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BOOST-INVARIANT VORTEX IN RELATIVISTIC FLUID*

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Hydrodynamic equations describing the Lorentz-boost-invariant fluid with azimuthal symmetry are generalized to include the rotational motion around the symmetry axis. The solution with no radial motion does not exist for fluids whose entropy is proportional to an arbitrary power of temperature.

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It was recently suggested by Bjorken [1] that the boost-invariant solutions of the hydrodynamic equations may be useful in description of the time development of the quark-gluon plasma, possibly formed in high-energy heavy ion collisions. In Ref. [1] only the longitudinal motion of the plasma was discussed. It was shown in Ref. [2] how to include the radial motion of the plasma for the system with full rotational symmetry around the collision axis. The discussion of the solutions in this more realistic case was given in Ref. [2] and [3]. The purpose of the present note is to show that the result of Ref. [2] can be generalized to include also the rotational motion of the plasma around the collision axis. As an illustration, we discuss the simplest solution of so obtained equations of motion: a boost-invariant vortex rotating around the axis of the collision. We show that such a solution does not exist for liquids whose equations of state, Eq. (21), belong to a class which includes the one of the black body radiation as a special case.

The hydrodynamic equations are obtained from the conservation law

$$\partial_\mu T^{\mu\nu} = 0 \tag{1}$$

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and the well-known formula for the energy-momentum tensor of the perfect fluid

$$T^{\mu\nu} = (\varepsilon + p)U^\mu U^\nu - pg^{\mu\nu}, \tag{2}$$

where ε and p are energy density and pressure of the fluid (in the local rest frame) and U^μ is its 4-velocity. Assuming that the considered system is independent of an arbitrary boost along the collision axis (z-axis) we conclude that ε and p can depend only on $\tau \equiv \sqrt{t^2 - z^2}$, r and ϕ where z, r and ϕ are cylindrical coordinates. Furthermore, we assume, following Ref. [1] and [2]

$$U^\mu = \gamma(1, \vec{v}, z/t), \tag{3}$$

where $\gamma = (1 - v^2 - z^2/t^2)^{-1/2}$ and $\vec{v}$ is the component of the velocity of the fluid orthogonal to the z-axis. $\vec{v}$ can, in general, depend on τ, r and ϕ.

In the following it shall be convenient to split $\vec{v}$ into radial and transverse parts

$$\vec{v} = \vec{v}_r + \vec{v}_t, \tag{4}$$

where

$$\vec{v}_r = v_r \frac{\vec{r}}{r}, \tag{5}$$

$$\vec{v}_t = v_t \vec{n} \times \frac{\vec{r}}{r}. \tag{6}$$

Here $\vec{n}$ is a unit vector in the direction of the z axis.

Following Ref. [2] we shall assume that there is no ϕ dependence of the scalar quantities considered, i.e. that the system exhibits full rotational symmetry around the z axis.

Substituting Eq. (2) into (1) we obtain

$$U^\nu[U^\mu \partial_\mu(\varepsilon + p) + (\varepsilon + p)\partial_\mu U^\mu] + (\varepsilon + p)U^\mu \partial_\mu U^\nu - \partial^\nu p = 0. \tag{7}$$

Multiplication by U_ν gives

$$U^\mu \partial_\mu \varepsilon + (\varepsilon + p)\partial_\mu U^\mu = 0 \tag{8}$$

and thus Eq. (7) can be written as

$$U^\nu U^\mu \partial_\mu p - \partial^\nu p + (\varepsilon + p)U^\mu \partial_\mu U^\nu = 0. \tag{9}$$

Using Eqs. (3)–(6) it is not difficult to show that for any function $\psi(\tau, r)$ of τ and r one has

$$U^\mu \partial_\mu \psi = \gamma \frac{\tau}{t} \dot{\psi} + \gamma v_r \psi', \tag{10}$$

and that

$$\partial_\mu U^\mu = \frac{\tau}{t} \dot{\gamma} + \frac{1}{t} \gamma + \gamma \left(v_t' + \frac{v_r}{r} \right) + \gamma' v_r, \tag{11}$$

where dot denotes partial derivative with respect to time t and prime that with respect to r. These formulae, substituted into Eq. (8) give

$$\gamma\dot{\varepsilon}+\gamma v_r\varepsilon'+(\dot{\gamma}+(\gamma v_r)'+\gamma v_r/r+\gamma/t)(\varepsilon+p) = 0. \tag{12}$$

