THE ACTION PRINCIPLE FOR THE LONGITUDINAL ELECTROMAGNETIC FIELD III

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The aim of this paper is to remove the contradiction between two previously given values of the coefficient γ with which the longitudinal part of the electromagnetic field enters into the total action.

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1. Introduction

In [1] we considered the action of the electromagnetic field including the longitudinal part,

$$-\frac{1}{16\pi}\int d^4x\left\{F_{\mu\nu}F^{\mu\nu}+2\gamma\left(\partial^\mu A_\mu+\frac{1}{e}\Box S\right)^2\right\},\tag{1}$$

and gave an argument that $\gamma = e^2/4\pi$. Here e is the elementary charge and S is the phase so that the action is gauge invariant. In [2] another argument led us to conclude that $\gamma = e^2/(4\pi+e^2)$. The aim of the present paper is to remove the contradiction and to indicate its origin.

The first argument, given in [1], runs as follows. The electric current calculated from (1) is

$$j_\mu = -\frac{\gamma}{4\pi}\partial_\mu\left(\partial^\nu A_\nu+\frac{1}{e}\Box S\right).$$

The improper gauge transformation

$$A_\mu \to A'_\mu = A_\mu,$$

$$S \to S' = S-2\pi\,\mathrm{sign}\,(x^0)\Theta(xx),$$

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Θ being the Heaviside step function, creates an additional charge $e' = (4\pi/e)\gamma$. Assuming that $e' = e$ one has $\gamma = e^2/4\pi$. This argument is so simple and transparent that we can accept it or reject it but we cannot change it.

The second argument, given in [2], is based on the definition of phase associated with the electromagnetic field $A_\mu(x)$:

$$S(x) = \frac{e}{4\pi}\int d^4y A_\mu(x+y)\partial^\mu\delta(yy). \tag{2}$$

This definition is obviously correct for an "arbitrary" field $A_\mu(x)$ but for a field fulfilling equations of motion it has to be handled with some care, as shown in the next section.

2. The commutator of two phases expressed as a Fourier integral

We have from (2)

$$[S(x), S(y)] = \frac{e}{4\pi}\int d^4\xi\partial^\mu(\xi\xi)\frac{e}{4\pi}\int d^4\zeta\partial^\nu\delta(\zeta\zeta)\,[A_\mu(x+\xi), A_\nu(y+\zeta)],$$

where

$$[A_\mu(x), A_\nu(y)] = \frac{i}{2}\left(g_{\mu\nu}\Box + \frac{1-\gamma}{\gamma}\,\partial_{\mu\nu}\right)\text{sign}\,(x^0-y^0)\Theta[(x-y)\,(x-y)].$$

Using the Fourier transforms

$$\delta(xx) = -\frac{1}{4\pi^3}\int d^4k\frac{1}{kk}e^{-ikx},$$

$$\text{sign}\,(x^0)\Theta(xx) = \frac{i}{\pi^2}\int d^4k\,\text{sign}\,(k^0)\delta'(kk)e^{-ikx},$$

one finds

$$[S(x), S(y)] = \frac{e^2}{2\pi^2}\int d^4k\,\text{sign}\,(k^0)\delta'(kk)\frac{k^\mu k^\nu}{(kk)^2}\left(g_{\mu\nu}kk + \frac{1-\gamma}{\gamma}\,k_\mu k_\nu\right)e^{-ik(x-y)}$$

$$= \frac{e^2}{2\pi^2}\int d^4k\,\text{sign}\,(k^0)\delta'(kk)\frac{1}{(kk)^2}\,(kk)^2\,\frac{1}{\gamma}\,e^{-ik(x-y)}. \tag{3}$$

It is well known that the product of distributions which appears above is ambiguous. If we put

$$\delta'(kk)\frac{1}{(kk)^2}(kk)^2 = \delta'(kk)\left\{\frac{1}{(kk)^2}(kk)^2\right\} = \delta'(kk),$$

then

$$[S(x), S(y)] = \frac{e^2}{2i\gamma} \operatorname{sign}(x^0 - y^0)\Theta[(x-y)(x-y)]$$

and we recover the free phase commutator from [1],

$$[S(x), S(y)] = \frac{2\pi}{i} \operatorname{sign}(x^0 - y^0)\Theta[(x-y)(x-y)]$$

for

$$\gamma = \frac{e^2}{4\pi}.$$

3. *Conclusions*

It is clear that the contradiction between the two values of the coefficient γ given in [1] and [2] respectively is traceable to the singular nature of the integral (3). The calculation given in [2] is unambiguous but it is just one way to define the product of distributions in question. For this reason we think that the method given in [1], which is both elementary and unambiguous, is to be preferred and that

$$\gamma = \frac{e^2}{4\pi}$$

is the distinguished value of the coefficient γ.

It was noted in [2] that for a restricted gauge transformation

$$\delta A_\mu(x) = \partial_\mu f(x), \qquad \Box f = 0,$$

we have for the phase (2)

$$\delta S(x) = -\frac{e}{2} f(x)$$

instead of $-ef(x)$. It is clear that the unexpected appearance of the factor 1/2 can also be traced to the same origin, namely to the ambiguity of the product $(1/x)\delta(x)$. Schwinger writes (Eq. (1.85) in [3])

$$\frac{\delta(x)}{x} = -\delta'(x) \tag{4}$$

while Antosik, Mikusiński and Sikorski [4] have

$$\frac{1}{x}\delta(x) = -\tfrac{1}{2}\delta'(x). \tag{5}$$

The calculation given in the Appendix in [2] shows that if one calculates $\delta S(x)$ from (2) using the Fourier transform, one has to apply (5) rather than (4).

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