NONSINGULAR COSMOLOGY AND THE GRAVITATIONAL LAGRANGIANS IN THE GAUGE THEORY OF GRAVITY

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Within the framework of the gauge approach to gravitation, including terms in the lagrangian quadratic in the curvature and torsion tensors, restrictions on the indefinite parameters of the lagrangians are obtained. It is shown, that the simultaneous consideration of the cosmological problem, quantisation of gravity and Birkhoff's theorem reduces to the two 5-parametric gravitational lagrangians.

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At present various gauge theories of gravity are known, which differ one from another by choice of the gauge group of transformations. In this paper we consider the gauge theory of gravity, for which the group of the gauge transformations is the direct product of the group of general coordinate transformations and the group of tetrad Lorentz transformations (see, for example, [1–4]). Then the gravity is described by the field of tetrads $h^i{}_\mu$ and the rotating Ricci coefficients $A^{ik}{}_\mu$[1]. Due to the dynamical independence of the fields $h^i{}_\mu$ and $A^{ik}{}_\mu$ the space-time have both the curvature and the torsion. Usually one connects the torsion with the spin angular momentum of the matter, because in the simplest gauge theory of gravity — the Einstein-Cartan theory — there exists an algebraical dependence between the spin angular momentum and the torsion, and the vanishing of the spin leads to the vanishing of the torsion too. However, as it was shown in [5], the torsion of the space-time can play an important role in the case of the spinless matter with the extremal high mass densities by including terms in the gravitational lagrangian quadratic in the curvature and torsion tensors. By some restrictions on the parameters of the gravitational lagrangian this theory leads to the conclusion about the existence of the limiting mass density in the nature and this allows one to solve the problem of the gravitational singularity [5, 6].

In the gauge theory of gravity the choice of the gravitational lagrangian is important. The principle of the local gauge invariance itself gives only the form of the gauge fields

[1] $i, k, \ldots$ are anholonomic Lorentz indices, $\mu, \nu, \ldots$ are holonomic coordinate indices, the signature is equal to -2, the light velocity $c = 1$.

and their strengths from the gauge group. However this principle does not allow to determine the explicit form of lagrangian for the gauge field.

In the present paper the restrictions on the parameters of the gravitational lagrangian in the gauge theory of gravity are considered. We use the most general expression for the gravitational lagrangian

$$L_g = h[f_0 F + F^{\alpha\beta\mu\nu}(f_1 F_{\alpha\beta\mu\nu} + f_2 F_{\alpha\mu\beta\nu} + f_3 F_{\mu\nu\alpha\beta})$$

$$+ F^{\mu\nu}(f_4 F_{\mu\nu} + f_5 F_{\nu\mu}) + f_6 F^2 + S^{\alpha\mu\nu}(a_1 S_{\alpha\mu\nu} + a_2 S_{\nu\mu\alpha}) + a_3 S^{\alpha}{}_{\mu\alpha} S_{\beta}{}^{\mu\beta}],$$

$$(F^{\mu\nu} = F^{\alpha}{}_{\mu\alpha\nu}, \quad F = F^{\mu}{}_{\mu}, \quad h = \det(h^{i}{}_{\mu})), \tag{1}$$

where

$$F^{ik}{}_{\mu\nu} = 2\partial_{[\mu} A^{ik}{}_{\nu]} + 2A^{il}{}_{[\mu} A^{k}{}_{|l|\nu]},$$

$$S^{i}_{\mu\nu} = \partial_{[\nu} h^{i}{}_{\mu]} - h_{k[\mu} A^{ik}{}_{\nu]}$$

and f_0, f_i ($i = 1, 2, \ldots, 6$), a_k ($k = 1, 2, 3$) are indefinite constants. One of the constants f_3, f_5 and f_6 can be excluded according to Gauss-Bonnet theorem [3, 7], therefore we assume further that $f_6 = 0$. One can obtain the restrictions for the parameters of the lagrangian (1), requiring that the corresponding gravitational equations should lead to the most satisfactory physical consequences. At first we note that in the case of the material systems with the rather small mass densities, the principle of the correspondence with General Relativity (GR) leads to $f_0 = \dfrac{1}{16\pi G}$, where G is the Newton's gravitational constant [5, 6]. The presence of the term linear in the curvature in L_g is essential from the point of view of the quantum theory of gravity [8, 3] and it is also necessary to avoid the unphysical spherically symmetric solutions, analogous to the unphysical solutions in the Yang theory of gravity (see, for example, [9]).

