TENSOR FORM OF THE BREIT EQUATION: PART TWO

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The Breit equation for a system of two Dirac particles, which was recently represented in the tensor form, is reduced explicitly in the case of equal masses to a considerably simple system of radial equations. The vector and scalar spin-dependent central potentials are considered. For the reduction the multipole technique is used.

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In this note we make use of the recently introduced tensor representation [1] of the Breit equation for two Dirac particles [2] to reduce this equation to a considerably simple system of radial equations which may be of a practical importance in the problem of fermionium (and, in particular, of quarkonium). To this end we consider the Breit equation

$$\left\{E-V-\left[\vec{\alpha}_1\cdot\vec{p}+\beta_1\left(m_1+\frac{S}{2}\right)\right]-\left[-\vec{\alpha}_2\cdot\vec{p}+\beta_2\left(m_2+\frac{S}{2}\right)\right]\right\}\psi(\vec{r})=0 \qquad (1)$$

(where $\vec{r}\equiv\vec{r}_1-\vec{r}_2$ and $\vec{p}\equiv\vec{p}_1=-\vec{p}_2$) with a vector potential V and a scalar potential S which beside the particle distance $\vec{r}$ may depend on the spin $s=0, 1$ of the system of two Dirac particles:

$$V=P_0V_0+P_1V_1, \qquad S=P_0S_0+P_1S_1, \qquad (2)$$

P_s denoting the projection operators on states with spin s:

$$P_0=\tfrac{1}{4}(1-\vec{\sigma}_1\cdot\vec{\sigma}_2), \qquad P_1=\tfrac{1}{4}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)=\tfrac{1}{2}\vec{S}^2 \qquad (3)$$

(where $\vec{S}=\frac{1}{2}(\vec{\sigma}_1+\vec{\sigma}_2)$ is the spin operator). We will assume that the potentials are static and central, $V_s=V_s(r)$ and $S_s=S_s(r)$, the latter property being physically consistent with the former. Note that non-static effects would introduce into the potentials the velocities $\vec{\alpha}_1$ and $\vec{\alpha}_2$ which, if treated (on the level of Breit equation) more exactly than in the first perturbation order, would cause considerable errors [2, 3] related to the fact that

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the hole theory is not consistent with the Breit equation. It is in contrast with, say, the Salpeter equation [4, 5] which, however, is much more complicated in practical calculations.

In our tensor representation, the Breit equation (1) takes the form of the system of equations (12a) and (12b) given in Ref. [1], where now m_1 and m_2 should be replaced properly by $m_1+\frac{1}{2}S_s$ and $m_2+\frac{1}{2}S_s$, and V_s^I and V_s^R should be put V_s and zero, respectively. In the case of equal masses $m_1 = m_2 \equiv m$ (to which we will restrict ourselves for the sake of simplicity) this system splits into two subsystems of equations describing in a relativistic way the parafermionium ($s = 0$) and orthofermionium ($s = 1$). They are given respectively in Eqs. (13) and (14) in Ref. [1], where now the same insertions should be made as in Eqs. (12a) and (12b).

These two subsystems of equations can be conveniently reduced to first-order radial equations by means of the multipole technique used already in Ref. [1] in order to derive the second-order radial equations (23) and (25E), (26E), (27) from Eqs. (13) and (14), respectively. These second-order radial equations are valid in the case of $V_0 = V_1 \equiv V$ and $S_0 = S_1 = 0$.

In the multipole technique we make use of the expansion

$$\vec{\chi}(\vec{r}) = \hat{r}\chi_{\text{el}}(\vec{r}) - \frac{\partial}{\partial\hat{r}}\frac{\chi_{\text{lon}}(\vec{r})}{j(j+1)} - \left(\hat{r}\times\frac{\partial}{\partial\hat{r}}\right)\frac{\chi_{\text{mag}}(\vec{r})}{j(j+1)}, \tag{4}$$

where

$$\begin{aligned}
\chi_{\text{el}}(\vec{r}) &= \hat{r}\cdot\vec{\chi}(\vec{r}) = \chi_{\text{el}}(r)Y_{jm}(\hat{r}),\\
\chi_{\text{lon}}(\vec{r}) &= \left(\frac{\partial}{\partial\hat{r}} - 2\hat{r}\right)\cdot\vec{\chi}(\vec{r}) = \chi_{\text{lon}}(r)Y_{jm}(\hat{r}),\\
\chi_{\text{mag}}(\vec{r}) &= \left(\hat{r}\times\frac{\partial}{\partial\hat{r}}\right)\cdot\vec{\chi}(\vec{r}) = \chi_{\text{mag}}(r)Y_{jm}(\hat{r})
\end{aligned} \tag{5}$$

