ON THE GENERAL VACUUM AND STIFF MATTER SOLUTIONS FOR "DIAGONAL" BIANCHI TYPE-VI_0 AND TYPE-VII_0 MODELS

By D. Lorenz-Petzold

Fakultät für Physik, Universität Konstanz*

(Received August 16, 1983)

Einstein's field equations are solved in the case of Bianchi type-VI_0 and type-VII_0 models. In these solutions appears a particular form of the Painlevé transcendent of type III.

PACS numbers: 04.20.Jb, 98.80.Dr

1. Introduction

Spatially homogeneous space-times with electromagnetic fields obeying the source-free Maxwell equations and (or) perfect fluid matter are widely used to study different properties of solutions of Einstein's field equations in the general theory of relativity. These space-times belong either to the Bianchi types I–IX or to the Kantowski-Sachs class and have been investigated by many authors over the last three decades since the basic work of Taub (1951). (An almost complete list of exact solutions of Einstein's field equations is given by Kramer et al. (1980).)

It is well known that the field equations of general relativity form in general a system of ten second-order quasilinear partial differential equations. However, for the spatially homogeneous space-times these equations reduce to a system of ordinary differential equations with the independent variable being temporal. Even in this case they have not been completely discussed because of their remaining high analytic complexity. In papers previously published we presented new exact solutions of the Einstein-Maxwell equations for various Bianchi types and the Kantowski-Sachs models which are not included in Ref. Kramer et al. (1980). (See Lorenz (1983a) for complete references.) Among the highly intractable Bianchi types are the type-VI_0 ($n^{\beta}{}_{\beta} \neq 0$), as has been pointed out by Collins (1971), and type-VII_0 models, where $n^{\beta}{}_{\beta}$ denotes the anti-symmetric part in the Ellis-MacCallum (1969) decomposition of the structure constants which characterize the Bianchi types I–IX. It is the purpose of this paper to discuss the corresponding field equations

* Address: Fakultät für Physik, Universität Konstanz, D-7750 Konstanz, West Germany.

and to obtain exact solutions in the vacuum and the stiff matter case for non-LRS models.

In what follows we obtain exact solutions in terms of Painlevé transcendental functions of type III (Painlevé 1902; Ince 1956). We note that the appearance of Painlevé transcendents as solutions of the Einstein-Maxwell equations is well known (Marek 1968; Chitre et al. 1975; Maartens and Nel 1978; Marcilhacy 1979; Léauté and Marcilhacy 1979a, b; Lorenz 1981). One knows that there are in general six nonlinear second-order differential equations which define the corresponding Painlevé transcendents. They were discovered towards the end of the 19th century, exploited for about forty years and then (apparently) forgotten; quite recently they have re-emerged in a somewhat different form. It has been shown that they are related to evolution equations such as the sin (sinh)-Gordon or the Korteweg-de Vries equations solvable by inverse scattering transforms (Ablowitz 1978; Segur 1980). More recently it has been shown by Ablowitz et al. (1980), Ablowitz and Segur (1981) and Flaschka and Newell (1980) that certain nonlinear differential equations can be solved via linear integral equations or arise as integrability conditions for certain kinds of deformations of linear equations. In particular their works are closely connected with the solutions of the sinh (sin)-Gordon equations of the special kind of Painlevé transcendents of type III which occur precisely in our present work (see also McCoy et al. 1977; Ablowitz and Segur 1977; Airault 1979, 1981; Kaliappan and Lakshmanan 1979; Osborne and Stuart 1980; Morris and Dodd 1980; Léauté et al. 1981).

2. *Field equations and solutions*

In choosing a local orthonormal basis σ^μ, we can put the metric of "diagonal" space--times in the form

$$ds^2 = \eta_{\mu\nu}\sigma^\mu\sigma^\nu, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. For a spatially homogeneous model, we take

$$\sigma^0 = \omega^0 = dt, \qquad \sigma^i = R_i\omega^i \quad \text{(no sum)}, \tag{2}$$

where ω^i are time-independent differential one-forms (Kramer et al. 1980) and where, because of homogeneity, the cosmic scale-functions R_i are functions of t only. (Here and henceforth Latin indices assume the values 1, 2, 3 whereas Greek indices assume the values 0, 1, 2, 3).

