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THE CORRESPONDENCE CONSIDERATIONS IN RELATIVISTIC QUANTUM MECHANICS

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The correspondence of the previously given relativistic quantum mechanics of spinless or spin 1/2 particle with the quantum nonrelativistic and the classical relativistic mechanics is proved. In the Galilean quantum case the appearance of the familiar transformation phase factor is explained and the origin of the mass superselection rule indicated. In the relativistic classical case the equation for trajectories and the covariant equation of motion for spin are obtained. The conditions are found under which the usual theory of relativistic wave equations is of relevance (in some particular reference frame).

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1. *Introduction*

The present article is a direct continuation of two papers [1, 2] devoted to the study of relativistic quantum mechanics. We refer to [1] for general idea and all denotations. Here (in Section 2) we show how the proper time relativistic quantum mechanics of [1] reproduces the theory of the Dirac or the Klein-Gordon wave equations when the mass spread of the particle can be regarded as small and the physical time can be regarded as a parameter. This approximation is essentially the case of small $\hbar/c^2$. Moreover, this derivation sheds some light on the non-covariancy of the usual single body quantum mechanics of these wave equations. In Section 3 and 4 we show that by accepting the additional assumptions about smallness of $1/c$ or $\hbar$ one is led to the non-relativistic quantum mechanics or the relativistic classical mechanics respectively. All three limits will constitute the lowest orders in some asymptotic expansions in powers of small parameters. We shall not investigate rigorously the conditions for correctness of these expansions. Nevertheless our procedure will be justified heuristically (as is the case in most correspondence considerations).

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2. *Expansion around definite mass and space-like hypersurface*

Two fundamental parts of quantum theory are: the evolution equation of states and the formula for computing physical quantities, and these two elements will now be considered successively.

Let $S(x)$ be some real function of space-time variables x^μ, and let $\Phi(\tau, x) \in H$ (the Hilbert space of the model) for all τ. We introduce the following transformation $\Phi \to \Psi$:

$$\Phi(\tau, x) = \Psi(\tau - S(x), x). \tag{2.1}$$

If Φ is a physical state as defined in [1] then Φ satisfies the evolution equation

$$M\Phi(\tau, x) = \mathcal{M}\Phi(\tau, x), \tag{2.2}$$

where[1] we have denoted $M \equiv -\frac{i\hbar}{c^2}\frac{\partial}{\partial\tau}$. When expressed in terms of $\Psi(\tau, x)$ this equation becomes

$$[\mathcal{M}(\pi_\mu + c^2\partial_\mu S \cdot M) - M]\Psi(\tau, x) = 0 \tag{2.3}$$

and when further Fourier-transformed according to

$$\Psi(\tau, x) = \left(2\pi\frac{\hbar}{c^2}\right)^{-1/2} \int e^{\frac{ic^2}{\hbar}\tau m} \hat{\Psi}(m, x) dm \tag{2.4}$$

it yields

$$[\mathcal{M}(\pi_\mu + mc^2\partial_\mu S) - m]\hat{\Psi}(m, x) = 0. \tag{2.5}$$

The existence of the Fourier transform $\hat{\Psi}$ follows from the definition of the physical state (see postulate IV in [1], Section 4).

Let us now assume that the extent of $\Phi(\tau, x)$ in the time-like direction is small for some large interval of τ values, i.e. for every such τ $\Phi(\tau, x)$ is concentrated around some space-like hypersurface. We choose $S(x)$ so that with flowing τ these hypersurfaces are described by the equation

$$S(x) = \tau. \tag{2.6}$$

$\Psi(\tau, x)$ is then peaked around zero of the first argument, with some spread, say $\Delta\tau$. We assume also that the state Φ is concentrated in mass around some value m_0; hence $\hat{\Psi}(m, x)$ is also peaked in the first argument, around $m = m_0$, with spread Δm. The product $\Delta\tau \cdot \Delta m$ is of order of the factor $\hbar/c^2$ in the exponent in the Fourier transformation rule (2.4). This factor we assume to be small and we shall expand equations and formulae in increasing order of smallness. It will be convenient to replace $\hbar/c^2$ by $\varepsilon^2\hbar/c^2$, investigate the asymptotic expansion for $\varepsilon \to 0$ and at the end set $\varepsilon = 1$. We assume the state to be minimal in the sense that $\Delta\tau \sim \varepsilon$ and $\Delta m \sim \varepsilon$. To this end let us set

$$\hat{\Psi}(m, x) = \hat{f}(m - m_0)\psi(m, x) \tag{2.7}$$

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[1] We recall that $c\mathcal{M} \equiv c\mathcal{M}(\pi_\mu) = \frac{\alpha}{\varrho c}\pi^2 + \varrho c\alpha^\dagger$ in the spin 0 case and $c\mathcal{M} \equiv c\mathcal{M}(\pi_\mu) = \gamma^\mu\pi_\mu$ for spin 1/2.

