CAN REALLY THE ELECTROMAGNETIC TRANSVERSE WAVES WITH $\vec{E} \parallel \vec{B}$ AND FINITE ENERGY EXIST?

BY S. MALINOWSKI

Institute of Physics, University of Łódź*

(*Received July 5, 1983*)

It is shown that the total field energy for general solutions of the sourceless Maxwell's equations with $\vec{E} \parallel \vec{B}$ is infinite. Moreover the action and/or the "pseudoscalar charge" must be infinite too in this case. Therefore the expected similarity to the instanton or meron solutions of nonabelian gauge theories is illusory.

PACS numbers: 03.50.De, 41.10.Hv

Recently Chu and Okhawa [1] have shown that there exists a class of unexpected solutions of the sourceless Maxwell's equations with $\vec{E}(t, \vec{x})$ parallel to $\vec{B}(t, \vec{x})$. It is interesting that for such solutions the energy density is time-independent because Poynting's vector vanishes. The solution due to Chu and Okhawa [1] is in fact the monochromatic standing wave:

$$\vec{E} = \vec{C}(\vec{x}) \sin(\omega t + \alpha)$$
$$\vec{B} = \vec{C}(\vec{x}) \cos(\omega t + \alpha), \tag{1}$$

where

$$\vec{C}(\vec{x}) = \vec{D}(\vec{x}) + \frac{1}{\omega} \vec{\nabla} \times \vec{D}(\vec{x}),$$

with

$$\vec{\nabla}\vec{D}(\vec{x}) = 0 \quad \text{and} \quad (\Delta + \omega^2)\vec{D}(\vec{x}) = 0.$$

Very recently Khare and Pradhan [2] have tried to give the explicit example of the transverse electromagnetic wave with $\vec{E} \parallel \vec{B}$ which is in general nonmonochromatic and nonstanding. They claim that their solution has finite total energy as well as the action and the "pseudocharge" and for this reason it resembles the instanton solutions of nonabelian

* Address: Instytut Fizyki, Uniwersytet Łódzki, Nowotki 149/153, 90-236 Łódź, Poland.

gauge theories. However there are serious errors in their paper and consequently the conclusions are wrong [3].

We will show now that for the electromagnetic waves with $\vec{E}||\vec{B}$ the total energy diverges. The proof is quite analogous to the one given by Deser [4] and Pagels [5] in the case of nonabelian theories. Let us assume for the moment that the energy

$$\varepsilon = \frac{1}{2}\int d^3x(\vec{E}^2+\vec{B}^2) \tag{2}$$

is finite for the solution with $\vec{E}||\vec{B}$. From the conformal invariance of Maxwell equations it follows that the dilatation current $x^\nu\theta_{\mu\nu}$ is conserved

$$\partial^\mu(x^\nu\theta_{\mu\nu}) = 0.$$

Here $\theta_{\mu\nu}$ is the traceless electromagnetic energy-momentum tensor. Thus the dilatation charge

$$D = \int d^3x x^\mu\theta_{0\mu} \tag{3}$$

if it exists, is the constant of motion

$$\frac{dD}{dt} = 0. \tag{4}$$

But due to the condition $\vec{E}||\vec{B}$ Poynting's vector θ_{0k} vanishes. Therefore the finiteness of the energy ε and Eq. (3) implies

$$D = t\varepsilon$$

and consequently from Eq. (4) $\varepsilon = 0$ so that $\vec{E} = \vec{B} = 0$. Thus the nontrivial solutions of the sourceless Maxwell equations with $E||\vec{B}$ have the infinite total energy.

Now, let us take into account the identity

$$(\vec{E}^2+\vec{B}^2)^2 = (\vec{E}^2-\vec{B}^2)^2+(2|\vec{E}|\cdot|\vec{B}|)^2,$$

which implies the triangle inequality

$$\frac{1}{2}(\vec{E}^2+\vec{B}^2) \leqslant \frac{1}{2}|\vec{E}^2-\vec{B}^2|+|\vec{E}|\cdot|\vec{B}|$$

and consequently

$$\frac{1}{2}\int d^3x(\vec{E}^2+\vec{B}^2) \leqslant \frac{1}{2}\int d^3x|\vec{E}^2-\vec{B}^2|+\int d^3x|\vec{E}|\cdot|\vec{B}|. \tag{5}$$

We see that the divergence of the left-hand side implies the divergence of $\int d^3x|\vec{E}^2-\vec{B}^2|$ and/or $\int d^3x|\vec{E}|\cdot|\vec{B}|$. On the other hand, for multidimensional improper integrals convergence is equivalent to the absolute convergence (i.e. for $n \geqslant 2$, $|\int_{-\infty}^{\infty} d^nxf(x)| < \infty$ is equivalent to $\int_{-\infty}^{\infty} d^nx|f(x)| < \infty$). Therefore the divergence of $\varepsilon = \frac{1}{2}\int d^3x(\vec{E}^2+\vec{B}^2)$ implies the divergence of the action $S = \frac{1}{2}\int d^4x(\vec{E}^2-\vec{B}^2)$ and/or the integral $\int d^4x|\vec{E}|\cdot|\vec{B}|$. However in the case under consideration ($\vec{E}||\vec{B}$) convergence of this last integral is equivalent to the convergence of $q = -\int d^4x\vec{E}\cdot\vec{B}$.

Summarizing, for the solutions of the sourceless Maxwell equations with $\vec{E}||\vec{B}$
(*i*) the total field energy $\varepsilon = \frac{1}{2}\int d^3x(\vec{E}^2+\vec{B}^2)$ diverges,
(*ii*) the action $s = \frac{1}{2}\int d^4x(\vec{E}^2-\vec{B}^2)$ and/or the "pseudoscalar charge" $q = -\int d^4x\vec{E}\cdot\vec{B}$ diverges too.
Therefore the signalized similarity to the instanton or meron solutions [2] is illusory.

I thank my colleagues S. Giler, P. Kosiński and J. Rembieliński for interesing discussion.

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