(and analogical expansions for other $s = 1$ components: $\vec{\chi}^0(\vec{r})$, $\vec{\phi}(\vec{r})$ and $\vec{\phi}^0(\vec{r})$). Also $\chi^0(\vec{r}) = \chi^0(\vec{r})Y_{jm}(r)$ (and analogically for other $s = 0$ components: $\phi(\vec{r})$ and $\phi^0(\vec{r})$). Here, $\hat{r} = \frac{\vec{r}}{r}$ and $\frac{\partial}{\partial\hat{r}} = r\frac{\partial}{\partial\vec{r}} - \vec{r}\frac{\partial}{\partial r}$ implying $\hat{r}^2 = 1$, $\hat{r}\cdot\frac{\partial}{\partial\hat{r}} = 0$, $\frac{\partial}{\partial\hat{r}}\cdot\hat{r} = 2$, $\left(\frac{\partial}{\partial\hat{r}}\right)^2 = -\vec{L}^2 = \left(\frac{\partial}{\partial\hat{r}} - 2\hat{r}\right)^2$, $\hat{r}\times\frac{\partial}{\partial\hat{r}} = \vec{r}\times\frac{\partial}{\partial\vec{r}} = i\vec{L}$, $\hat{r}\times\vec{L} = i\frac{\partial}{\partial\hat{r}}$ and $\frac{\partial}{\partial\hat{r}}\times\frac{\partial}{\partial\hat{r}} = i\vec{L}$. Note that $\left(\frac{\partial}{\partial\hat{r}}\right)^+ = -\left(\frac{\partial}{\partial\hat{r}} - 2\hat{r}\right)$ and $\left(\hat{r}\times\frac{\partial}{\partial\hat{r}}\right)^+ = -\left(\hat{r}\times\frac{\partial}{\partial\hat{r}}\right)$. Of course, $\vec{J} = \vec{L}+\vec{S}$ and $j = 0, 1, 2, \ldots$, while $l = 0, 1, 2, \ldots$ and $s = 0, 1$. The wave-function components with $j > 0$ and $s = 1$ and with the determined orbital angular momentum $l = j-1$, $+1$, j are

$$\chi_{l=j-1} = \sqrt{\frac{j}{2j+1}}\chi_{\text{el}} - \sqrt{\frac{j+1}{2j+1}}\frac{\chi_{\text{lon}}}{\sqrt{j(j+1)}},$$

$$\chi_{l=j+1} = \sqrt{\frac{j+1}{2j+1}}\chi_{\text{el}} + \sqrt{\frac{j}{2j+1}}\frac{\chi_{\text{lon}}}{\sqrt{j(j+1)}},$$

$$\chi_{l=j} = \frac{\chi_{\text{mag}}}{\sqrt{j(j+1)}} \tag{6}$$

(and analogically for the other $s = 1$ components: $\vec{\chi}^0$, $\vec{\phi}$ and $\vec{\phi}^0$). If $j = 0$ then $\chi_{\text{lon}} = 0$ and $\chi_{\text{mag}} = 0$ so that $\chi_{l=j-1} = 0$, $\chi_{l=j+1} = \chi_{\text{el}}$, $\chi_{l=j} = 0$ (and similarly for $\vec{\chi}^0$, $\vec{\phi}$ and $\vec{\phi}^0$). Note that for norms we have

$$\|\vec{\chi}\|^2 = \|\chi_{l=j-1}\|^2 + \|\chi_{l=j+1}\|^2 + \|\chi_{l=j}\|^2 = \|\chi_{\text{el}}\|^2 + \frac{\|\chi_{\text{lon}}\|^2 + \|\chi_{\text{mag}}\|^2}{j(j+1)}$$