The one-forms σ^i, ω^i obey the relations

$$d\omega^i = -\tfrac{1}{2}\, C_{kl}{}^i\omega^k \wedge \omega^l, \qquad d\sigma^i = -\tfrac{1}{2}\,\gamma_{\alpha\beta}{}^i\sigma^\alpha \wedge \sigma^\beta, \tag{3}$$

where the $C_{kl}{}^i$ are the structure constants, $\gamma_{\alpha\beta}{}^i$ are the commutation coefficients and $\wedge$ denotes the exterior product. The nonzero structure constants are given by

$$C_{23}{}^1 = 1, \qquad C_{13}{}^2 = n, \tag{4}$$

where one has a Bianchi type-VI_0 or type-VII_0 model in case of $n = 1$ or $n = -1$. By using Cartan's calculus of differential forms one easily obtains the components of the Ricci tensor $R_{\mu\nu}$ (see Lorenz 1980a for further details).

The Einstein equations considered here are

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}T^{\lambda}_{\lambda}), \tag{5}$$

where

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu + \eta_{\mu\nu}p \tag{6}$$

is the energy-momentum tensor, u_μ is the velocity four-vector, and ε and p are, respectively the density and pressure of the perfect fluid. The units are chosen that $\kappa = 8\pi G/c^4 = 1$. The perfect fluid matter is characterized by the equation of state

$$p = (\gamma - 1)\varepsilon, \qquad 1 \leqslant \gamma \leqslant 2. \tag{7}$$

In the local inertial frame determined by (2), an observer comoving with the fluid is assumed to have four-velocity $u^\alpha = \delta^\alpha_0$, i.e. we are considering only non-tilted models. The field equations reduce to the following equations

$$(\ln R_i^2)'' = n_j R_j^4 + n_k R_k^4 - n_i R_i^4 + 2n\delta_{3i}(R_1R_2)^2 + \varepsilon(2-\gamma)(R_1R_2R_3)^2, \tag{8a}$$

$$[\ln (R_1R_2R_3)^2]'' = 4[(\ln R_3)'(\ln R_1R_2)' + (\ln R_1)'(\ln R_2)'] - \varepsilon(3\gamma - 2)(R_1R_2R_3)^2, \tag{8b}$$

where $n_1 = n_2 = 1, n_3 = 0$, (i, j, k) are in cyclic order, δ_{3i} denotes the Kronecker symbol, $(\ )' = d/d\eta$ and $dt = R_1R_2R_3 d\eta$.

The conservation law for the energy-momentum tensor (6)

$$T^{\mu}{}_{\nu;\mu} = 0 \tag{9}$$

gives

$$\varepsilon = m^\gamma(R_1R_2R_3)^{-\gamma}, \qquad m = \text{const.} \tag{10}$$

In considering first the LRS-case $R_1 = R_2$, the Bianchi type-VII$_0$ model reduces to a special Bianchi type-I model. The corresponding field equations can be easily integrated for all values of γ by one of the methods described by Stewart and Ellis (1969), Jacobs (1969a, b), Vajk and Eltgroth (1970), Lorenz (1980b) and Lorenz and Zimmermann (1981). Very recently we have also discussed the incorporation of an electromagnetic field with non-vanishing cosmological constant Λ into the LRS-Bianchi type-I space-time (Lorenz 1982c). The field equations for the LRS Bianchi type-VI$_0$ ($n^\beta{}_\beta \neq 0$) model reduce to those of the type-VI$_0$ ($n^\beta{}_\beta = 0$) by a simple change of the variables R_i. We thus obtain the Ellis-MacCallum (1969) stiff matter ($\gamma = 2$) solution and also the special solutions for $1 \leqslant \gamma < 2$ given by Collins (1971), Dunn and Tupper (1976) and Evans (1978). Some electromagnetic generalizations have been discussed by Lorenz (1982a, b) for both orthogonal and tilted Bianchi type-VI$_0$ ($n^\beta{}_\beta = 0$) models. The vacuum solution with $\Lambda \neq 0$ has been reduced by us (Lorenz 1983b) recently to an Abel first-order differential equation and a transcendental solution (not of the Painlevé types) was given.

The non-LRS case has been first considered by Lukash (1973). However, only a study of the asymptotic behaviour of the model for various equations of state was given. By taking $m = 0$ (vacuum) or $\gamma = 2$ (stiff matter) it follows that the terms depending on ε in (8a) vanishes. We are looking in this paper for exact solutions of (8a, b) in these cases. Introducing the new variables $r_i = r_i(\eta)$ by

$$R_i = \exp(r_i), \qquad r = 2(r_1 - r_2) \tag{11}$$

and assuming $(r_1 + r_2)' \neq 0$ the field equations (8a, b) can be decoupled and partial integrated to give

$$r_1 + r_2 = a(\eta - \eta_0), \tag{12a}$$

$$r'' + 4 \exp[2a(\eta - \eta_0)] \sinh(r) = 0, \tag{12b}$$

$$ar_3' = -r_1' r_2' + \tfrac{1}{4}[\exp(2r_1) + n \exp(2r_2)]^2 + m^2, \tag{12c}$$

where a, η_0 are constants of integration. The field equations for $i = 3$ will not be written out for in the case considered here it is a consequence of the system (12a–c). The case $(r_1 + r_2)' = 0$ is only possible for the Bianchi type-VII$_0$ model and gives the trivial solutions $r_1 = r_2 = \text{const.}$, $r_3 = b(\eta - \eta_1)$, b, $\eta_1 = \text{const.}$ and $m = 0$.