where $f \in \mathscr{S}$ (the Schwartz class of functions); f, $\hat{f}$ are concentrated around zero and $\psi(m, x)$ is slowly changing in m, compared to $\hat{f}$. f is assumed to have the following properties:

$$\int |f(\tau)|^2 d\tau = \int |\hat{f}(m)|^2 dm = 1,$$
$$\int |f(\tau)|^2 \tau d\tau = \int |\hat{f}(m)|^2 m dm = 0, \tag{2.8}$$

and the "moments" defined as

$$\kappa_{kln} = (-1)^l \left(\frac{i\hbar}{c^2}\right)^{k+l} \int \overline{f^{(k)}} f^{(l)}(\tau) \tau^n d\tau \tag{2.9}$$

are rapidly decreasing with the growing order of $k+l+n$. For the sake of expansion we replace $\hat{\Psi}$ by

$$\hat{\Psi}_\varepsilon(m, x) = \hat{f}\left(\frac{m-m_0}{\varepsilon}\right) \psi(m, x) \tag{2.10}$$

and Φ by[2]

$$\Phi_\varepsilon(\tau, x) = \left(2\pi\varepsilon^2 \frac{\hbar}{c^2}\right)^{-1/2} \int e^{\frac{ic^2}{\varepsilon^2 \hbar}(\tau - S(x))m} \hat{\Psi}_\varepsilon(m, x) dm. \tag{2.11}$$

Obviously $\hat{\Psi}_{\varepsilon=1} = \hat{\Psi}$, $\Phi_{\varepsilon=1} = \Phi$. $\hat{\Psi}_\varepsilon$ satisfies the same equation (2.5) as $\hat{\Psi}$. With the substitution $m' = \dfrac{m-m_0}{\varepsilon}$ it can be stated as

$$\hat{f}(m')\,[\mathscr{M}(\pi_\mu + (m_0 + \varepsilon m')c^2 \partial_\mu S) - (m_0 + \varepsilon m')]\,\psi(m_0 + \varepsilon m', x) = 0. \tag{2.12}$$

As $\hat{f}(m')$ confines strongly the scope of m', the last equation can be expanded in powers of $\varepsilon m'$, which yields

$$[\mathscr{M}(\pi_\mu + m_0 c^2 \partial_\mu S) - m_0]\,\psi(m_0, x) = 0 \tag{2.13}$$

in the lowest order; the higher order equations are

$$\begin{aligned}
&[\mathscr{M}(\pi_\mu + m_0 c^2 \partial_\mu S) - m_0]\,\psi^{(n+1)}(m_0, x) \\
&= (n+1)\left[\left(1 - \frac{\partial}{\partial m_0} \mathscr{M}(\pi_\mu + m_0 c^2 \partial_\mu S)\right) \psi^{(n)}(m_0, x)\right. \\
&\quad \left. - \frac{n}{2} \frac{\partial^2}{\partial m_0^2} \mathscr{M}(\pi_\mu + m_0 c^2 \partial_\mu S) \cdot \psi^{(n-1)}(m_0, x)\right],
\end{aligned} \tag{2.14}$$

where $\psi^{(k)}(m_0, x) \equiv \dfrac{\partial^k}{\partial m_0^k} \psi(m_0, x)$. In the spin 1/2 case (2.13) is the Dirac equation (the term $m_0 c^2 \partial_\mu S$ changes the electromagnetic gauge only). In the spin 0 case, if after the equations are written down the arbitrary mass scale ϱ in $\mathscr{M}$ is chosen as $\varrho = m_0$ (which

[2] For $\varepsilon \neq 1$ Φ_ε satisfies equation slightly modified as compared with (2.2), but at the end we shall put $\varepsilon = 1$.

can always be accomplished — see Appendix 1) the equation (2.13) is equivalent to the projection equation[3] $\xi_-\psi(m_0, x) = 0$ and the Klein-Gordon equation for $\xi_+\psi(m_0, x)$ (see Appendix 2). Both the Dirac and the Klein-Gordon equation need the initial conditions to be specified on a space-like hypersurface, which is consistent with the confinement of $\Phi(\tau, x)$ for $\tau = \text{const}$ on (around) $S(x) = \text{const.}$ (cf. formula (2.19) below).

Let us now turn to the calculation of physical quantities. We note first that if Φ is a state and q is an observable then the action of q on Φ can be transformed onto $\psi(m, x)$ with the use of formulae (2.1), (2.4), (2.7), and we denote this transformed operator by q_ψ. Only in the case of $q = \pi_\mu$ (or p_μ) does the operator q_ψ differ from q and

$$q_\psi = \pi_\mu + mc^2\partial_\mu S. \tag{2.15}$$

The formulae (2.10) and (2.11) remain valid for $\Phi_1 = q\Phi$ (with corresponding $\hat{\Psi}_1$ and $\psi_1(m, x) = q_\psi\psi(m, x)$. Therefore we can expand the form $\langle\Phi_\varepsilon|\Phi_{1_\varepsilon}\rangle = (\Phi_\varepsilon, \eta\Phi_{1_\varepsilon})$ used for determination of physical quantities (see [1]). To this end let us denote

$$\Psi_\varepsilon(\tau, x) = \left(2\pi\varepsilon^2\frac{\hbar}{c^2}\right)^{-1/2}\int e^{\frac{ic^2}{\varepsilon\hbar}\tau m}\hat{\Psi}_\varepsilon(m, x)dm \tag{2.16}$$

so that

$$\Phi_\varepsilon(\tau, x) = \Psi_\varepsilon\left(\frac{\tau - S(x)}{\varepsilon}, x\right). \tag{2.17}$$

