S}^{\mathrm{P}} \end{pmatrix} \tag{22}$$

is the total spin. Its quantum number takes the values $s = 1/2$ and $s = 3/2$. The corresponding projection operators can be written as follows:

$$P_{1/2} = \tfrac{1}{3}(\tfrac{15}{4} - \vec{S}^2) = P_0 + \tfrac{1}{3}[1 - \tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{\sigma}_3]P_1,$$

$$P_{3/2} = \tfrac{1}{3}(\vec{S}^2 - \tfrac{3}{4}) = \tfrac{1}{3}[2 + \tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{\sigma}_3]P_1. \tag{23}$$

Here,

$$P_0 = \tfrac{1}{2}(2 - \vec{S}^{(12)2}) = \tfrac{1}{4}(1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2),$$

$$P_1 = \tfrac{1}{2}\vec{S}^{(12)2} = \tfrac{1}{4}(3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2) \tag{24}$$

are the projection operators corresponding to the values $s_{12} = 0$ and $s_{12} = 1$ of the quantum number related to the partial spin

$$\vec{S}^{(12)} = \tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2). \tag{25}$$

Note the useful identities:

$$(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{\sigma}_3 P_{3/2} = 2P_{3/2}, \quad (\vec{\sigma}_1-\vec{\sigma}_2)\cdot\vec{\sigma}_3 P_{3/2} = 0 \tag{26}$$

and

$$(\vec{\sigma}_1-\vec{\sigma}_2)P_{0,1} = P_{1,0}(\vec{\sigma}_1-\vec{\sigma}_2), (\vec{\sigma}_1-\vec{\sigma}_2)\left[(\vec{\sigma}_1\pm\vec{\sigma}_2)P_{1,0}\cdot\vec{a}\right] = \begin{cases}-2i(\vec{\sigma}_1-\vec{\sigma}_2)P_1\times\vec{a},\\ 4P_0\vec{a},\end{cases} \tag{27}$$

the last for any $\vec{a}$, where $\pm$ corresponds to 1, 0, respectively.

In terms of the wave-function components

$$f^{\pm} = \frac{\psi_+ \pm \psi_-}{\sqrt{2}} = \psi_{\mathrm{R,L}}, \tag{28}$$

representing ψ in the van der Waerden representation where

$$\gamma^5 = \begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}, \quad \beta = \begin{pmatrix}0, & 1\\ 1, & 0\end{pmatrix}, \quad \sigma_i = \begin{pmatrix}\vec{\sigma}_i^{\mathrm{P}}, & 0\\ 0, & \vec{\sigma}_i^{\mathrm{P}}\end{pmatrix}, \quad \psi = \begin{pmatrix}\psi_{\mathrm{R}}\\ \psi_{\mathrm{L}}\end{pmatrix}, \tag{29}$$

we obtain from Eq. (19) or (21):

$$(E-3V\mp 2\vec{S}^{\mathrm{P}}\cdot\vec{p})f^{\pm} = 3mf^{\mp}. \tag{30}$$

From the wave equation in this form we can derive directly the following second-order equation:

$$\left[(E-3V)^2-(2\vec{S}^{\mathrm{P}}\cdot\vec{p})^2-(3m)^2\pm 6iS_r^{\mathrm{P}}\frac{dV}{dr}\right]f^{\pm} = 0, \tag{31}$$

where $S_r^{\mathrm{P}} = \hat{r}\cdot\vec{S}^{\mathrm{P}}$ with $\hat{r} = \vec{r}/r$. In the case of $s = 1/2$ we get here $(2\vec{S}^{\mathrm{P}}\cdot\vec{p})^2 = \vec{p}^2$ (cf. Eq. (32) below)[1].

3. *Wave equations for $s = 1/2$ and $s = 3/2$*

It is easily seen from the wave equation (19) or its forms (21) and (30) that $\vec{S}^2$ and $\vec{S}_{12}^2$ are constants of motion. Thus, s and s_{12} are "good" quantum numbers and, consequently, the wave equation splits into three independent parts with ($s = 1/2$, $s_{12} = 0$), ($s = 1/2$, $s_{12} = 1$) and ($s = 3/2$, $s_{12} = 1$) corresponding to the projections with $P_{1/2}P_0 = P_0, P_{1/2}P_1$ and $P_{3/2}P_1 = P_{3/2}$, respectively. In these parts we can write

$$\vec{S}^{\mathrm{P}} = \begin{cases}\frac{1}{2}\vec{\sigma}^{\mathrm{P}} & \text{for} \quad s = 1/2 \quad (\text{and } s_{12} = 0 \text{ or } 1),\\ \frac{1}{2}\vec{\Sigma}^{\mathrm{P}} & \text{for} \quad s = 3/2 \quad (\text{and } s_{12} = 1),\end{cases} \tag{32}$$

[1] We can see from Eq. (31) that the non-relativistic approximation applied to Eq. (19) leads to the kinetic energy $3m+(2\vec{S}^{\mathrm{P}}\cdot\vec{p})^2/6m$ which gives $3(m+\vec{p}^2/2m)$ only in the state with equal three helicities (otherwise giving $3m+\vec{p}^2/2(3m)$, as for $s = 1/2$). On the other hand, the form $3(m+\vec{p}^2/2m)$ follows from the non-relativistic approximation applied to Eq. (16). So, the models based on Eqs. (16) and (19) differ considerably, at least in the non-relativistic limit.

where

$$\sigma_1^{\mathrm{P}} = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}, \quad \sigma_2^{\mathrm{P}} = \begin{pmatrix} 0, & -i \\ i, & 0 \end{pmatrix}, \quad \sigma_3^{\mathrm{P}} = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix} \tag{33}$$

and [2]

$$\Sigma_1^{\mathrm{P}} = \begin{pmatrix} 0, & \sqrt{3}, & 0, & 0 \\ \sqrt{3}, & 0, & 2, & 0 \\ 0, & 2, & 0, & \sqrt{3} \\ 0, & 0, & \sqrt{3}, & 0 \end{pmatrix}, \quad \Sigma_2^{\mathrm{P}} = \begin{pmatrix} 0, & -i\sqrt{3}, & 0, & 0 \\ i\sqrt{3}, & 0, & -2i, & 0 \\ 0, & 2i, & 0, & -i\sqrt{3} \\ 0, & 0, & i\sqrt{3}, & 0 \end{pmatrix},$$

$$\Sigma_3^{\mathrm{P}} = \begin{pmatrix} 3, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & -3 \end{pmatrix}. \tag{34}$$