The investigation of homogeneous isotropic cosmological models with torsion [5] leads to some restrictions for the gravitational lagrangian. The generalised cosmological Friedmann equations obtained in [5] by the assumption of the invariance with respect to spacial inversions depend on the following two parameters: $a = 2a_1 + a_2 + 3a_3$ and $f = f_1 + \dfrac{f_2}{2} + f_3 + f_4 + f_5 + 3f_6$. If $f \to 0$, these equations transform into the ordinary cosmological equations of GR. Therefore it is necessary to require $f \neq 0$ and to keep in the gravitational lagrangian terms, quadratic in the curvature tensor. When $a = 0$ ($f \neq 0$) the generalised cosmological Friedmann equations lead to the conclusion about the existence of the limiting mass density and allow one to solve the problem of cosmological gravitational singularity. Nonsingular in the metric solutions exist also when $a \neq 0$, and they were investigated in [6]. However, it should be noted, that when $a \neq 0$ singular solutions exist in general together with nonsingular solutions [13]. The sign of the parameter f depends of the state equation and in the case $p < \dfrac{\varrho}{3}$ (ϱ — mass density, p —

pressure) f must be negative[2]. Thus, we have

$$a = 0, \quad f < 0. \tag{2}$$

The requirement $a = 0$ agrees with the Birkhoff's theorem. As was shown in [11, 12], by fulfilment of the restrictions

$$2a_1 + a_2 = a_3 = 0 \tag{3}$$

and, moreover, of one of the following two conditions

$$4f_1 + f_2 + 2f_4 = 0, \tag{4}$$

$$f_2 + 4f_3 - f_4 + f_5 = 0 \tag{5}$$

the Birkhoff's theorem with spatial reflection invariance takes place, i.e. in the vacuum spherical-symmetrical case the metric of the space-time coincides with the Schwarzschild metric and the torsion equals to zero. The fulfilment of the Birkhoff's theorem is essential due to the fact that any vacuum solution of the Einstein's equations with zero torsion satisfies also the gravitational field equations corresponding to the lagrangian (1) [3, 5]. In this connection the theorem given provides for the absence of the unphysical spherically symmetric vacuum solutions in the theory of gravity which is being considered.

The conditions (2) lead to the lagrangians with the 8 independent parameters. Some additional restrictions on the parameters of the gravitational lagrangian can be obtained from the quantisation of the gravitational field [8, 3]. The requirement of the absence of ghosts and tachyons leads to 12 types of six-parameter gravitational lagrangians [8]. Only one of them agrees with the conditions (2). The parameters of a given gravitational lagrangian must satisfy the following conditions

$$2a_1 + a_2 = 0, \quad a_3 = 0, \quad 4f_1 + f_2 = 0, \quad f_0 > 0, \quad f < 0, \tag{6}$$

$$a_1 > 0, \quad f_1 < 0, \quad 4(f_1 - f_3) + f_4 - f_5 > 0. \tag{7}$$

The condition $f_0 > 0$ determines the presence of the massless graviton by quantisation and inequalities (7) are connected with the presence of the massive tordions with spin and parity $J^P \doteq 1^+, 0^-$. Taking into account the equalities (6) the conditions of the presence of these particles have the following form:

$$(0^-): f_1 < 0, \quad 3a_1 - f_0 > 0, \tag{8}$$

$$(1^+): 4(f_1 - f_3) + f_4 - f_5 > 0, \quad a_1(3a_1 + f_0) > 0. \tag{9}$$

The additional restrictions (4) and (5) connected with the Birkhoff's theorem, lead to two five-parameter gravitational lagrangians. One of them is defined by the conditions

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[2] Note, that in the framework of quantum chromodynamics for superhigh mass densities $p < \frac{\varrho}{3}$ [10].