Expanding (2.16) formally one obtains

$$\Psi_\varepsilon(\tau, x) = e^{\frac{ic^2}{\varepsilon\hbar}m_0\tau}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\hbar}{c^2}\varepsilon\right)^n f^{(n)}(\tau)\psi^{(n)}(m_0, x). \tag{2.18}$$

With the help of (2.17) $\langle\Phi_\varepsilon|\Phi_{1_\varepsilon}\rangle$ takes the form

$$\int \Phi_\varepsilon^\dagger\eta\Phi_{1_\varepsilon}(\tau, x)d^4x = \varepsilon\int \Psi_\varepsilon^\dagger\eta\Psi_{1_\varepsilon}(\tau', x)\delta(\tau - S(x) - \varepsilon\tau')d\tau'd^4x.$$

The expansion of δ and substitution of (2.18) for Ψ_ε results in

$$\langle\Phi_\varepsilon|\Phi_{1_\varepsilon}\rangle = \varepsilon\sum_{k,l,n=0}^{\infty}\frac{\varepsilon^{k+l+n}}{k!l!n!}(-1)^n\kappa_{kln}$$
$$\times\frac{d^n}{d\tau^n}\int\psi^{(k)\dagger}\eta\psi_1^{(l)}(m_0, x)\delta(\tau - S(x))d^4x, \tag{2.19}$$

where κ_{kln} are the "moments" (2.9). The confinement of Φ around $S(x) = \tau$ is explicit. In order to secure the reality of physical quantities (which could possibly be lost in each

[3] In both spin cases we denote $\xi_\pm \equiv \frac{1}{2}(1 \pm \eta)$. $\xi_\pm$ are projection operators.

order separately) we regard half the sum of (2.19) and its complex cojugate as the proper expansion of $\langle q\rangle_{\Phi_\varepsilon}\langle\Phi_\varepsilon|\Phi_\varepsilon\rangle$:

$$\langle q\rangle_{\Phi_\varepsilon}\langle\Phi_\varepsilon|\Phi_\varepsilon\rangle = \tfrac{1}{2}\,\varepsilon\sum_{k,l,n=0}^{\infty}\frac{\varepsilon^{k+l+n}}{k!l!n!}(-1)^n\kappa_{kln}$$

$$\frac{d^n}{d\tau^n}\int(\psi^{(k)\dagger}\eta(q_\psi\psi)^{(l)}+(q_\psi\psi)^{(k)\dagger}\eta\psi^{(l)})\,(m_0,x)\delta(\tau-S(x))d^4x,\ (\overline{\kappa_{lkn}} = \kappa_{kln}). \tag{2.20}$$

It is quite obvious that the whole expansion (both the evolution equation and the expansion (2.20)) is Poincare covariant in each order separately (all $\psi^{(k)}$ are transformed in the same way as the corresponding Φ, and S is a scalar function).

In the lowest order (2.20) yields ($\varepsilon = 1$)

$$\langle q\rangle_\Phi\langle\Phi|\Phi\rangle = \tfrac{1}{2}\int(\psi^\dagger\eta q_\psi\psi+(q_\psi\psi)^\dagger\eta\psi)\,(m_0,x)\delta(\tau-S(x))d^4x. \tag{2.21}$$

The next to lowest order vanishes (because of (2.8)). The function $S(x)$, as an operator, is a measure of time, and in the considered approximation we have

$$\langle(\mathcal{M}-m_0)^k\rangle_\Phi = 0,\qquad \langle(S(x)-\tau)^k\rangle_\Phi = 0,\qquad (k\geqslant 1)$$

in particular

$$\langle\mathcal{M}\rangle_\Phi = m_0,\qquad \langle S(x)\rangle_\Phi = \tau. \tag{2.22}$$