Note that in the case of $s_{12} = 0$ we get $\vec{\sigma}^{\mathrm{P}} = \vec{\sigma}_3^{\mathrm{P}}$. It is not difficult to see that the components $s_{12} = 0$ and $s_{12} = 1$ of the state $s = 1/2$ are spinors of the form

$$\frac{1}{\sqrt{2}} \begin{pmatrix} |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle \\ |\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle \end{pmatrix} \tag{35}$$

and

$$\tfrac{1}{2} \begin{pmatrix} |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle \\ -|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle + 2|\downarrow\downarrow\uparrow\rangle \end{pmatrix}, \tag{36}$$

respectively. They belong, obviously, to two orthogonal mixed representations of the symmetric group of three elements [3] and are, respectively, antisymmetric and symmetric under permutations of two elements $i = 1, 2$. In the quark theory of the nucleon these mixed representations for spin are multiplied by the corresponding mixed representations for isospin and then added together in order to form an overall symmetric representation for spin and isospin.

We can conclude that in the case of $s = 1/2$ (like e.g. for the nucleon) the wave equation (19) reduces to the Dirac equation (but now describing the *internal* motion):

$$\left[\vec{\alpha} \cdot \vec{p} + 3\beta m + 3V\left(\frac{2}{\sqrt{3}}r\right)\right] \psi_{1/2} = E\psi_{1/2}, \tag{37}$$

where

$$\vec{\alpha} = \gamma^5 \vec{\sigma}, \quad \vec{\sigma} = \begin{pmatrix} \vec{\sigma}^{\mathrm{P}}, & 0 \\ 0, & \vec{\sigma}^{\mathrm{P}} \end{pmatrix} \tag{38}$$

and β and γ^5 are given in Eq. (20) or (29) with

$$\mathbf{1} = \mathbf{1}^{\mathrm{P}} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}. \tag{39}$$

So, in this case we are meeting familiar calculatory problems.

In particular, for a Coulombic attraction $V(r_{ij}) = -\alpha_c/r_{ij}$ such a Dirac equation leads to energy levels described by the Sommerfeld formula

$$E = 3m\left\{1+\left[\frac{3\alpha_{\rm eff}}{n+\sqrt{(j+1/2)^2-(3\alpha_{\rm eff})^2}}\right]^2\right\}^{-1/2} \tag{40}$$

with $n_r = 0, 1, 2, \ldots, j = 1/2, 3/2, 5/2, \ldots$ and $\alpha_{\rm eff} = \alpha_c\sqrt{3}/2$. Here, the critical value of α_c is $\alpha_c = 2/3\sqrt{3} \simeq 0.385$. Thus, taking for example $\alpha_c = 2\alpha_s/3$ as in the case of one--gluon exchange between two quarks in the nucleon, we get for α_s the critical value $\alpha_s = 1/\sqrt{3} \simeq 0.577$. For such α_s the ground-state energy (with $n_r = 0$ and $j = 1/2$) would become zero, independently of the mass m.

In the case of $s = 3/2$ (like e.g. for the Δ baryon) the wave equation (19) reduces to a more complicated equation because of the higher-than-two dimension of $\vec{\Sigma}^{\rm P}$ matrices given in Eq. (34). But, formally, it can be written down in the way analogical to Eq. (37):

$$\left[\vec{A}\cdot\vec{p}+3\beta m+3V\left(\frac{2}{\sqrt{3}}r\right)\right]\psi_{3/2} = E\psi_{3/2}, \tag{41}$$

where

$$\vec{A} = \gamma^5\vec{\Sigma}, \quad \vec{\Sigma} = \begin{pmatrix}\vec{\Sigma}^{\rm P}, & 0\\ 0, & \vec{\Sigma}^{\rm P}\end{pmatrix} \tag{42}$$

and β and γ^5 are given in Eq. (20) or (29) with

$$\mathbf{1} = \begin{pmatrix}1, & 0, & 0, & 0\\ 0, & 1, & 0, & 0\\ 0, & 0, & 1, & 0\\ 0, & 0, & 0, & 1\end{pmatrix}. \tag{43}$$

Of course, Eq. (41) can be represented as in Eq. (21) or (30) with $\vec{S}^{\rm P} = \vec{\Sigma}^{\rm P}/2$, when Eq. (20) or (29) holds. But other representations of β and γ^5 may be also used.

Sometimes in the case of $s = 3/2$ it may be convenient to operate with a new vector $\otimes$ bispinor wave function $\vec{\psi}_{3/2}$ instead of the wave function $\psi_{3/2} = P_{3/2}\psi$ being a spinor $\otimes$ spinor $\otimes$ bispinor reduced to $s = 3/2$:

$$(\vec{\sigma}_1+\vec{\sigma}_2+\vec{\sigma}_3)\psi_{3/2} = \vec{\Sigma}\psi_{3/2}. \tag{44}$$

The way of constructing such a wave function follows from the formula

$$(\vec{\sigma}_1-\vec{\sigma}_2)P_{3/2} = P_0(\vec{\sigma}_1-\vec{\sigma}_2)P_{3/2} \tag{45}$$

being a consequence of the definition (23) of $P_{3/2}$ and the first identity (27). Thus, the operator (45) acting on ψ makes it a scalar with respect to the spinor indices α_1 and α_2 of particles $i = 1, 2$, introducing instead a vector index. Then, in fact, the dependence of ψ on the indices α_1 and α_2 factorizes out in the form of a constant matrix $2i(\sigma_2)_{\alpha_1\alpha_2}$ and so can be simply dropped or rather summed as $\mathrm{Tr}[(i\sigma_2)^{\mathrm{T}}\,\frac{1}{2}\,(\vec{\sigma}_1-\vec{\sigma}_2)\psi_{3/2}]$ (in the final equations, where α_1 and α_2 become free). Hence

$$\vec{\psi}_{3/2} = \frac{\vec{\sigma}_1-\vec{\sigma}_2}{2}\,\psi_{3/2}\;(\psi_{3/2} = P_{3/2}\psi) \tag{46}$$

is the new function, subject to the constraint

$$\vec{\sigma}_3\cdot\vec{\psi}_{3/2} = 0 \tag{47}$$

that follows from the second identity (26) and expresses the reduction to $s = 3/2$. Note that the constraint (47) implies

$$\vec{\psi}_{3/2} = i\vec{\sigma}_3\times\vec{\psi}_{3/2} \tag{48}$$

due to the identity

$$\vec{a} = \vec{\sigma}_3(\vec{\sigma}_3\cdot\vec{a})+i\vec{\sigma}_3\times\vec{a} \tag{49}$$

valid for any $\vec{a}$.