After solving Eq. (12b) to give $r = r(\eta)$ the most general vacuum or stiff matter solution for Bianchi type-VI$_0$ and type-VII$_0$ would arise, since the expressions for r_i could be easily obtained from the remaining field equations. We will now show how the solutions can be expressed in terms of a particular form of the third Painlevé transcendents. Introducing the time variable ξ by

$$\xi = \frac{2}{a} \exp[a(\eta - \eta_0)], \tag{13}$$

we can transform the system (12) to obtain

$$r_1 + r_2 = \ln(a\xi/2), \tag{14a}$$

$$\ddot{r} + \dot{r}/\xi + \sinh(r) = 0, \tag{14b}$$

$$\dot{r}_3 = [(m/a)^2 - 1/4]/\xi + \xi[\dot{r}^2 + 2(\cosh(r) + n)]/16, \tag{14c}$$

where $(\ )^{\cdot} = d/d\xi$. Eq. (14b) is a particular case of a sinh-Gordon type equation (Ablowitz 1978; Segur 1980; Ablowitz et al. 1980; Flaschka and Newell 1980; Ablowitz and Segur 1977). We notice that a purely imaginary solution of the sinh-Gordon equation gives a purely real solution of the sin-Gordon equation and vice versa. Both equations arise in many areas of theoretical physics and applied mathematics.

If we put

$$w = \exp(r), \qquad z = \xi^2/4, \qquad w = w(z), \qquad (\ )' = d/dz, \tag{15}$$

Eq. (14b) becomes

$$w'' = w'^2/w - [w' + (w^2 - 1)/2]z. \tag{16}$$

This equation is a particular form of the non-linear equation of second-order which defines the third Painlevé transcendent (Ince 1956; Painlevé 1902). Inspection of the above mentioned works of Ablowitz et al. (1980) and Flaschka and Newell (1980) shows that we can apply (up to a simple change of the considered sinh-Gordon equations) their methods to obtain solutions of Eq. (16). The solutions would be completed by

$$r_1 = \ln (a^2 zw)^{1/4}, \qquad r_2 = \ln (a^2 zw^{-1})^{1/4}, \tag{17}$$

$$r_3 = [(m/a)^2 - 1/4]2z + [zw'/w + 2(\cosh (\ln w) + n)]/8. \tag{18}$$

It therefore follows that the most general vacuum and stiff matter solutions (in terms of Painlevé transcendents) have been found. In this connection we mention also the works of Belinskii et al. (1969, 1970) and Lifshitz and Khalatnikov (1970) on the diagonal Bianchi type-VIII and type-IX models. It was shown that near the initial singularity the function r_3 may be neglected compared with the two others (r_1, r_2). Thus their approximative field equations for types-VIII and IX are the same as our exact field equations for types-VI_0 and VII_0. (Note that there are some minor errors in the above mentioned papers). The asymptotic behaviour of the solutions of Eq. (14) in the regions $\xi \gg 1$ and $\xi \ll 1$ is discussed in the vaccum ($m = 0$) case. The incorporation of an electromagnetic field into the Bianchi type-VI_0 (VII_0) space-times has been given by Ruban (1978) and some asymptotic considerations was made. We finally mention the work of Barnes (1978) on electromagnetic Bianchi type-VI_0 and type-VII_0 models using the Newman-Penrose spin-coefficient formalism. It would be interesting to discuss the relation to our solutions.

REFERENCES

Ablowitz, M. J., in: *Nonlinear Evolution Equations Solvable by the Spectral Transform*, edited by F. Calogero, Pitman, London-San Francisco-New York 1978.

Ablowitz, M. J., Segur, H., *Phys. Rev. Lett.* **38**, 1103 (1977).

Ablowitz, M. J., Segur, H., *Solitons and the Inverse Spectral Transform*, SIAM, Philadelphia 1981.

Ablowitz, M. J., Ramani, A., Segur, H., *J. Math. Phys.* **21**, 1006 (1980).

Airault, H., *Studies Appl. Math.* **61**, 31 (1979).