In general even the lowest order approximation is not self-contained if $S(x)$ is not known. Let us, however, assume that $S(x)$ is such that in some reference frame it depends on the time variable only: $S = S(t)$, and let $T = S^{-1}$. Then

$$\delta(\tau-S(t)) = \delta(t-T(\tau))\frac{dT}{d\tau}$$

and the lowest order approximation reads now

$$\langle q\rangle_\Phi(\tau) = \frac{\int(\psi^\dagger\eta q_\psi\psi+(q_\psi\psi)^\dagger\eta\psi)\,(m_0,cT(\tau),\vec{x})d^3x}{2\int\psi^\dagger\eta\psi(m_0,cT(\tau),\vec{x})d^3x}, \tag{2.23}$$

where the factor $\frac{dT}{d\tau}$ has been cancelled. $T(\tau)$ is equal to the physical time which becomes a parameter and all reference to τ can be dropped. The term $m_0c^2\partial_\mu S(x)$ (here proportional to $\delta_{0\mu}$), which adds to π_μ in accordance with (2.15), can be absorbed into[4] A_μ and we obtain a self-contained theory (here $\langle q\rangle(t)$ indicates dependence on physical time)

spin = 1/2

$$(\gamma^\mu\pi_\mu-m_0c)\psi(m_0,x) = 0,$$

$$\langle q\rangle(t) = \frac{\int(\psi^\dagger\gamma^0q\psi+(q\psi)^\dagger\gamma^0\psi)\,(m_0,ct,\vec{x})d^3x}{2\int\psi^\dagger\gamma^0\psi(m_0,ct,\vec{x})d^3x}, \tag{2.24}$$

[4] More properly, compensated by suitable phase transformation.

spin $= 0$

$$(\pi^2 - m_0^2c^2)\varphi(m_0, x) = 0,$$

$$\langle q\rangle(t) = \frac{\int (\bar{\varphi}(\xi_+ q\xi_+)\varphi + \overline{(\xi_+ q\xi_+)\varphi}\cdot\varphi)(m_0, ct, \vec{x})d^3x}{2\int \bar{\varphi}\varphi(m_0, ct, \vec{x})d^3x}, \tag{2.25}$$

where $\varphi = \xi_+\psi$. The rules for computing physical quantities differ from these usually employed, but for eigenvectors of observables proportional to matrix **1** (such as energy, momentum etc.) they yield the same values. To illustrate other applications of these formulae we compute the velocity and the energy:

spin $= 1/2$

$$\frac{d}{dt}\langle\vec{z}\rangle(t) = c\,\frac{\left\langle\dfrac{d\vec{z}}{d\tau}\right\rangle}{\left\langle\dfrac{dz^0}{d\tau}\right\rangle} = c\,\frac{\int \psi^\dagger\gamma^0\vec{\gamma}\psi(m_0, ct, \vec{x})d^3x}{\int \psi^\dagger\gamma^0\psi(m_0, ct, \vec{x})d^3x}$$

$$\times\left(\frac{\int \psi^\dagger\gamma^0\gamma^0\psi(m_0, ct, \vec{x})d^3x}{\int \psi^\dagger\gamma^0\psi(m_0, ct, \vec{x})d^3x}\right)^{-1} = c\,\frac{\int \psi^\dagger\gamma^0\vec{\gamma}\psi(m_0, ct, \vec{x})d^3x}{\int \psi^\dagger\psi(m_0, ct, \vec{x})d^3x},$$

$$\langle E\rangle(t) \equiv c\left\langle \pi^0 + \frac{e}{c}A^0\right\rangle = c\left\langle m_0c\gamma^0 + \gamma^0\vec{\gamma}\cdot\vec{\pi} + \frac{e}{c}A^0\right\rangle$$

$$= m_0c\left\langle\frac{dz^0}{d\tau}\right\rangle + e\langle A^0\rangle$$

$$= \frac{m_0c^2\int \psi^\dagger\psi(m_0, ct, \vec{x})d^3x + e\int A^0\psi^\dagger\gamma^0\psi(m_0, ct, \vec{x})d^3x}{\int \psi^\dagger\gamma^0\psi(m_0, ct, \vec{x})d^3x}, \tag{2.26}$$

spin $= 0$

$$\frac{d}{dt}\langle\vec{z}\rangle(t) = c\,\frac{\left\langle\dfrac{d\vec{z}}{d\tau}\right\rangle}{\left\langle\dfrac{dz^0}{d\tau}\right\rangle} = c\,\frac{\left\langle\dfrac{2}{m_0}\alpha\vec{\pi}\right\rangle}{\left\langle\dfrac{2}{m_0}\alpha\pi^0\right\rangle}$$

$$= \frac{2\int \bar{\varphi}\vec{\pi}\varphi(m_0, ct, \vec{x})d^3x}{\int (\bar{\varphi}\pi^0\varphi + \overline{\pi^0\varphi}\varphi)(m_0, ct, \vec{x})d^3x},$$

$$\langle E\rangle(t) \equiv c\left\langle \pi^0 + \frac{e}{c}A^0\right\rangle = m_0c\left\langle\frac{dz^0}{d\tau}\right\rangle + e\langle A^0\rangle$$

$$= \frac{c\,\frac{1}{2}\int (\bar{\varphi}\pi^0\varphi + \overline{\pi^0\varphi}\varphi)(m_0, ct, \vec{x})d^3x + e\int A^0\bar{\varphi}\varphi(m_0, ct, \vec{x})d^3x}{\int \bar{\varphi}\varphi(m_0, ct, \vec{x})d^3x}$$

$$(\xi_+\alpha\xi_+ = \tfrac{1}{2}\xi_+). \tag{2.27}$$

Note, for instance, that the formula for velocity of the particle with spin is the same as usual; on the other hand, here an eigenvector of $\gamma^0\gamma^k$ is not the eigenvector of velocity, so that the familiar difficulty with velocity eigenvalues $\pm c$ does not arise.