In order to obtain the wave equation for $\vec{\psi}_{3/2}$ we multiply Eq. (19) from the left by $\frac{1}{2}\,(\vec{\sigma}_1-\vec{\sigma}_2)P_{3/2}$ and make use of the second identity (27) (with the upper sign):

$$(E-3\beta m-3V-\gamma^5\vec{\sigma}_3\cdot\vec{p})\vec{\psi}_{3/2} = 2i\gamma^5\vec{p}\times\vec{\psi}_{3/2}, \tag{50}$$

where the constraint (47) holds. Here, using Eq. (48) we can write

$$i\vec{p}\times\vec{\psi}_{3/2} = (\vec{\sigma}_3\cdot\vec{p})\vec{\psi}_{3/2}-\vec{\sigma}_3(\vec{p}\cdot\vec{\psi}_{3/2}), \tag{51}$$

so the wave equation (50) takes the following final form

$$\left(\frac{E}{3}-\beta m-V-\gamma^5\vec{\sigma}_3\cdot\vec{p}\right)\vec{\psi}_{3/2} = -\tfrac{2}{3}\gamma^5\vec{\sigma}_3(\vec{p}\cdot\vec{\psi}_{3/2}) \tag{52}$$

with the constraint (47). Here,

$$\vec{\sigma}_3 = \begin{pmatrix}\vec{\sigma}_3^{\mathrm{P}}, & 0\\ 0, & \vec{\sigma}_3^{\mathrm{P}}\end{pmatrix} \tag{53}$$

and β and γ^5 are given in Eq. (20) or (29) with

$$\mathbf{1} = \mathbf{1}^{\mathrm{P}} = \begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix}, \tag{54}$$

but other representations may be also used.

The equations (52) and (47) for $\vec{\psi}_{3/2}$ are in an analogy with the Rarita–Schwinger equations for spin $s = 3/2$ [4], but describe the *internal* motion. Formally, they differ from the latter equations by the absence of time component of the wave function $\vec{\psi}_{3/2}$ in contrast to the Rarita-Schwinger wave function (being a four-vector $\otimes$ bispinor) and also by the fact that the right-hand-side of Eq. (52) is $\neq 0$. Both facts are related to the absence of explicit relativistic covariance of internal motion within a composite particle, in contrast to its external motion (at least if the former is described by the conventional one-time equation like (12)). In the conventional one-time description of internal motion there is no relative time nor the corresponding energy (the energy E being connected with the usual time measured in the centre-of-mass frame). On the other hand, the relative coordinates and corresponding momenta appear, of course.

4. Separation of angular coordinates for $s = 3/2$

While the separation of angular coordinates for $s = 1/2$ from Eq. (37) (having the Dirac form) is a familiar problem [5], such a separation for $s = 3/2$ from Eq. (41) or (52) (with the constraint (47) in the case of the latter equation) deserves a description. Here, we will present such a separation from Eq. (52) (and Eq. (47)), based on a multipole technique worked out previously in the case of Breit equation [6].

The method consists of two steps. Firstly, we expand the wave function $\vec{\psi}$ into three parts: "electric", "longitudinal" and "magnetic" defined by means of three linearly independent (but not all normalized and Hermitian) vector operators:

$$\vec{e}_{\mathrm{E}} = \hat{r}, \quad \vec{e}_{\mathrm{L}} = -\frac{\partial}{\partial \hat{r}} \frac{1}{\vec{L}^2}, \quad \vec{e}_{\mathrm{M}} = -\left(\hat{r} \times \frac{\partial}{\partial \hat{r}}\right) \frac{1}{\vec{L}^2} \gamma^5, \tag{55}$$

where

$$\hat{r} = \frac{\vec{r}}{r}, \quad \frac{\partial}{\partial \hat{r}} = r \frac{\partial}{\partial \vec{r}} - \vec{r} \frac{\partial}{\partial r}. \tag{56}$$

The operators (56) have the following calculatory properties:

$$r^2 = 1, \quad \hat{r} \cdot \frac{\partial}{\partial \hat{r}} = 0, \quad \frac{\partial}{\partial \hat{r}} \cdot \hat{r} = 2, \quad \left(\frac{\partial}{\partial \hat{r}}\right)^2 = -\vec{L}^2 \tag{57}$$

and

$$\hat{r} \times \frac{\partial}{\partial \hat{r}} = \vec{r} \times \frac{\partial}{\partial \vec{r}} = i\vec{L}, \quad \hat{r} \times \vec{L} = i\left(\frac{\partial}{\partial \hat{r}} - 2\hat{r}\right), \quad \frac{\partial}{\partial \hat{r}} \times \frac{\partial}{\partial \hat{r}} = i\vec{L}. \tag{58}$$

Note that the contragradient operator basis is given by

$$\vec{e}^{\,\mathrm{E}} = \hat{r}, \quad \vec{e}^{\,\mathrm{L}} = -\frac{\partial}{\partial \hat{r}}, \quad \vec{e}^{\,\mathrm{M}} = -\left(\hat{r} \times \frac{\partial}{\partial \hat{r}}\right) \gamma^5 \tag{59}$$

since then

$$\vec{e}^{\,A+} \cdot \vec{e}_B = \delta^A_B \quad (A, B = \mathrm{E}, \mathrm{L}, \mathrm{M}) \tag{60}$$

due to the formulae

$$\vec{e}^{\,\mathrm{E}+} = \hat{r}, \quad \vec{e}^{\,\mathrm{L}+} = \frac{\partial}{\partial \hat{r}} - 2\hat{r}, \quad \vec{e}^{\,\mathrm{M}+} = \left(\hat{r} \times \frac{\partial}{\partial \hat{r}}\right) \gamma^5 \tag{61}$$

and Eqs. (57) and (58). Thus, we can write

$$\vec{\psi} = \sum_A \vec{e}_A \psi^A, \tag{62}$$

where

$$\psi^A = \vec{e}^{A+} \cdot \vec{\psi} \quad (A = \mathrm{E, L, M}). \tag{63}$$