and

$$\|\psi\|^2 = \|\chi\|^2 + \|\chi^0\|^2 + \|\phi\|^2 + \|\phi^0\|^2 + \|\vec{\chi}\|^2 + \|\vec{\chi}^0\|^2 + \|\vec{\phi}\|^2 + \|\vec{\phi}^0\|^2,$$

where $\psi(\vec{r})$ is the whole many-component wave function for our system of two Dirac particles. The extra $s = 0$ component $\chi(\vec{r})$ vanishes in the case of equal masses $m_1 = m_2 \equiv m$.

In this way we obtain in the case of equal masses $m_1 = m_2 \equiv m$ two following subsystems of first-order radial equations including the "large-large" components with spin $s = 0$ and $s = 1$, respectively: for $s = 0$ the system

$$\frac{E-V_0}{2}\phi^0 + i\left(\frac{d}{dr} + \frac{2}{r}\right)\phi_{\text{el}} + i\frac{1}{r}\phi_{\text{lon}} = \left(m + \frac{S_0}{2}\right)\phi,$$

$$\frac{E-V_0}{2}\phi = \left(m + \frac{S_0}{2}\right)\phi^0,$$

$$\frac{E-V_1}{2}\phi_{\text{el}} + i\frac{d}{dr}\phi^0 = 0,$$

$$\frac{E-V_1}{2}\phi_{\text{lon}} - i\frac{j(j+1)}{r}\phi^0 = 0,$$

$$\frac{E-V_1}{2}\phi_{\text{mag}} = 0 \tag{7}$$

and for $s = 1$ two independent systems

$$\frac{E-V_0}{2}\chi^0 + i\left(\frac{d}{dr} + \frac{2}{r}\right)\chi_{\text{el}} + i\frac{1}{r}\chi_{\text{lon}} = 0,$$

$$\frac{E-V_1}{2}\chi_{\text{el}} + i\frac{d}{dr}\chi^0 = \left(m + \frac{S_1}{2}\right)\chi^0_{\text{el}},$$

$$\frac{E-V_1}{2}\chi_{\text{lon}}-i\frac{j(j+1)}{r}\chi^0 = \left(m+\frac{S_1}{2}\right)\chi^0_{\text{lon}},$$

$$\frac{E-V_1}{2}\chi^0_{\text{el}}+\frac{1}{r}\phi^0_{\text{mag}} = \left(m+\frac{S_1}{2}\right)\chi_{\text{el}},$$

$$\frac{E-V_1}{2}\chi^0_{\text{lon}}-\left(\frac{d}{dr}+\frac{1}{r}\right)\phi^0_{\text{mag}} = \left(m+\frac{S_1}{2}\right)\chi_{\text{lon}},$$

$$\frac{E-V_1}{2}\phi^0_{\text{mag}}+\frac{j(j+1)}{r}\chi^0_{\text{el}}+\left(\frac{d}{dr}+\frac{1}{r}\right)\chi^0_{\text{lon}} = 0 \tag{8a}$$

and

$$\frac{E-V_1}{2}\chi_{\text{mag}} = \left(m+\frac{S_1}{2}\right)\chi^0_{\text{mag}},$$

$$\frac{E-V_1}{2}\chi^0_{\text{mag}}+\frac{j(j+1)}{r}\phi^0_{\text{el}}+\left(\frac{d}{dr}+\frac{1}{r}\right)\phi^0_{\text{lon}} = \left(m+\frac{S_1}{2}\right)\chi_{\text{mag}},$$

$$\frac{E-V_1}{2}\phi^0_{\text{el}}+\frac{1}{r}\chi^0_{\text{mag}} = 0,$$

$$\frac{E-V_1}{2}\phi^0_{\text{lon}}-\left(\frac{d}{dr}+\frac{1}{r}\right)\chi^0_{\text{mag}} = 0. \tag{8b}$$

The sixteenth component is zero: $\chi = 0$.