However, the theory of (2.24) and (2.25) is not covariant as the scalar function $S(x)$ has to be transformed with the change of reference frame and we must return to the general case of (2.21): the state which is confined around $t =$ const. in some reference frame loses this property when viewed from another frame. This is, in our opinion, the reason why the usual (first quantized) quantum theory of relativistic wave equations fails to be Poincare covariant (though the equations themselves are invariant).

The assumptions of the present section (smallness of $\hbar/c^2$ and "minimality" of states as described before (2.7)) are consistent with both the Galilean and the classical limits. These two will be now considered.

3. *Expansion around the Galilean limit*

Let us again choose some reference frame and assume that in that frame $S = S(t)$, so that the physical time is a parameter and as a function of proper time is $T(\tau)$. Following the classical analogy we assume that the physical time is equal to the proper time i.e. $T(\tau) = \tau + t_0$, or $S(t) = t - t_0$. We repeat now the procedure of Section 2 replacing, however, consequently $1/c$ by ε/c (in agreement with the substitution $\frac{\hbar}{c^2} \to \varepsilon^2 \frac{\hbar}{c^2}$ performed in the preceding section). In that way instead of (2.12) the equation

$$\hat{f}\left(\frac{m-m_0}{\varepsilon}\right)\left[\mathcal{M}\left(mc+\frac{1}{c}E_k, \vec{\pi}\right)-m\right]\Bigg|_{\frac{1}{c}\to\frac{\varepsilon}{c}} \psi(m, t, \vec{x}) = 0 \tag{3.1}$$

is obtained ($E_k \equiv i\hbar \frac{\partial}{\partial t} - eA^0$). In order that A^0, $\vec{A}$ do not fall out from E_k, $\vec{\pi}$ in the limit, we assume that A^0, $\frac{1}{c}\vec{A}$ have asymptotic expansions in powers of $\frac{1}{c^2}$ beginning with terms of zeroth order. This assumption is consistent, on the formal level, with the Maxwell equations and the Lorentz transformation laws. The lowest order so obtained forms the Galilean limit of electrodynamics discussed and called "magnetic" by Le Bellac and Lévy-Leblond in [3].

In contrast to the preceding section here $\psi(m, t, \vec{x})$ cannot be regarded as independent of ε. Denoting $\varphi \equiv \xi_+\psi$ for spin $= 0$ and $\varphi \equiv \xi_+\psi + \frac{1}{\varepsilon}\xi_-\psi$ for spin $= 1/2$ one obtains from (3.1):

spin $= 1/2$

$$\left(\frac{1}{c}E_k\xi_+ - \left(2mc + \frac{\varepsilon^2}{c}E_k\right)\xi_- - \vec{\gamma}\cdot\vec{\pi}\right)\varphi(m, t, \vec{x}) = 0, \tag{3.2}$$

spin = 0

$$(m+\varrho)\xi_-\psi(m,t,\vec{x}) = (m-\varrho)(2\alpha-1)\varphi(m,t,\vec{x}),$$

$$\left(2mE_k-\vec{\pi}^2+\frac{\varepsilon^2}{c^2}E_k^2\right)\varphi(m,t,\vec{x}) = 0 \tag{3.3}$$

(for spin 0 case see Appendix 2). The expansion of φ in powers of ε^2:

$$\varphi = \sum_{n=0}^{\infty}\frac{(\varepsilon^2)^n}{n!}\varphi_n \tag{3.4}$$

is consistent with the above equations and transforms them into infinite series of equations. Each of these will be in turn expanded around $m = m_0$ in analogy with (2.13), (2.14). The idea of the procedure being clear, we shall write down the lowest order equations only (choosing $\varrho = m_0$ in the scalar case):
spin = 1/2

$$\left(\frac{1}{c}E_k\zeta_+-2m_0c\zeta_--\vec{\gamma}\cdot\vec{\pi}\right)\varphi_0(m_0,t,\vec{x}) = 0, \tag{3.5}$$

spin = 0

$$\zeta_-\psi_0(m_0,t,\vec{x}) = 0, \tag{3.6}$$

$$(2m_0E_k-\vec{\pi}^2)\varphi_0(m_0,t,\vec{x}) = 0. \tag{3.7}$$

(3.7) is the Schrödinger and (3.5) the Lévy-Leblond equation [4]. For the calculation of physical quantities the formula (2.20) should be used, where for ψ's their expansions are to be substituted. We leave it to the reader to convince himself that the lowest and next to lowest order terms yield ($\varepsilon = 1$)
spin = 1/2