Following Ref. [1], this equation can be rewritten as an equation for the entropy of the plasma. Using

$$\varepsilon+p = Ts; \quad d\varepsilon/ds = T, \tag{13}$$

we obtain at the plane $z = 0$

$$(rs\gamma t)^{\cdot}+(rs\gamma tv_r)' = 0. \tag{14}$$

This equation expresses conservation of entropy [4]. For any other plane $z \neq 0$ the corresponding equation is obtained by Lorentz boost.

For $v = v_r$ (i.e. $v_t = 0$), Eq. (14) is identical to that obtained in Ref. [2].

Second equation can be obtained by multiplication of Eq. (9) by the 4-vector

$$(1, 0, 0, z/t). \tag{15}$$

One obtains, after some algebraic manipulations

$$v^2\gamma\dot{p}+\gamma v_r p'+(\varepsilon+p)(\dot{\gamma}+v_r\gamma') = 0, \tag{16}$$

which, using Eq. (13) gives

$$v(T\gamma v)^{\cdot}+v_r(T\gamma)' = 0. \tag{17}$$

For $v = v_r$ it is identical with the second equation of Ref. [2].

The last equation is obtained by multiplication of Eq. (9) by the 4-vector $(0, \vec{v}_t, 0)$. The result is

$$(rT\gamma v_t)^{\cdot}+v_r(rT\gamma v_t)' = 0. \tag{18}$$

We thus obtained 3 equations (14) (17) and (18) for 4 unknowns: T, s, v_r, v_t. They should be supplemented by equation of state and then the system is complete.

For illustration, we shall now discuss the solutions of Eqs. (14), (17) and (18) in the case when there is no radial motion, i.e. for the fluid which is simply rotating around the collison axis. Using the condition $v_r = 0$ we obtain from Eq. (14)

$$s\gamma t = A(r), \tag{19}$$

where $A(r)$ is an arbitrary function of r. Similarly, it follows from Eqs. (17) and (18)

$$T\gamma v = B(r), \tag{20}$$

where $B(r)$ is another arbitrary function of r. Assuming the equation of state in the form

$$s = \Omega T^{\kappa}, \tag{21}$$

we can easily obtain T and v in terms of A and B. First we notice that

$$v^{\kappa}\gamma^{\kappa-1} = A(r)t, \tag{22}$$

where $\Lambda = \Omega B^{\kappa}/A$. Eq. (22) implies that v depends only on Λt and is not difficult to solve. However, these observations contradict the original equations (17) and (18). To see this, let us observe that in the limit $v_r \to 0$ but for $v_r \neq 0$, Eqs. (17) and (18) imply a consistency condition

$$\frac{1}{v}(T\gamma)' = \frac{1}{r}(rT\gamma v)', \tag{23}$$

which, using Eqs. (19)–(21) can be rewritten as

$$\frac{1}{v}(B/v)' = \frac{1}{r}(rB)' \tag{24}$$

(we still work in the limit $v_r \to 0$). Eq. (24) has far-reaching consequences if we remember that v is the function of Λt and that B is a function of r. To satisfy that we have to require

$$\Lambda'/\Lambda = a/r \qquad B'/B = b\Lambda'/\Lambda, \tag{25}$$

where a and b are constants. Consequently, we obtain

$$\Lambda = (r/r_0)^a \qquad B = (r/r_1)^{ab} \tag{26}$$

with arbitrary constants r_0, r_1.

Eq. (24) can now be solved for v and we obtain

$$v^2 = (\Lambda t)^{2b}/[\lambda + c^2(\Lambda t)^{2b}], \tag{27}$$

where λ is another constant and $c^2 = 1+1/ab$.

It is easy to see that Eq. (27) is inconsistent with Eq. (22) and thus a boost invariant vortex does not exist in ideal fluids with equations of state given by Eq. (21). Consequently, an arbitrarily small fluctuation with $v_r \neq 0$ changes discontinuously the solution with $v_r = 0$, which is therefore unstable under such fluctuations.

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