The (non-zero) wave-function components involved in the systems (7), (8a) and (8b) have the total parity $P = \eta(-1)^j$, $P = \eta(-1)^{j+1}$ and $P = \eta(-1)^j$, respectively, where $\eta^2 = 1$. For a *fermion* + *fermion* or *fermion* + *antifermion* system we get $\eta = +1$ or -1, respectively, at least if the considered Dirac particles are of the same kind. The "large-large" components with $s = 0$ are contained in ϕ and ϕ^0 and those with $s = 1$ – in $\vec{\chi}$ and $\vec{\chi}^0$ (i.e., in $\chi_{\text{el}}, \chi_{\text{lon}}, \chi_{\text{mag}}$ and $\chi^0_{\text{el}}, \chi^0_{\text{lon}}, \chi^0_{\text{mag}}$). Thus the "large-large" components involved in the systems (7), (8a) and (8b) correspond to $l = j$ with $s = 0$, $l = j\mp 1$ with $s = 1$ and $l = j$ with $s = 1$, respectively, so that (in the usual spectroscopic notation ${}^{2s+1}l_j$) these systems of first-order radial equations describe in a relativistic way the states ${}^1j_j$, ${}^3(j\mp 1)_j$ and ${}^3j_j$ with the corresponding total parity $P = \eta(-1)^j$, $P = \eta(-1)^{j+1}$ and $P = \eta(-1)^j$ (the states ${}^3(j\mp 1)_j$ being superposed in the way determined dynamically by Eq. (8a) and kinematically by Eq. (6)). If $j = 0$ there are only the states ${}^1S_0$ and ${}^3P_0$.

The system (7) for parafermionium reduces in the practically important case of $V_0 = V_1$ and $S_0 \neq 0$ to the second-order radial equation

$$\left[\left(\frac{E-V_0}{2}\right)^2+\frac{1}{r}\frac{d^2}{dr^2}r-\frac{j(j+1)}{r^2}-\left(m+\frac{S_0}{2}\right)^2+\frac{\frac{dV_0}{dr}}{E-V_0}\frac{d}{dr}\right]\phi^0 = 0. \tag{9}$$

The system (8b) for orthofermionium with $P = \eta(-1)^j$ can be reduced in the general case to the second-order equation

$$\left[\left(\frac{E-V_1}{2}\right)^2 + \frac{1}{r}\frac{d^2}{dr^2}r - \frac{j(j+1)}{r^2} - \left(m+\frac{S_1}{2}\right)^2 + \frac{\frac{dV_1}{dr}}{E-V_1}\frac{1}{r}\frac{d}{dr}r\right]\chi^0_{\rm mag} = 0. \qquad (10)$$

The system (8a) for orthofermionium with $P = \eta(-1)^{j+1}$ is more complicated, leading in the general case to a rather involved system of two second-order radial equations for, say, $\chi^0_{\rm el}$ and $\chi^0_{\rm lon}$ (although in the case of $V_0 = V_1$ and $S_1 = 0$ the latter is considerably simple, consisting of Eqs. (25E) and (26E) in Ref. [1]). So, in the general case, the system of first-order radial equations (8a) seems to be more adequate to practical calculations. In the practically important case of $V_0 = V_1$ and $S_1 \neq 0$, when eliminating from Eq. (8a) the components $\chi^0_{\rm el}$ and $\chi_{\rm lon}$ by the algebraic equations contained in Eq. (8a), we get for orthofermionium with $P = \eta(-1)^{j+1}$ the following system of four first-order radial equations which is considerably simple for numerical calculations:

$$\left[\left(\frac{E-V_1}{2}\right)^2 - \frac{j(j+1)}{r^2}\right]i\chi^0 - \frac{E-V_1}{2}\left(\frac{d}{dr}+\frac{2}{r}\right)\chi_{\rm el} - \left(m+\frac{S_1}{2}\right)\frac{1}{r}\chi^0_{\rm lon} = 0,$$

$$\left[\left(\frac{E-V_1}{2}\right)^2 - \left(m+\frac{S_1}{2}\right)^2\right]\chi_{\rm el} + \frac{E-V_1}{2}\frac{d}{dr}i\chi^0 + \left(m+\frac{S_1}{2}\right)\frac{1}{r}\phi^0_{\rm mag} = 0,$$