Using the convenient representation where

$$\beta = \begin{pmatrix} \mathbf{1}^\mathrm{P}, & 0 \\ 0, & -\mathbf{1}^\mathrm{P} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0, & i\mathbf{1}^\mathrm{P} \\ -i\mathbf{1}^\mathrm{P}, & 0 \end{pmatrix}, \quad \vec{\sigma}_3 = \begin{pmatrix} \vec{\sigma}_3^\mathrm{P}, & 0 \\ 0, & \vec{\sigma}_3^\mathrm{P} \end{pmatrix}$$

$$\vec{\psi} = \begin{pmatrix} \vec{\psi}_+ \\ \vec{\psi}_- \end{pmatrix}, \quad \psi^A = \begin{pmatrix} \psi_+^A \\ \psi_-^A \end{pmatrix}, \tag{64}$$

we get

$$\psi_\pm^\mathrm{E} = \hat{r} \cdot \vec{\psi}_\pm, \quad \psi_\pm^\mathrm{L} = \left(\frac{\partial}{\partial \hat{r}} - 2\hat{r}\right) \cdot \vec{\psi}_\pm, \quad \psi_\pm^\mathrm{M} = \pm i \left(\hat{r} \times \frac{\partial}{\partial \hat{r}}\right) \cdot \vec{\psi}_\mp. \tag{65}$$

Because of γ^5 appearing in the definition of $\vec{e}_\mathrm{M}$, all $\psi_\pm^A$ have the same intrinsic parity equal to ± 1, respectively.

Secondly, we separate the angular coordinates $\hat{r} = (\theta, \phi)$ from the radial coordinate r by the substitution

$$\psi_\pm^A(\vec{r}) = \psi_\pm^A(r) \left\langle \hat{r} \middle| j \mp \frac{\varepsilon}{2} jm \right\rangle, \tag{66}$$

where the angular part is a Pauli spinor given by

$$\langle \hat{r} | j_{12} jm \rangle = \frac{1}{2j_{12}+1} \begin{pmatrix} \sqrt{j_{12}+\frac{1}{2}\pm m}\; Y_{j_{12}m-\frac{1}{2}}(\hat{r}) \\ \mp \sqrt{j_{12}+\frac{1}{2}\mp m}\; Y_{j_{12}m+\frac{1}{2}}(\hat{r}) \end{pmatrix}, \tag{67}$$

the upper/lower signs in Eq. (67) corresponding to $j_{12} = j \pm \frac{1}{2}$, respectively. Here, $\varepsilon = +1$ or -1 is determined by the total parity

$$P = (-1)^{j-\varepsilon/2}, \tag{68}$$

while $j = 1/2, 3/2, 5/2, \ldots$ is the quantum number of total angular momentum

$$\vec{J} = \vec{S} + \vec{L} = \tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2 + \vec{\sigma}_3) + \vec{L}. \tag{69}$$

This can be considered in two ways as either $\vec{J} = \vec{J}^{(12)} + \frac{1}{2}\vec{\sigma}_3$ or $\vec{J} = \vec{S}^{(12)} + \vec{J}^{(3)}$ with

$$\vec{J}^{(12)} = \vec{S}^{(12)} + \vec{L} = \tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2) + \vec{L}, \quad \vec{J}^{(3)} = \tfrac{1}{2}\vec{\sigma}_3 + \vec{L}. \tag{70}$$

Note the important relations

$$J_k^{(12)} \vec{e}_A = \vec{e}_A L_k, \quad J_k \vec{e}_A = \vec{e}_A J_k^{(3)} \tag{71}$$

valid in the case of $s_{12} = 1$, where we can use the vector representation

$$(\vec{S}^{(12)})_{lm} = -i\vec{i}_k \varepsilon_{klm} \; (k, l, m = 1, 2, 3). \tag{72}$$

They follow from the rotation-group commutation relations

$$[L_k, \vec{e}_A] = i\varepsilon_{klm}\vec{i}_l e_{Am} = -S_k^{(12)}\vec{e}_A, \tag{73}$$

where the matrix multiplication

$$(S_k^{(12)}\vec{e}_A)_l = (S_k^{(12)})_{lm} e_{Am} = -i\varepsilon_{klm} e_{Am} \tag{74}$$

is applied.

We can see from the substitution (66) (with (67)) that

$$\vec{L}^2 \psi_\pm^A = j_{12}(j_{12}+1)\psi_\pm^A \quad \text{where} \quad j_{12} = j \mp \frac{\varepsilon}{2} \tag{75}$$

(respectively), and

$$\vec{J}^{(3)2}\psi^A = j(j+1)\psi^A. \tag{76}$$

From Eqs. (75) and (76) we infer via the relations (71) that

$$\vec{J}^{(12)2}\vec{e}_A\psi_\pm^A = j_{12}(j_{12}+1)\vec{e}_A\psi_\pm^A \quad \text{where} \quad j_{12} = j \mp \frac{\varepsilon}{2} \tag{77}$$

(respectively), and

$$\vec{J}^2\vec{e}_A\psi^A = j(j+1)\vec{e}_A\psi^A. \tag{78}$$

Thus, summing Eqs. (77) and (78) over A = E, L, M we can conclude that functions $\vec{\psi}$ of the form given by Eqs. (62), (64), (66) and (67) (or their components $\vec{\psi}_\pm$ in the representation (64)) are eigenfunctions of $\vec{J}^2$ (or $\vec{J}^{(12)2}$) corresponding to the eigenvalues $j(j+1)$ $\left(\text{or } j_{12}(j_{12}+1) \text{ where } j_{12} = j \mp \frac{\varepsilon}{2}\right)$. They are also eigenfunctions of J_z and of the total parity with the eigenvalues m and P, respectively. So, using $\vec{\psi}$ of this form we can separate out the angular coordinates $\hat{r} = (\theta, \phi)$ from the rotational covariant wave equation (52) (and the constraint (47)), obtaining in this way a set of radial equations for $\psi_\pm^A(r)$ (A = E, L, M).

