DIRAC'S ELECTRIC MONOPOLE

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(Received June 29, 1984)

It is shown that an improper gauge transformation, similar to that applied by Dirac in his theory of magnetic monopoles but performed in a null plane, produces a finite electric charge provided the shape of the corresponding string is appropriately chosen.

PACS numbers: 14.80.Hv

1. The Dirac monopole

Dirac [1] made the famous observation that principles of quantum mechanics determine a natural unit of a magnetic flux. This may be seen as follows. The quantity

$$eA_\mu + \partial_\mu S,$$

where e is the elementary charge, A_μ is the electromagnetic vector potential and S is the phase, is gauge invariant:

$$e\delta A_\mu + \partial_\mu \delta S = 0.$$

Performing an improper gauge transformation

$$\delta S = \varphi = \operatorname{arctg} \frac{x^2}{x^1}$$

one has

$$\delta A_\mu = -\frac{1}{e}\,\partial_\mu \varphi$$

and

$$\oint \delta A_\mu dx^\mu = -\frac{2\pi}{e},$$

which is the natural unit of a magnetic flux.

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2. *Alternative forms of the Dirac vector potential*

The potential

$$A_\mu = -\frac{1}{e}\,\partial_\mu \varphi$$

is equivalent to the potential

$$A_\mu = \frac{2\pi}{e}\,\Theta(x^1)\partial_\mu \Theta(x^2),$$

where Θ is the Heaviside step function; this is seen from the fact that both potentials give the same field

$$F_{12} = \frac{2\pi}{e}\,\delta(x^1)\delta(x^2)$$

other components being zero.

Now, in the theory of relativity one has three kinds of planes, spacelike, timelike and null; for example the Cartesian basis vectors e_1, e_2 span a spacelike plane, the vectors e_0, e_3 span a timelike plane, the vectors e_0+e_3, e_2 span a null plane. It is clear that an improper gauge transformation similar to the Dirac one but performed in a timelike or null plane gives a field physically different from that obtained in the spacelike case. This is seen in the table below.

The Dirac potential for	The corresponding field
spacelike plane spanned by e_1, e_2 $A_\mu = \frac{2\pi}{e}\,\Theta(x^1)\partial_\mu\Theta(x^2)$	$F_{12} = \frac{2\pi}{e}\,\delta(x^1)\delta(x^2)$
timelike plane spanned by e_0, e_3 $A_\mu = \frac{2\pi}{e}\,\Theta(x^0)\partial_\mu\Theta(x^3)$	$F_{03} = \frac{2\pi}{e}\,\delta(x^0)\delta(x^3)$
null plane spanned by e_0+e_3, e_2 $A_\mu = \frac{2\pi}{e}\,\Theta(x^2)\partial_\mu\Theta(x^0-x^3)$	$F_{20} = F_{32} = \frac{2\pi}{e}\,\delta(x^2)\delta(x^0-x^3)$

3. *The Dirac potential which has a finite charge*

Let us put forward the question: is it possible to create a finite electric charge by means of an improper gauge transformation similar to the Dirac one? It is obvious that the answer is negative in the spacelike case because a purely magnetic field cannot have an electric charge; it is also negative in the timelike case because, as seen from the table in the preceding section, the field created has an infinitely small duration while charge is something which has always an infinite duration.

In the null case, however, the answer is not obvious because the field created has an electric part and an infinite duration. The charge density associated with the potential

$$A_\mu = \frac{2\pi}{e}\,\Theta(x^2)\partial_\mu\Theta(x^0-x^3)$$

is

$$j_0 = -\frac{1}{2e}\,\delta(x^0-x^3)\delta'(x^2).$$

The total charge

$$Q = \int j_0 d^3x$$

must be put equal to zero on symmetry grounds; we shall see, however, that it is possible to deform the line of singularity in such a way that the total charge is well defined and finite.

Consider the potential

$$A_\mu = -\frac{2\pi}{e}\,\Theta[f(x^1, x^2)]\partial_\mu\Theta(x^0-x^3).$$

The total charge within the sphere of radius $r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$ is

$$Q(r) = \frac{1}{2e}\int d\Omega r\delta(x^0-x^3)\delta[f(x^1, x^2)]\left(x^1\frac{\partial f}{\partial x^1}+x^2\frac{\partial f}{\partial x^2}\right),$$

where $d\Omega$ is the measure of the solid angle on the sphere. Since the charge is conserved one can integrate for $x^0 = 0$; introducing the cylindrical coordinates

$$x^1 = \varrho\cos\varphi, \qquad x^2 = \varrho\sin\varphi,$$

one has

$$Q = \lim_{\varrho\to\infty}\frac{1}{2e}\int_0^{2\pi} d\varphi\delta[f(\varrho, \varphi)]\varrho\frac{\partial f}{\partial\varrho}.$$

It is clear from this formula that the line $f(\varrho, \varphi) = 0$ must extend to infinity, otherwise the charge vanishes. Investigating charges associated with simple lines like a straight line, a parabola or a hyperbola one finds $Q = 0$ in each case. It is clear that a line which approaches infinity in a simple way, like a parabola or a hyperbola, is equivalent, as far as the total charge is concerned, to a straight line or a collection of straight lines and therefore the total charge for such a line vanishes.

It turns out that the only line which gives a well defined and finite charge is the logarithmic spiral, the *spira mirabilis* of James Bernoulli [2]. This may be seen as follows.

For the logarithmic spiral

$$\varrho = \varrho_0 e^{\alpha\varphi}$$

where ϱ_0 and α are constants; hence

$$\ln\frac{\varrho}{\varrho_0} - \alpha\varphi = 0.$$

The function

$$f(\varrho, \varphi) = \operatorname{tg}\left(\frac{1}{2\alpha}\ln\frac{\varrho}{\varrho_0} - \frac{\varphi}{2}\right)$$

is a single valued, analytic function such that the equation

$$f(\varrho, \varphi) = 0$$

gives implicitly the logarithmic spiral in question. Putting the function $f(\varrho, \varphi)$ into the formula for the total charge we have

$$Q = \lim_{\varrho\to\infty}\frac{1}{4e\alpha}\int_0^{2\pi} d\varphi\,\frac{\delta\left[\operatorname{tg}\left(\frac{1}{2\alpha}\ln\frac{\varrho}{\varrho_0} - \frac{\varphi}{2}\right)\right]}{\cos^2\left(\frac{1}{2\alpha}\ln\frac{\varrho}{\varrho_0} - \frac{\varphi}{2}\right)}.$$

On the spiral ϱ is an increasing function of φ; therefore for a given ϱ there is only one φ such that the argument of the δ-function vanishes; hence

$$Q = \lim_{\varrho\to\infty}\frac{1}{2e\alpha} = \frac{1}{2e\alpha}.$$

The calculation above shows that the charge is not only well defined and finite but, moreover, it is concentrated on the null straight line $x^0 - x^3 = 0$, $x^1 = x^2 = 0$. In other words the potential

$$A_\mu = -\frac{2\pi}{e}\Theta\left[\operatorname{tg}\left(eQ\ln\frac{\varrho}{\varrho_0} - \frac{\varphi}{2}\right)\right]\partial_\mu\Theta(x^0 - x^3)$$

is a potential of a point charge Q which moves with the velocity of light along the z-axis.

REFERENCES

[1] P. A. M. Dirac, Proc. Roy. Soc. A133, 60 (1931). See also P. Goddard and D. I. Olive, *New Developments in the Theory of Magnetic Monopoles*, CERN preprint TH. 2445 (1977).

[2] H. Weyl, *Symmetry*, Princeton University Press, Princeton 1952.