$$\left[\left(\frac{E-V_1}{2}\right)^2 - \left(m+\frac{S_1}{2}\right)^2\right]\chi^0_{\rm lon} - \frac{E-V_1}{2}\left(\frac{d}{dr}+\frac{1}{r}\right)\phi^0_{\rm mag} - \left(m+\frac{S_1}{2}\right)\frac{j(j+1)}{r}i\chi^0 = 0,$$

$$\left[\left(\frac{E-V_1}{2}\right)^2 - \frac{j(j+1)}{r^2}\right]\phi^0_{\rm mag} + \frac{E-V_1}{2}\left(\frac{d}{dr}+\frac{1}{r}\right)\chi^0_{\rm lon} + \left(m+\frac{S_1}{2}\right)\frac{j(j+1)}{r}\chi_{\rm el} = 0. \qquad (11)$$

We hope that Eqs. (9), (10) and (11) will turn out to be useful in relativistic calculations for quarkonia.

Finally, we would like to remark that the tensor representation of the Breit equation (1) and the multipole technique of reducing it to radial equations (described in Ref. [1] and in the present note) must be, of course, equivalent to its usual *spinor* ⊗ *spinor* representation and the unitary-transformation method of reducing it to the radial form [6]. In fact, it can be seen that Eqs. (7), (8a) and (8b) form a system of radial equations which in the case of $V_0 = V_1 \equiv V$ and $S_0 = S_1 = 0$ is identical with that given in Table I in the last Ref. [6] when $m_1 = m_2 \equiv m$ there. The resulting relation between the wave-function components used in that Table and the components considered in the present paper is quite instructive for the interpretation of both:

$$f_1^+ = -i\phi, \quad f_2^+ = i\phi^0, \quad f_3^+ = \chi_{\rm el}, \quad f_4^+ = \chi^0_{\rm el},$$

$$f_1^- = -i\chi = 0, \quad f_2^- = i\chi^0, \quad f_3^- = \phi_{\rm el}, \quad f_4^- = \phi^0_{\rm el} \qquad (12a)$$

and

$$g_1^+ = \frac{i\chi_{\text{mag}}}{\sqrt{j(j+1)}}, \quad g_2^+ = -\frac{i\chi^{\upsilon}_{\text{mag}}}{\sqrt{j(j+1)}}, \quad g_3^+ = \frac{i\chi^{\upsilon}_{\text{lon}}}{\sqrt{j(j+1)}}, \quad g_4^+ = \frac{i\chi_{\text{lon}}}{\sqrt{j(j+1)}},$$

$$g_1^- = \frac{i\phi_{\text{mag}}}{\sqrt{j(j+1)}} = 0, \quad g_2^- = -\frac{i\phi^0_{\text{mag}}}{\sqrt{j(j+1)}}, \quad g_3^- = \frac{i\phi^0_{\text{lon}}}{\sqrt{j(j+1)}}, \quad g_4^- = \frac{i\phi_{\text{lon}}}{\sqrt{j(j+1)}}. \quad (12b)$$

Here, the upper $\pm$ signs denote the intrinsic parity $\pm\eta$ defined by the operator $\eta\beta_1\beta_2$ with $\eta^2 = 1$ (the "large-large" components superposed with the "small-small" components are obviously contained in the "+" components, in contrast with the "small-large" and "large-small" components which are superposed within the "$-$" components). The components f_1 and f_2 have spin $s = 0$, while the rest of f's and all g's $-$ spin $s = 1$. The total parity of "$\pm$" components is $P = \pm\eta(-1)^l$ which is equal to $P = \pm\eta(-1)^j$ for "1" and "2" components, all having $l = j$, and to $P = \pm\eta(-1)^{j+1}$ for the "3" and "4" components, all being superpositions of $l = j-1$ and $l = j+1$ determined kinematically by Eq. (6). Thus the components "$1\pm$" and "$2\pm$" as well as "$3\mp$" and "$4\mp$" have equal total parity $P = \pm\eta(-1)^j = \mp\eta(-1)^{j+1}$.

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