In practical calculations we are making use of the identity

$$\sigma_{3r}^{\mathrm{P}} \left\langle \hat{r} \middle| j \mp \frac{\varepsilon}{2} jm \right\rangle = -\left\langle \hat{r} \middle| j \pm \frac{\varepsilon}{2} jm \right\rangle \tag{79}$$

with $\sigma_{3r}^{\mathrm{P}} = \hat{r} \cdot \vec{\sigma}_3^{\mathrm{P}}$, which enables us to get the same angular parts in all terms of the wave equation, and then drop these parts. Also useful is the well known formula [5]

$$\vec{\alpha}_3 \cdot \vec{p} = -i\alpha_{3r}\left(\frac{\partial}{\partial r} + \frac{1}{r} - \beta\frac{A}{r}\right), \tag{80}$$

where $\vec{\alpha}_3 = \gamma^5\vec{\sigma}_3, \alpha_{3r} = \hat{r}\cdot\vec{\alpha}_3 = \gamma^5\sigma_{3r}$ and $\sigma_{3r} = \hat{r}\cdot\vec{\sigma}_3$, whereas

$$\Lambda = \beta(\vec{\sigma}_3\cdot\vec{L}+1) \tag{81}$$

commutes with α_{3r} (and $\vec{\alpha}_3\cdot\vec{p}$) and gives the relations

$$\vec{J}^{(3)2} = \Lambda^2-\tfrac{1}{4}, \quad \vec{L}^2 = \Lambda(\Lambda-\beta). \tag{82}$$

We can see from the first relation (82) and Eq. (76) that

$$\Lambda\psi^A = \lambda\psi^A, \tag{83}$$

where $j(j+1) = \lambda^2-\frac{1}{4}$, while the second relation (82) and Eq. (75) show that $j_{12}(j_{12}+1) = \lambda(\lambda\mp 1)$ with $j_{12} = j\mp\dfrac{\varepsilon}{2}$ (respectively). Jointly, we get

$$\lambda = \varepsilon(j+\tfrac{1}{2}). \tag{84}$$

We have also

$$\vec{L}^2\psi^A_\pm = \lambda(\lambda\mp 1)\psi^A_\pm, \quad (\vec{\sigma}^{\mathrm{P}}\cdot\vec{L}+1)\psi^A_\pm = \pm\lambda\psi^A_\pm. \tag{85}$$

Note that for norms we get

$$\|\vec{\psi}\|^2 = \|\psi^{\mathrm{E}}_+\|^2+\|\psi^{\mathrm{E}}_-\|^2+\frac{\|\psi^{\mathrm{L}}_+\|^2+\|\psi^{\mathrm{M}}_+\|^2}{\lambda(\lambda-1)}+\frac{\|\psi^{\mathrm{L}}_-\|^2+\|\psi^{\mathrm{M}}_-\|^2}{\lambda(\lambda+1)} \tag{86}$$

due to Eqs. (62), (55) and (85).

The multipole technique described above, when applied to the wave equation (52) and the constraint (47) for $\vec{\psi}(\vec{r}) \equiv \vec{\psi}_{3/2}(\vec{r})$, leads after lengthy algebraic calculations to the following set of eight radial equations for six radial functions $\psi^A_\pm(r)$ (A = E, L, M):

$$\left(\frac{E}{3}\mp m-V\right)\psi^{\mathrm{E}}_\pm = \mp\left(\frac{d}{dr}\pm\frac{\lambda\pm 2}{r}\right)\psi^{\mathrm{E}}_\mp\pm\tfrac{2}{3}\,d_\mp(\vec{\psi}),$$

$$\left(\frac{E}{3}\mp m-V\right)\psi^{\mathrm{L}}_\pm = \mp\left(\frac{d}{dr}\pm\frac{\lambda}{r}\right)\psi^{\mathrm{L}}_\mp\pm\left(\frac{d}{dr}\mp\frac{\lambda\mp 2}{r}\right)\psi^{\mathrm{E}}_\mp+\tfrac{2}{3}(\lambda\mp 1)d_\mp(\vec{\psi}),$$

$$\pm\left(\frac{E}{3}\mp m-V\right)\psi^{\mathrm{M}}_\mp = \left(\frac{d}{dr}\mp\frac{\lambda}{r}\right)\psi^{\mathrm{M}}_\pm\mp\left(\frac{d}{dr}\pm\frac{\lambda\pm 2}{r}\right)\psi^{\mathrm{E}}_\mp+\tfrac{2}{3}(\lambda\pm 1)d_\mp(\vec{\psi}) \tag{87}$$

and

$$\lambda\psi^{\mathrm{E}}_\pm\pm\psi^{\mathrm{L}}_\pm+\psi^{\mathrm{M}}_\mp = 0, \tag{88}$$

where the abbreviation

$$d_\pm(\vec{\psi}) \equiv \left(\frac{d}{dr}+\frac{2}{r}\right)\psi^{\mathrm{E}}_\pm+\frac{1}{r}\,\psi^{\mathrm{L}}_\pm \tag{89}$$

denotes the radial part of the divergence $i\vec{p} \cdot \vec{\psi}_{\pm}$, while Eq. (88) is the radial part of the constraint $\vec{\sigma}_3^{\mathrm{P}} \cdot \vec{\psi}_{\pm} = 0$ (multiplied by $-\lambda$). Here, $V = V\left(\frac{2}{\sqrt{3}} r\right)$. The set of radial equations (87) and (88) describes energy levels of our Lagrange triangle of Dirac particles with total spin $s = 3/2$, total angular momentum j and total parity $P = (-1)^{j-\varepsilon/2} (\lambda = \varepsilon(j+\frac{1}{2}))$.

The system of eight equations (87) and (88) for six unknown functions $\psi_{\pm}^{4}$ must be selfconsistent as it follows from its construction. Practically, this can be seen e.g. by adding the second and third of Eqs. (87), what after taking into account Eq. (88) leads to the first Eq. (87). So, only two of Eqs. (87), e.g. the first and second, supplemented by Eq. (88) are needed to solve the system.