(6), (7) and $f_4 = 0$ and it describes the massive tordions with $J^P = 1^+, 0^-$. Another lagrangian is defined by the conditions (6), (8) and $f_2+4f_3-f_4+f_5 = 0$ and it describes one massive tordion with $J^P = 0^-$. By excluding the massive tordions from a given five-parameter lagrangian by replacing the inequalities in (8) and (9) by the corresponding equalities, a four- and three-parameter lagrangian can be obtained. The four-parameter lagrangians are determined by means of (6) and one of the following set conditions:

$$f_1 = f_4 = 0, \quad 4f_3+f_5 < 0, \quad a_1(3a_1+f_0) > 0, \tag{10}$$

$$f_4 = 0, \quad a_1 = 0, \quad f_1 < 0, \tag{11}$$

$$f_4 = 0, \quad 3a_1+f_0 = 0, \tag{12}$$

$$f_2+4f_3-f_4+f_5 = 0, \quad 3a_1+f_0 = 0, \tag{13}$$

$$f_1 = 0, \quad 4f_3-f_4+f_5 = 0. \tag{14}$$

The corresponding three-parameter lagrangian satisfies the conditions (6) and

$$f_1 = f_4 = 0, \quad a_1 = 0. \tag{15}$$

In the theory with the four-parameter lagrangians, determined by means of the (6) and (10), the massive tordion with $J^P = 1^+$ are present, and in the theory with the lagrangian, satisfying the conditions (6) and (11) the massive tordion with $J^P = 0^-$ are present. The remaining four- and three-parameter lagrangians connected with (6) and one of the set conditions (12)–(15) do not contain the massive tordions. All other lagrangians, which satisfy the restrictions following from the simultaneous consideration of the cosmological problem, the quantisation and the Birkhoff's theorem are particular cases of the above obtained five-, four- and three-parameter lagrangians have additional conditions imposed on the parameter in the form of an equality.

Note that the conditions (2) were found from the generalised cosmological Friedmann equations, which were obtained by assumption of the invariance under spatial reflections. This led to exclusion of some components of the torsion tensor in the homogeneous isotropic space. It is interesting to obtain the gravitational equations for a homogeneous isotropic cosmology with torsion without the assumption of the spatial reflection invariance and to consider, by what restrictions for parameters of gravitational lagrangian this equations reduce to the generalised cosmological Friedmann equations [5].

In the homogeneous isotropic space the tetrad is chosen in the form:

$$h^i{}_\mu = \mathrm{diag}\left(1, \frac{R(t)}{\sqrt{1-\kappa r^2}}, R(t)r, R(t)r\sin\vartheta\right) \quad (\kappa = 0, \pm 1).$$

In this case the following components of the torsion tensor cannot be equal to zero

$$S^1{}_{10} = S^2{}_{20} = S^3{}_{30} \equiv S_1(t), \quad S_{123} = S_{231} = S_{312} \equiv S_2(t)\frac{R^3r^2}{\sqrt{1-\kappa r^2}}\sin\vartheta.$$

Then the gravitational equations corresponding to the lagrangian (1) for the homogeneous isotropic cosmological models have the following form [6]:

$$a\left(\frac{\dot{R}}{R}-S_1\right)S_1-2bS_2^2-2f_0A_2+4f(A_1^2-A_2^2)+2q_2(A_3^2-A_4^2) = -\frac{\varrho}{3}, \tag{16}$$

$$a\left(\dot{S}_1+2\frac{\dot{R}}{R}S_1-S_1^2\right)-2bS_2^2-2f_0(2A_1+A_2)4f(A_1^2-A_2^2)-2q_2(A_3^2-A_4^2) = p, \tag{17}$$