$$\langle q\rangle(t) = \frac{\int\varphi_0^\dagger(\xi_+q\zeta_++\xi_+q\xi_--\xi_-q\zeta_+)\varphi_0(m_0,t,\vec{x})d^3x}{\int\varphi_0^\dagger\zeta_+\varphi_0(m_0,t,\vec{x})d^3x}, \tag{3.8}$$

spin = 0

$$\langle q\rangle(t) = \frac{\int\bar{\varphi}_0(\xi_+q\xi_+)\varphi_0(m_0,t,\vec{x})d^3x}{\int\bar{\varphi}_0\varphi_0(m_0,t,\vec{x})d^3x} \tag{3.9}$$

and that the resulting formulae together with (3.5), (3.7) constitute the usual nonrelativistic theory.

Let us now similarly replace $\frac{1}{c}$ by $\frac{\varepsilon}{c}$ in the transformation law of ψ: $\psi'(m_0,t,\vec{x}) = W\psi(m_0,t',\vec{x}')$ where

$$t' = \gamma\left[t-b-\frac{\vec{v}}{c^2}\cdot(\vec{x}-\vec{a})\right],$$

$$x' = R^{-1}\left[\vec{x}-\vec{a}+(\gamma-1)\frac{(\vec{x}-\vec{a})\cdot\vec{v}}{v^2}\vec{v}-\gamma\vec{v}(t-b)\right]$$

and $W \equiv 1$ for spin 0, and W is the bispinor transformation for spin 1/2. This transformation law is consistent with the expansion (3.4) and in the lowest order we get

spin = 1/2

$$\varphi_0'(m_0, t, \vec{x}) = \left[\left(1 - \frac{\vec{v}}{2c} \cdot \xi_- \vec{\gamma} \xi_+\right) \xi_+ S(R) \xi_+ + \xi_- S(R) \xi_-\right] \times \varphi_0(m_0, t-b, R^{-1}(\vec{x} - \vec{a} - \vec{v}(t-b))), \tag{3.10}$$

spin = 0

$$\varphi_0'(m_0, t, \vec{x}) = \varphi_0(m_0, t-b, R^{-1}(\vec{x} - \vec{a} - \vec{v}(t-b))) \tag{3.11}$$

(for the matrix transformation in (3.10) see Appendix 3). Both transformations form representations of the Galilei group. We recall, however, that not only ψ but also S have to be transformed. In general this would break the Galilean invariance, but we again leave it to the reader to verify that in the considered approximation the change in S can be completely compensated by the translation in t_0: $t_0 \to t_0 + b$ and by placing the factor $e^{\frac{im}{\hbar}(-1/2\, v^2(t-b) + \vec{v}\cdot(\vec{x}-\vec{a}))}$ on the r.h.s. of transformations (3.10), (3.11). This appears to be the origin of that familiar factor, which is here the nonrelativistic remnant of the function $S(x)$. The theory of (3.5)–(3.9) is covariant with respect to transformations (3.10), (3.11) (with potentials A^μ transforming according to the magnetic limit of electrodynamics).

Finally, let us take the product of two states Φ_1, Φ_2 with different masses m_1, m_2. We obtain a formula similar to (2.19) in which κ_{kln} are replaced by

$$(-1)^l \left(\frac{i\hbar}{c^2}\right)^{k+l} \int \overline{f^{(k)}} f^{(l)}(\tau) \tau^n e^{\frac{ic^2}{\varepsilon\hbar}(m_1 - m_2)\tau} d\tau.$$

These coefficients, being the Fourier transforms of Schwartz functions, vanish with decreasing ε faster than any power of ε. This is how here the nonrelativistic mass superselection rule originates. Similar result is obtained if different values of t_0 are assumed.

4. Expansion around classical limit

In the present Section we follow the procedure of Section 4 once more, but here the Planck constant will consequently be assumed to be small.

The substitution of $e^{-\frac{i}{\hbar}K(x)}\varphi(m, x)$ for $\psi(m, x)$ in equation (2.12) yields

$$\hat{f}\left(\frac{m - m_0}{\varepsilon}\right)\left[\mathcal{M}\left(P_\mu + \partial_\mu(mc^2 S + K) - \frac{e}{c} A_\mu\right) - m\right] \varphi(m, x) = 0. \tag{4.1}$$

We assume now the expansion

$$\varphi = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \varphi_n \tag{4.2}$$

and replace P_μ in (4.1) by the operator $P_{\varepsilon\mu}(m)$, defined in the Appendix 4 and having the following properties:

$$\hat{f}\left(\frac{m-m_0}{\varepsilon}\right)P_{\varepsilon\mu}(m) \sim \varepsilon^2, \quad P_{\varepsilon=1\mu}(m) = P_\mu, \quad P_{\varepsilon\mu}(m_0) = \varepsilon^2 P_\mu.$$