5. *Relaxing the simplification* $\beta_1 = \beta_2 = \beta_3$

We are aware of a simplifying (or oversimplifying?) character of the assumption (17), though it makes practical the discussion of Lagrange triangle of Dirac particles, reducing the original number $4^3 = 64$ of Dirac wave-function components to $2 \cdot 2^3 = 16$ (what leads eventually to $2+4 = 6$ independent radial components for $s = 1/2$ and $s = 3/2$). So, for future purposes, it may be interesting to write down some split forms of the wave equation (16) before its simplification (17) is made, giving Eq. (19).

In the Dirac representation for three particles we can write

$$\beta_1 = \beta \otimes \mathbf{1}^{\mathrm{D}} \otimes \mathbf{1}^{\mathrm{D}}, \quad \beta_2 = \mathbf{1}^{\mathrm{D}} \otimes \beta \otimes \mathbf{1}^{\mathrm{D}}, \quad \beta_3 = \mathbf{1}^{\mathrm{D}} \otimes \mathbf{1}^{\mathrm{D}} \otimes \beta \tag{90}$$

and analogically for $\gamma_i^5, \vec{\sigma}_i$ and $\mathbf{1}$ (and $\vec{\alpha}_i = \gamma_i^5 \vec{\sigma}_i$), where $\beta, \gamma^5, \vec{\sigma}$ and $\mathbf{1}^{\mathrm{D}}$ (and $\vec{\alpha} = \gamma^5 \vec{\sigma}$) are the usual 4×4 Dirac matrices. We also have

$$\vec{\sigma}_1 = \begin{pmatrix} \vec{\sigma}^{\mathrm{P}}, & 0 \\ 0, & \vec{\sigma}^{\mathrm{P}} \end{pmatrix} \otimes \begin{pmatrix} \mathbf{1}^{\mathrm{P}}, & 0 \\ 0, & \mathbf{1}^{\mathrm{P}} \end{pmatrix} \otimes \begin{pmatrix} \mathbf{1}^{\mathrm{P}}, & 0 \\ 0, & \mathbf{1}^{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} \vec{\sigma}_1^{\mathrm{P}}, & 0 \\ 0, & \vec{\sigma}_1^{\mathrm{P}} \end{pmatrix}, \quad \text{etc.} \tag{91}$$

where $\vec{\sigma}_1^{\mathrm{P}} = \vec{\sigma}^{\mathrm{P}} \otimes \mathbf{1}^{\mathrm{D}} \otimes \mathbf{1}^{\mathrm{D}}$, etc., with $\vec{\sigma}^{\mathrm{P}}$ and $\mathbf{1}^{\mathrm{P}}$ being the usual 2×2 Pauli matrices. Splitting Eq. (16) in this representation and then combining the wave-function components $\psi_{\beta_1 \beta_2 \beta_3}$ ($\beta_i = \pm, i = 1, 2, 3$) into the new components

$$f_{\beta_3}^{\pm} = \frac{\psi_{++\beta_3} \pm \psi_{--\beta_3}}{\sqrt{2}}, \quad g_{\beta_3}^{\pm} = \frac{\psi_{+-\beta_3} \pm \psi_{-+\beta_3}}{\sqrt{2}}, \tag{92}$$

we obtain the following representation of the wave equation (16) (where we add to $3V\left(\frac{2}{\sqrt{3}} r\right)$ the scalar mutual interaction $(\sum_{i<j} \beta_i \beta_j)\, S\left(\frac{2}{\sqrt{3}} r\right)$ corresponding to $\sum_{i<j} \beta_i \beta_j$ $S(r_{ij})$):

$$[E-3V-S-(\vec{\alpha}_3 \cdot \vec{p}+\beta_3 m)] f^{\pm} \mp (\vec{\sigma}_1^{\mathrm{P}} \pm \vec{\sigma}_2^{\mathrm{P}}) \cdot \vec{p} g^{\pm} = 2(m+\beta_3 S) f^{\mp},$$

$$[E-3V+S-(\vec{\alpha}_3 \cdot \vec{p}+\beta_3 m)] g^{\pm} \mp (\vec{\sigma}_1^{\mathrm{P}} \pm \vec{\sigma}_2^{\mathrm{P}}) \cdot \vec{p} f^{\pm} = 0, \tag{93}$$

where

$$f^{\pm} = \begin{pmatrix} f_{+}^{\pm} \\ f_{-}^{\pm} \end{pmatrix}, \quad g^{\pm} = \begin{pmatrix} g_{+}^{\pm} \\ g_{-}^{\pm} \end{pmatrix} \tag{94}$$

are bispinors with respect to particle $i = 3$. Hence, for the wave-function components $f_{s_{12}}^{\pm} = P_{s_{12}}^{\mathrm{P}} f^{\pm}$ and $g_{s_{12}}^{\pm} = P_{s_{12}}^{\mathrm{P}} g^{\pm}$ with definite spin $s_{12} = 0$ or $s_{12} = 1$ of particles $i = 1, 2$ we get three independent subsets of equations:

$$(E-3V-S-D_3)f_0^{\pm} + \begin{Bmatrix} 0 \\ (\vec{\sigma}_1^{\mathrm{P}} - \vec{\sigma}_2^{\mathrm{P}}) \cdot \vec{p} g_1^{-} \end{Bmatrix} = 2(m+\beta_3 S) f_0^{\mp},$$

$$(E-3V+S-D_3)g_1^{-} + (\vec{\sigma}_1^{\mathrm{P}} - \vec{\sigma}_2^{\mathrm{P}}) \cdot \vec{p} f_0^{-} = 0 \tag{95}$$

and

$$(E-3V-S-D_3)f_1^{\pm} + \begin{Bmatrix} -(\vec{\sigma}_1^{\mathrm{P}} + \vec{\sigma}_2^{\mathrm{P}}) \cdot \vec{p} g_1^{+} \\ +(\vec{\sigma}_1^{\mathrm{P}} - \vec{\sigma}_2^{\mathrm{P}}) \cdot \vec{p} g_0^{-} \end{Bmatrix} = 2(m+\beta_3 S) f_1^{\mp},$$