$$f[(\dot{A}_1+\dot{A}_2)+4S_1(A_1+A_2)]+[q_2A_3+(2f-q_1)A_4]S_2+\left(f_0+\frac{a}{8}\right)S_1 = 0, \tag{18}$$

$$q_2[(\dot{A}_3+\dot{A}_4)+4S_1(A_3+A_4)]-[4fA_2+2(q_1+q_2)A_1+(f_0-b)]S_2 = 0, \tag{19}$$

where

$$A_1 = \frac{(\dot{R}-2RS_1)^{\cdot}}{R}, \quad A_2 = \frac{\kappa+(\dot{R}-2RS_1)^{\cdot}}{R^2}-S_2^2,$$

$$A_3 = 2\frac{\dot{R}-2RS_1}{R}S_2, \quad A_4 = \frac{(RS_2)^{\cdot}}{R},$$

$$b = a_2-a_1, \quad q_1 = f_2-2f_3+f_4+f_5+6f_6, \quad q_2 = 2f_1-f_2$$

and a dot denotes the differentiation with respect to time. It is easy to show that, by the fulfilment of the correspondence principle with GR for gravitating systems with rather small mass densities (functions A_1 and A_2 must approximate their values in GR), in the case

$$2f_1-f_2 = 0 \tag{20}$$

the function S_2 vanishes due to equation (19), and equations (16)–(18) reduce to the generalised cosmological Friedmann equations. The condition (20) together with (6) and (9) leads to the four-parameter lagrangian.

Thus, as the above analysis shows, the consideration of the cosmological problem reduces to a restriction on the parameters of the gravitational lagrangian in the gauge theory of gravity.

REFERENCES

[1] T. W. B. Kibble, *J. Math. Phys.* **2**, 212 (1961); A. M. Brodskii, D. Ivanenko, H. A. Sokolik, *Acta Phys. Hungar.* **14**, 21 (1962); A. Trautman, *Ann. N. Y. Acad. Sci.* **262**, 241 (1975).

[2] F. W. Hehl, P. von der Heyde, G. D. Kerlick, J. M. Nester, *Rev. Mod. Phys.* **48**, 393 (1976).

[3] K. Hayashi, T. Shirafuji, *Progr. Theor. Phys.* **64**, 866; 883; 1435; 2222 (1980), **65**, 525 (1981); **66**, 318 (1981).

[4] V. Ponomariov, A. Tseytlin, Abstracts 9th International Conference on General Relativity and Gravitation, Jena, **3**, 641 (1980).

[5] A. V. Minkevich, *Izv. Akad. Nauk BSSR Ser. Fiz.-Mat.* no. 2, 87 (1980); *Phys. Lett.* **80A**, 232 (1980).

[6] V. I. Kudin, A. V. Minkevich, F. I. Fedorov, *Izv. Akad. Nauk BSSR Ser. Fiz.-Mat.* no. 4, 59 (1981).
[7] H. T. Nieh, *J. Mat. Phys.* **21**, 1439 (1980).
[8] E. Sezgin, P. van Nieuwenhuizen, *Phys. Rev.* **D21**, 3269 (1980); E. Sezgin, *Phys. Rev.* **D24**, 1677 (1981).
[9] A. H. Thompson, *Phys. Rev. Lett.* **34**, 507 (1975); R. Pavelle, *Phys. Rev. Lett.* **34**, 1114 (1975).
[10] E. V. Shuryak, *Phys. Repts.* **61**, 71 (1980); O. K. Kalashnikov, V. V. Klimov, *Yad. Fiz.* **33**, 1572 (1981).
[11] V. I. Kudin, A. V. Minkevich, *On Birkhoff's theorem in the gauge theory of gravity*, Dep. in VINITI, reg. Dep. N 4971–80 (1980).
[12] R. Rauch, H. T. Nieh, *Phys. Rev.* **D24**, 2029 (1981).
[13] F. Müller-Hoissen, *Phys. Lett.* **92A**, 433 (1982).