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EXACT BRANS-DICKE-BIANCHI TYPE-VIII AND TYPE-IX VACUUM SOLUTIONS

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The field equations of the Brans-Dicke scalar-tensor theory are investigated. Exact vacuum solutions are given for Bianchi type-VIII and type-IX models.

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1. *Introduction*

By now, much is known about exact solutions of the general theory of relativity (GRT) for spatially homogeneous but anisotropic space-times (Kramer et al. 1980; Lorenz 1983). These space-times belong either to the Bianchi types I–IX or to the Kantowski-Sachs class and are often interpreted as cosmological models (Ryan and Shepley 1975). However, only a restricted number of exact Bianchi solutions are known of the Brans-Dicke theory (BDT) (Ruban and Finkelstein 1972; Belinskii and Khalatnikov 1972; Nariai 1972a, b; Matzner et al. 1973; Rauchaudhuri 1975, 1979; Ruban and Finkelstein 1975; Johri and Goswami 1980, 1981; Banerjee and Santos 1982a, b; Chakravarti and De 1983; Lorenz-Petzold 1983a).

Amongst the various modifications of the GRT the BDT is treated most seriously (Weinberg 1972; Wesson 1980). The BDT is consistent with local observations in the solar system as long as the coupling parameter ω between the scalar and tensor components is about equal to or greater than 500 (Will 1980). In the limit $\omega \to \infty$ the BDT reduces to the GRT for a constant BDT-scalar field ϕ.

Dicke (1964) has claimed that the BDT is Machian in the sense that there are no solutions at all for an empty space and that space closes about a localized mass configuration. However, the first point has been proved to be wrong (O'Hanlon and Tupper 1972) for the Brans-Dicke-Friedmann-Robertson-Walker metric. The Bianchi types are the simplest generalizations of the FRW-models. The vacuum Brans-Dicke-Bianchi type-I solution

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has been first given by Ruban and Finkelstein (1972) (see also Belinskii and Khalatnikov (1972)). The vacuum Bianchi type-II solution has been found by us recently (Lorenz--Petzold 1983b). In this paper we consider the Bianchi type-VIII and type-IX models.

2. Field equations and their solutions

In choosing a local orthonormal basis σ^μ, we can put the metric of space-time in the form

$$ds^2 = \eta_{\mu\nu}\sigma^\mu\sigma^\nu, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. For a spatially homogeneous model we take

$$\sigma^0 = \omega^0 = dt, \qquad \sigma^i = R_i\omega^i \quad \text{(no sum)}, \tag{2}$$

where ω^i are time-independent differential one-forms and where, because of homogeneity, the R_i are functions of t only. The BDT-equations to be considered are (Lorenz 1980):

$$\dot{H}_i + 3HH_i + \frac{1}{2R^6}\left[R_i^4 - R_k^4 - R_l^4 + 2\delta n_i R_k^2 R_l^2\right] + (\ln\phi)^{\cdot}(\ln R_i)^{\cdot} = 0, \quad i = 1, 2, 3, \tag{3}$$

$$H_1H_2 + H_1H_3 + H_2H_3 = \tfrac{1}{2}\omega(\ln\phi)^{\cdot 2} - 3H(\ln\phi)^{\cdot}$$
$$+\tfrac{1}{4}\left[R_1^4 + R_2^4 + R_3^4 - 2\delta(R_2^2R_3^2 + R_1^2R_3^2 + \delta R_1^2R_2^2)\right]R^{-6}, \tag{4}$$

$$(R^3\dot{\phi})^{\cdot} = 0, \tag{5}$$

where i, k, l are in cyclic order, $H_i = \dot{R}_i/R_i$ are the Hubble parameters, $3H = \sum H_i$, $R^3 = R_1R_2R_3$, $n_1 = n_2 = n_3\delta = 1$, $\delta = -1, 1$ (VIII, IX) and $(\ )^{\cdot} = d/dt$.

We consider the locally rotationally symmetric (LRS)-case $R := R_1 = R_2$, $S := R_3$. (It can easily be seen that Eqs. (3) do not turn into each other under any permutation of the indices i, k, l for type-VIII, whereas for type-IX the intrinsic geometry of three-space does not privilege any direction of space. For type-VIII we can only equate R_1 with R_2 obtaining a symmetry about the third axis.) Introducing the new time variable η by $dt = SR^2 d\eta$, we obtain from Eq. (5)

$$\phi = a(\eta - \eta_0), \qquad a, \eta_0 = \text{const.}, \tag{6}$$

whereas the linear combination of Eq. (3) yields

$$(\ln y)'' + (\ln\phi)'(\ln y)' + 2\delta y = 0, \tag{7}$$

where $y = (RS)^2$ and $(\ )' = d/d\eta$. Eq. (7) can be solved to give

$$y = q^2 f \sinh^{-2} q(\eta - \eta_1 + g), \qquad \text{type-VIII}, \tag{8a}$$

$$y = q^2 f \cosh^{-2} q(\eta - \eta_1 + g), \qquad \text{type-IX}, \tag{8b}$$

where

$$f = c_1^2\phi^{-2}, \qquad g = \ln\phi^{c_1/a} - \eta + c_2 \tag{9}$$

and $q, c_i, \eta_1 = $ const. Substitution of (8a, b) into Eq. (4) and setting $u = S^2$ we obtain the differential equation

$$u'^2 + 2(\ln\phi)'uu' + u^4 + 4Au^2\phi^{-2} = 0, \tag{10}$$

where

$$A = [(\tfrac{1}{2}\,\omega+1)a^2 - q^2 c_1^2]. \tag{11}$$

After some calculations we obtain the general solution

$$S^2 = B\phi^{-1}\cosh^{-1}(\ln \phi^{B/a}), \tag{12}$$

where

$$B = [4q^2c_1^2 - (2\omega \pm 3)a^2]^{1/2}. \tag{13}$$

In conclusion we have derived the BDT-vacuum solutions for the Bianchi type-VIII and type-IX space-times, which are defined by Eqs. (6), (8a, b), (9), (12) and (13). Our solutions are the generalizations of the Taub (1951) GRT-vacuum solutions (note however, that Taub considers only the Bianchi type-IX model). However, it should be noted that in the limit $\phi' = 0$ we do not obtain the corresponding GRT-vacuum solutions. Furthermore, it seems to be impossible to incorporate a perfect fluid matter with stiff equation of state into the BDT Bianchi type-VIII and type-IX models. This is in contrast to the results obtained in the GRT (Kramer et al. 1980; Lorenz 1980).

Note added in proof: The stiff matter solutions have been found by us very recently (Lorenz-Petzold 1984, *Astrophys. Space Sci.*) as well as some more ganeral Bianchi type-VIII and type-IX vacuum solutions.

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