$$(E-3V+S-D_3)g_1^{+} - (\vec{\sigma}_1^{\mathrm{P}} + \vec{\sigma}_2^{\mathrm{P}}) \cdot \vec{p} f_1^{+} = 0,$$

$$(E-3V+S-D_3)g_0^{-} + (\vec{\sigma}_1^{\mathrm{P}} - \vec{\sigma}_2^{\mathrm{P}}) \cdot \vec{p} f_1^{-} = 0 \tag{96}$$

and

$$(E-3V+S-D_3)g_0^{+} = 0. \tag{97}$$

Here, $D_3 = \vec{\alpha}_3 \cdot \vec{p} + \beta_3 m$ is the Dirac kinetic energy of particle $i = 3$. Note that "large--large" components with respect to particles $i = 1, 2$ are contained in $f_{s_{12}}^{\pm}$ (and are absent from $g_{s_{12}}^{\pm}$). Thus, the subsets of equations (95) and (96), as involving $f_{s_{12}}^{\pm}$ with $s_{12} = 0$ and $s_{12} = 1$, respectively, describe in a relativistic way states with these spins.

Now, in place of the wave-function components $f_1^{\pm}$ and $g_1^{\pm}$, each of them being spinor ⊗ spinor with respect to particles $i = 1, 2$ reduced to $s_{12} = 1$, we can introduce new vector components given by

$$\vec{f}^{\pm} = \frac{\vec{\sigma}_1^{\mathrm{P}} - \vec{\sigma}_2^{\mathrm{P}}}{2} f_1^{\pm}, \quad \vec{g}^{\pm} = \frac{\vec{\sigma}_1^{\mathrm{P}} - \vec{\sigma}_2^{\mathrm{P}}}{2} g_1^{\pm} \tag{98}$$

(for an analogy cf. Eq. (46)). In terms of these components the subsets of equations (95) and (96) take the form

$$(E-3V-S-D_3)f_0^{\pm} + \begin{Bmatrix} 0 \\ 2\vec{p} \cdot \vec{g}^{-} \end{Bmatrix} = 2(m+\beta_3 S) f_0^{\mp},$$

$$(E-3V+S-D_3)\vec{g}^{-} + 2\vec{p} f_0^{-} = 0 \tag{99}$$

and

$$(E-3V-S-D_3)\vec{f}^{\pm} + \begin{Bmatrix} -2i\vec{p} \times \vec{g}^{+} \\ 2\vec{p} g_0^{-} \end{Bmatrix} = 2(m+\beta_3 S)\vec{f}^{\mp},$$

$$(E-3V+S-D_3)\vec{g}^{+} - 2i\vec{p} \times \vec{f}^{+} = 0,$$

$$(E-3V+S-D_3)g_0^{-} + 2\vec{p} \cdot \vec{f}^{-} = 0, \tag{100}$$

respectively. Here, the wave-function components $f_0^{\pm}$ and $g_0^{\pm}$ are both scalars with respect to particles $i = 1, 2$ and bispinors with respect to particle $i = 3$, while the components $\vec{f}^{\pm}$ and $\vec{g}^{\pm}$ are both vectors and bispinors (note that g_0^{+} satisfying Eq. (97) is separated and so can be put equal to zero). Altogether there are $4^3 = 64$ components (minus 4).

To Eqs. (99) and (100) one can apply the multipole technique (described in Section 4) in order to derive the corresponding sets of radial equations for $s_{12} = 0$ and $s_{12} = 1$ (in the sense of "large-large" components with respect to particles $i = 1, 2$).

APPENDIX

The problem of three Dirac particles in general

The wave equation (12) for a system of three Dirac particles without the equilateral-triangle constraint (1) can be written in the center-of-mass frame as follows:

$$\left[\left(-\frac{\vec{\alpha}_1+\vec{\alpha}_2}{2}+\vec{\alpha}_3\right)\cdot\vec{p}+(\vec{\alpha}_1-\vec{\alpha}_2)\cdot\vec{\pi}+(\beta_1+\beta_2+\beta_3)m+V\right]\psi(\vec{r},\vec{\varrho}) = E\psi(\vec{r},\vec{\varrho}), \tag{A1}$$

where $V = \sum_{i<j} V(r_{ij})$ and it is assumed for simplicity that $m_1 = m_2 = m_3 (\equiv m)$. Here, $\vec{p} = -i\partial/\partial\vec{r}$ and $\vec{\pi} = -i\partial/\partial\vec{\varrho}$, and the definitions (3) and (4) are used for the centre-of-mass momenta and coordinates. In contrast to the case of Lagrange triangle of Dirac particles, the wave function ψ depends now on two vectors $\vec{r}$ and $\vec{\varrho}$ describing relative coordinates. But its dependence on three Dirac bispinor indices is here not more general than that discussed already in Section 5.

So, in the Dirac representation where ψ is described by the components $\psi_{\beta_1\beta_2\beta_3}$ ($\beta_i = \pm, i = 1, 2, 3$), we can introduce (as via Eq. (92)) the components $f_{\beta_3}^{\pm}$ and $g_{\beta_3}^{\pm}$ and build of them (as in Eq. (94)) the components $f^{\pm}$ and $g^{\pm}$ being bispinors with respect to particle $i = 3$. Then, we obtain the following representation of the wave equation

$$[E-V-(\vec{\alpha}_3\cdot\vec{p}+\beta_3 m)]f^{\pm}\pm\left[\frac{\vec{\sigma}_1^{\mathrm{P}}\pm\vec{\sigma}_2^{\mathrm{P}}}{2}\cdot\vec{p}-(\vec{\sigma}_1^{\mathrm{P}}\mp\vec{\sigma}_2^{\mathrm{P}})\cdot\vec{\pi}\right]g^{\pm} = 2mf^{\mp},$$

$$[E-V-(\vec{\alpha}_3\cdot\vec{p}+\beta_3 m)]g^{\pm}\pm\left[\frac{\vec{\sigma}_1^{\mathrm{P}}\pm\vec{\sigma}_2^{\mathrm{P}}}{2}\cdot\vec{p}-(\vec{\sigma}_1^{\mathrm{P}}\mp\vec{\sigma}_2^{\mathrm{P}})\cdot\vec{\pi}\right]f^{\pm} = 0. \tag{A2}$$