These properties can be interpreted as realization of $\hbar \to \varepsilon^2\hbar$. The equation (4.1) yields in the lowest order in ε for $m = m_0$

$$\left[\mathcal{M}\left(\partial_\mu(m_0c^2S+K)-\frac{e}{c}A_\mu\right)-m_0\right]\varphi_0(m, x) = 0, \tag{4.3}$$

(see Appendix 4). With the denotation $v_\mu \equiv \frac{1}{m_0}\left(\partial_\mu(m_0c^2S+K)-\frac{e}{c}A_\mu\right)$ this is equiva-ent to

$$v^2 = c^2 \tag{4.4}$$

and for spin = 1/2

$$(\gamma^\mu v_\mu - c)\varphi_0(m_0, x) = 0, \tag{4.5}$$

for spin = 0

$$\xi_-\varphi_0(m_0, x) = 0. \tag{4.6}$$

The lowest order of the continuity equation which follows from (4.1) can be transformed with the help of (4.5), (4.6) into

$$\partial_\mu(v^\mu F)(x) = 0 \tag{4.7}$$

where $F \equiv (\xi_+\varphi_0)^\dagger\xi_+\varphi_0(m_0, x)$ for spin = 0 and $F \equiv \varphi_0^\dagger\gamma^0\varphi_0(m_0, x) = \frac{c}{v^0}\varphi_0^\dagger\varphi_0(m_0, x)$ for spin = 1/2. It is simple matter to show that in the considered approximation

$$\langle \dot{z}^\mu\rangle(\tau) = \frac{\int v^\mu F(x)\delta(\tau-S(x))d^4x}{\int F(x)\delta(\tau-S(x))d^4x}.$$

If the support of F was of linear dimensions of order of ε at $\tau = 0$ (i.e. on hypersurface $S(x) = 0$) then due to (4.7) F remains concentrated on every hypersurface $S(x) = \tau$, hence in the lowest order

$$\langle \dot{z}^\mu\rangle(\tau) = v^\mu(\langle z\rangle(\tau)), \tag{4.8}$$

i.e. the trajectory is a line tangent to the field v^μ. (4.4) now shows that in the present limit τ is indeed the usual classical proper time. Due to (4.8) there is $\frac{d}{d\tau}G(\langle z\rangle(\tau)) = (v^\mu\partial_\mu G)(\langle z\rangle(\tau))$. Setting $G = v^\mu$ and taking (4.4) into account we arrive at the classical equation of motion:

$$m_0\frac{d^2}{d\tau^2}\langle z^\mu\rangle(\tau) = \frac{e}{c}F^{\mu\nu}(\langle z\rangle(\tau))\frac{d\langle z_\nu\rangle}{d\tau}. \tag{4.9}$$

Finally, we give the equation of motion of spin along the trajectory, which results from application of the above approximation procedure to the original equation of motion for $S^{\mu\nu}$ given in [1].

$$\frac{d}{d\tau}\langle S^{\mu\nu}\rangle = \frac{e}{m_0 c}(F^{\mu}_{\cdot\lambda}(\langle z\rangle)\langle S^{\lambda\nu}\rangle - F^{\nu}_{\cdot\lambda}(\langle z\rangle)\langle S^{\lambda\mu}\rangle). \tag{4.10}$$

Among the quantities $\langle S^{\mu\nu}\rangle$ only spin variables $\langle S^{i}\rangle = \frac{1}{2}\varepsilon^{ijk}\langle S^{jk}\rangle$ are independent, as due to (4.5) there is

$$\langle S^{0k}\rangle = -\left(\frac{\langle\vec{v}\rangle}{\langle v^0\rangle}\times\langle\vec{S}\rangle\right)^{k}. \tag{4.11}$$

The independent equations resulting from (4.10) take in the rest frame of the particle the form

$$\frac{d}{d\tau}\langle\vec{S}\rangle = \frac{e}{m_0 c}\langle\vec{S}\rangle\times\vec{H}(\langle z\rangle). \tag{4.12}$$

It should be stressed that here the covariant equations for covariantly transforming physical quantities in the classical limit are obtained.

APPENDIX 1

The arbitrariness of mass scale

The generator of the evolution $\mathcal{M}$ for a spinless particle [1] must employ, for dimensional reasons, a mass scale ϱ:

$$\mathcal{M} = \frac{\alpha}{\varrho c^2}\pi^2 + \varrho\alpha^{\dagger}. \tag{A1.1}$$

Apart from the metric operator $\eta = \alpha+\alpha^{\dagger}$ the matrices α, $\alpha^{\dagger}$ enter into the theory only through $\frac{\alpha}{\varrho}$ and $\varrho\alpha^{\dagger}$. We shall show that the change of the mass scale is equivalent to certain transformation conserving the metric:

$$U^{\dagger}\eta U = \eta. \tag{A1.2}$$

Indeed, the matrix transformation

$$U = \sqrt{\frac{\varrho}{\varrho'}}\,\alpha^{\dagger}\alpha + \sqrt{\frac{\varrho'}{\varrho}}\,\alpha\alpha^{\dagger} \tag{A1.3}$$

fulfills (A1.2), leaves all operators proportional to matrix **1** unchanged, and

$$U^{-1}\frac{\alpha}{\varrho}U = \frac{\alpha}{\varrho'},\qquad U^{-1}\varrho\alpha^{\dagger}U = \varrho'\alpha^{\dagger} \tag{A1.4}$$

(the anticommutation rule for α, $\alpha^{\dagger}$ has been used).