Airault, H., *Studies Appl. Math.* **64**, 183 (1981).

Barnes, A., *J. Phys. A: Math. Gen.* **11**, 1303 (1978).

Belinskii, V. A., Khalatnikov, I. M., *Zh. Eksp. Teor. Fiz.* **56**, 1701 (1969); *Sov. Phys. JETP* **29**, 911 (1969).

Belinskii, V. A., Khalatnikov, I. M., Lifshitz, E. M., *Adv. Phys.* **19**, 525 (1970).

Collins, C. B., *Commun. Math. Phys.* **23**, 137 (1971).

Chitre, D. M., Güven, R., Nutku, Y., *J. Math. Phys.* **16**, 475 (1975).

Dunn, K. A., Tupper, B. O. P., *Astrophys. J.* **204**, 322 (1976).

Ellis, G. F. R., MacCallum, M. A. H., *Commun. Math. Phys.* **12**, 108 (1969).

Evans, A. B., *Mon. Not. Roy. Astron. Soc.* **183**, 727 (1978).

Flaschka, H., Newell, A. C., *Commun. Math. Phys.* **76**, 67 (1980).

Ince, E. L., *Ordinary Differential Equations*, Dover Publications, New York 1956.

Jacobs, K. C., Ph. D. Thesis, California Institute of Technology, Pasadena 1969a.

Jacobs, K. C., *Astrophys. J.* **155**, 379 (1969b).

Kaliappan, P., Lakshmanan, M., *J. Phys. A: Math. Gen.* **12**, L249 (1979).

Kramer, D., Stephani, H., MacCallum, M. A. H., Herlt, E., *Exact Solutions of Einstein's Field Equations*, VEB Deutscher Verlag der Wissenschaften, Berlin 1980.
Léauté, B., Marcilhacy, G., *Lett. Nuovo Cimento* **26**, 183 (1979a).
Léauté, B., Marcilhacy, G., *Ann. Inst. H. Poincaré* **31**, 363 (1979b).
Léauté, B., Marcilhacy, G., *Phys. Lett.* **87**, 159 (1982).
Léauté, B., Marcilhacy, G., Skinazi, J., *Lett. Nuovo Cimento* **32**, 13 (1981).
Lifshitz, E. M., Khalatnikov, I. M., *Pisma Zh. Eksp. Teor. Fiz.* **11**, 200 (1970); *JETP Lett.* **11**, 123 (1970).
Lorenz, D., *Phys. Rev.* **D22**, 1848 (1980a).
Lorenz, D., *Phys. Lett.* **80A**, 235 (1980b).
Lorenz, D., *Phys. Lett.* **83A**, 155 (1981).
Lorenz, D., *Astrophys. Space Sci.* **85**, 63 (1982a).
Lorenz, D., *Astrophys. Space Sci.* **85**, 69 (1982b).
Lorenz, D., *Phys. Lett.* **92A**, 118 (1982c).
Lorenz, D., *J. Phys. A: Math. Gen.* **16**, 575 (1983a).
Lorenz, D., *Acta Phys. Pol.* **B14**, 479 (1983b).
Lorenz, D., Zimmermann, R. E., *Lett. Nuovo Cimento* **31**, 603 (1981).
Lukash, V. N., *Astron. Zh.* **51**, 281 (1973); *Sov. Astron.* **18**, 164 (1974).
Maartens, R., Nel, S. D., *Commun. Math. Phys.* **59**, 273 (1978).
Marcilhacy, G., *Phys. Lett.* **73A**, 157 (1979).
Marek, J. J. J., *Proc. Camb. Phil. Soc.* **64**, 167 (1968).
McCoy, M., Tracy, C. A., Wu, T. T., *J. Math. Phys.* **18**, 1058 (1977).
Morris, H. C., Dodd, R. K., *Phys. Lett.* **75A**, 249 (1980).
Osborne, A. D., Stuart, A. E. G., *Phys. Lett.* **76A**, 5 (1980).
Painlevé, P., *Acta Math.* **25**, 1 (1902).
Ruban, V. A., Preprint No. 411, Leningrad Institute of Nuclear Physics, B. P. Konstantinova 1978.
Segur, H., in: *Lecture Notes in Phys.* **130**, edited by J. A. De Santo, A. W. Sáenz, W. W. Zachary, Springer Verlag, Berlin 1980.
Stewart, J. M., Ellis, G. F. R., *J. Math. Phys.* **9**, 1072 (1968).
Taub, A. H., *Ann. Math.* **53**, 472 (1951).
Vajk, J. G., Eltgroth, P. G., *J. Math. Phys.* **11**, 2212 (1970).