Hence, for the components $f_{s_{12}}^{\pm} = P_{s_{12}}^{\mathrm{P}} f^{\pm}$ and $g_{s_{12}}^{\pm} = P_{s_{12}}^{\mathrm{P}} g^{\pm}$, corresponding to definite spin $s_{12} = 0$ or $s_{12} = 1$ of particles $i = 1, 2$, we get this time an unsplit set of equations, viz.:

$$(E-V-D_3)f_0^{\pm}-\begin{Bmatrix}(\vec{\sigma}_1^{\mathrm{P}}-\vec{\sigma}_2^{\mathrm{P}})\cdot\vec{\pi}g_1^{+}\\ \frac{1}{2}(\vec{\sigma}_1^{\mathrm{P}}-\vec{\sigma}_2^{\mathrm{P}})\cdot\vec{p}g_1^{-}\end{Bmatrix} = 2mf_0^{\mp}$$

$$(E-V-D_3)g_0^{\pm}-\begin{Bmatrix}(\vec{\sigma}_1^{\mathrm{P}}-\vec{\sigma}_2^{\mathrm{P}})\cdot\vec{\pi}f_1^{+}\\ \frac{1}{2}(\vec{\sigma}_1^{\mathrm{P}}-\vec{\sigma}_2^{\mathrm{P}})\cdot\vec{p}f_1^{-}\end{Bmatrix} = 0,$$

$$(E-V-D_3)f_1^{\pm} - \left\{ \begin{matrix} -\frac{1}{2}(\vec{\sigma}_1^{P}+\vec{\sigma}_2^{P})\cdot\vec{p}g_1^{+} + (\vec{\sigma}_1^{P}-\vec{\sigma}_2^{P})\cdot\vec{\pi}g_0^{+} \\ +\frac{1}{2}(\vec{\sigma}_1^{P}-\vec{\sigma}_2^{P})\cdot\vec{p}g_0^{-} - (\vec{\sigma}_1^{P}+\vec{\sigma}_2^{P})\cdot\vec{\pi}g_1^{-} \end{matrix} \right\} = 2mf_1^{\mp},$$

$$(E-V-D_3)g_1^{\pm} - \left\{ \begin{matrix} -\frac{1}{2}(\vec{\sigma}_1^{P}+\vec{\sigma}_2^{P})\cdot\vec{p}f_1^{+} + (\vec{\sigma}_1^{P}-\vec{\sigma}_2^{P})\cdot\vec{\pi}f_0^{+} \\ +\frac{1}{2}(\vec{\sigma}_1^{P}-\vec{\sigma}_2^{P})\cdot\vec{p}f_0^{-} - (\vec{\sigma}_1^{P}+\vec{\sigma}_2^{P})\cdot\vec{\pi}f_1^{-} \end{matrix} \right\} = 0. \tag{A3}$$

Here, $D_3 = \vec{\alpha}_3 \cdot \vec{p} + \beta_3 m$.

Also in the present general case in place of the components $f_1^{\pm}$ and $g_1^{\pm}$ we can introduce new vector components $\vec{f}^{\pm}$ and $\vec{g}^{\pm}$ given as in Eq. (98). In terms of these components the set of equations (A3) takes the form

$$(E-V-D_3)f_0^{\pm} - \left\{ \begin{matrix} 2\vec{\pi}\cdot\vec{g}^{+} \\ \vec{p}\cdot\vec{g}^{-} \end{matrix} \right\} = 2mf_0^{\mp},$$

$$(E-V-D_3)g_0^{\pm} - \left\{ \begin{matrix} 2\vec{\pi}\cdot\vec{f}^{+} \\ \vec{p}\cdot\vec{f}^{-} \end{matrix} \right\} = 0,$$

$$(E-V-D_3)\vec{f}^{\pm} - \left\{ \begin{matrix} -i\vec{p}\times\vec{g}^{+} + 2\vec{\pi}g_0^{+} \\ \vec{p}g_0^{-} - 2i\vec{\pi}\times\vec{g}^{-} \end{matrix} \right\} = 2m\vec{f}^{\mp},$$

$$(E-V-D_3)\vec{g}^{\pm} - \left\{ \begin{matrix} -i\vec{p}\times\vec{f}^{+} + 2\vec{\pi}f_0^{+} \\ \vec{p}f_0^{-} - 2i\vec{\pi}\times\vec{f}^{-} \end{matrix} \right\} = 0. \tag{A4}$$

Here, the components $f_0^{\pm}$ and $g_0^{\pm}$ are scalars with respect to particles $i = 1, 2$ and bispinors with respect to particle $i = 3$, whilst the components $\vec{f}^{\pm}$ and $\vec{g}^{\pm}$ are vectors and bispinors. The number of all components is $4^3 = 64$.

Since each of these components depends on two vectors $\vec{r}$ and $\vec{\varrho}$, the multipole technique (presented in Section 4) cannot be applied directly to the separation of all angular coordinates in this general case.

It may be convenient to note that there is a simple mnemonic of obtaining from Eqs (A3) and (A4) the corresponding results for the Breit equation

$$[\vec{\alpha}_1\cdot\vec{p}_1 + \vec{\alpha}_2\cdot\vec{p}_2 + (\beta_1+\beta_2)m + V]\psi = E\psi \tag{A5}$$

which in the centre-of-mass frame assumes the form

$$[(\vec{\alpha}_1-\vec{\alpha}_2)\cdot\vec{p} + (\beta_1+\beta_2)m + V]\psi(\vec{r}) = E\psi(\vec{r}). \tag{A6}$$

We can see, comparing Eqs. (A1) and (A6), that to this end it is enough to put

$$D_3 \to 0, \quad \vec{p} \to 0, \quad \vec{\pi} \to \vec{p} \tag{A7}$$

in Eq. (A3) or (A4). Of course, in the case of Breit equation the multipole technique works to separate angular coordinates [6].

Note also that in order to get the quark-diquark model of the nucleon with a closely bounded diquark we may put approximately

$$\vec{\pi} \to 0 \tag{A8}$$

in the wave equation (A1) as well as in its split form (A3) or (A4) [7]. In this case the multipole technique also works, leading to radial equations.

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