APPENDIX 2

Spinless particle and the Klein-Gordon equation

The equation

$$\left(\frac{\alpha}{\varrho c^2}\pi^2+\varrho\alpha^\dagger-m\right)\psi(m,x)=0 \tag{A2.1}$$

which appeared in connection with the spinless particle is equivalent to

$$c^2(\varrho m-m^2\alpha-\varrho^2\alpha^\dagger)\psi(m,x)=\alpha(\pi^2-m^2c^2)\psi(m,x). \tag{A2.2}$$

Combining this with the square of (A2.1) we see that (A2.1) is equivalent to the set

$$\begin{aligned}(\pi^2-m^2c^2)\psi(m,x)&=0,\\ (\varrho m-m^2\alpha-\varrho^2\alpha^\dagger)\psi(m,x)&=0.\end{aligned} \tag{A2.3}$$

With the use of equalities

$$\alpha^\dagger\xi_+=(\mathbf{1}-\alpha)\xi_+,\quad \alpha^\dagger\xi_-=-(\mathbf{1}+\alpha)\xi_- \tag{A2.4}$$

the last equation is shown to be equivalent to

$$(m+\varrho)\xi_-\psi(m,x)=(m-\varrho)(2\alpha-\mathbf{1})\xi_+\psi(m,x). \tag{A2.5}$$

APPENDIX 3

The Galilean limit of the bispinor representation $S(\Lambda)$

We apply to $S(\Lambda)=S(B)S(R)$ (R — rotation, B — boost) the ε-limiting process. The substitution of $\frac{\varepsilon}{c}$ for $\frac{1}{c}$ in B yields in the two lowest orders

$$B^\mu{}_\nu=\delta^\mu_\nu+\varepsilon^\mu_{\cdot\nu},\quad \varepsilon_{0k}=\frac{\varepsilon v^k}{c},\quad \varepsilon_{jk}=0. \tag{A3.1}$$

It follows that $S(B)$ in the same order is

$$S(B)=e^{-\frac{i}{2\hbar}\varepsilon_{\mu\nu}S^{\mu\nu}}=\mathbf{1}+\frac{\varepsilon\vec{v}}{2c}\cdot\gamma^0\vec{\gamma}. \tag{A3.2}$$

Substituting now $\psi=\xi_+\varphi+\varepsilon\xi_-\varphi$ in

$$\psi'=S(B)S(R)\psi$$

we obtain in the lowest order in ε^2

$$\begin{pmatrix}\xi_+\varphi_0\\ \xi_-\varphi_0\end{pmatrix}'=\begin{pmatrix}S_+(R) & 0\\ -\frac{\vec{v}}{2c}\cdot\xi_-\vec{\gamma}\xi_+S_+(R) & S_-(R)\end{pmatrix}\begin{pmatrix}\xi_+\varphi_0\\ \xi_-\varphi_0\end{pmatrix} \tag{A3.3}$$

where $S_\pm(R)=\xi_\pm S(R)\xi_\pm$. The transformation (A3.3) is a representation of the Galilei group [4]. It can be checked that higher orders are consistent with the expansion of φ.

APPENDIX 4

The evolution equation in the limit $\hbar \to 0$

Wishing to obtain the expansion of the equation (4.1) when $\hbar \to 0$ it is convenient to write P_μ as

$$P_\mu = [a_1(m-m_0)^2 + a_2(m-m_0) + (1 - a_1(m-m_0)^2 - a_2(m-m_0))]P_\mu, \quad a_i \in \boldsymbol{R}$$

and substitute

$$P_\mu \to P_{\varepsilon\mu}(m) = [a_1(m-m_0)^2 + \varepsilon a_2(m-m_0) + \varepsilon^2(1 - a_1(m-m_0)^2 - a_2(m-m_0))]P_\mu.$$

It is immediately seen that $P_{\varepsilon\mu}(m)$ has the properties stated in Section 4. The simplest choice would be $a_1 = a_2 = 0$, but it would lead to contradiction; since in the lowest order in ε we would obtain an algebraic equation for $\varphi_0(m, x)$ which could not be satisfied for different values of m simultaneously. If on the other hand $a_1 \neq 0$ the lowest order equation reads

$$\left[\mathcal{M}\left(a_1(m-m_0)^2\, P_\mu + \partial_\mu(mc^2 S + K) - \frac{e}{c} A_\mu\right) - m\right] \varphi_0(m, x) = 0$$

and for $m = m_0$ simplifies to the algebraic equation (4.3).

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