SOFT RADIATION IN QCD JETS*

By G. Marchesini

Dipartimento di Fisica, Università di Parma INFN, sezione di Milano**

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A self contained description of the formalism for dealing with soft gluon emission in QCD jets is given.

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1. Introduction

Perturbative QCD not only predicts the emergence of hadron jets in hard collisions, such as in e^+e^- annihilation and in large p_t events in hadron hadron collision, but also predicts very peculiar features for these jets. For long time the analysis was centred in the study [1] of the jet radiation in the region of phase space of fast emitted hadrons. Only recently [2–6] it has been discovered how to extend this analysis to the phase space region of the soft radiation.

In order to describe the problems arising in the soft region consider the most characteristic quantity for the description of the jet structure: the fragmentation function $D(Q, x)$ and its moments

$$D_N(Q) = \int_0^1 \frac{dx}{x} x^N D(Q, x) \tag{1.1}$$

which, apart for inverse power corrections, in perturbative QCD are given by

$$D_N(Q) = D_N(Q_0) \exp \int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \gamma_N(\alpha(q^2)). \tag{1.2}$$

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** Address: Università degli Studi di Parma, Istituto di Fisica, Via D'Azeglio 85, Italy.

This representation is the result [1] of theorems on factorization of collinear singularities in QCD. The anomalous dimensions $\gamma_N(\alpha)$ are in general matrices (with quark and gluon indices) which can be computed by perturbation theory and have the general expansion

$$\gamma_N(\alpha) = \sum_{l=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^l A_N^l. \tag{1.3}$$

At large Q, due to asymptotic freedom

$$\alpha(Q^2) \simeq \frac{1}{b \ln (Q^2/\Lambda^2)}, \qquad 12\pi b = 11C_A - 2N_f, \tag{1.4}$$

only the first term in (1.3) are in general relevant. This is the leading collinear log approximation which have been largely discussed in the past. This approximation however is asymptotically correct only in the region of x finite, i.e. for distribution of the fast hadrons within the jet. In fact the region $x \simeq 0$ corresponds to $N \simeq 1$, where in general all A_N^l in (1.3) become singular. In fact for the gluon-gluon anomalous dimension one finds [1, 7]

$$\gamma_N(\alpha) \simeq \frac{\alpha C_A}{\pi(N-1)}\left(1 - \frac{2\alpha C_A}{\pi(N-1)^2} + \dots\right) = \frac{\alpha}{N-1} f\left(\frac{\alpha}{(N-1)^2}\right), \tag{1.5}$$

which leads for the fragmentation function to an expansion of the form

$$xD(Q, x) \simeq \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n \sum_{p=0}^{\infty} C_{n,p} (\ln (Q^2/Q_0^2))^{n-p} \left(\ln \frac{1}{x}\right)^{n+p-1} \tag{1.6}$$

This shows that in the phase space region

$$x < \left(\frac{\Lambda}{Q}\right)^{\varrho}, \qquad \varrho < 1, \tag{1.7}$$

the $\ln (1/x)$ powers are as important as the collinear $\ln (Q/Q_0)$ powers. Correspondingly all terms in (1.5) equally contribute thus spoiling the leading collinear log approximation. On the other hand the analysis of this phase space region is very important for the study of the jet structure. Note in fact that the hadron multiplicity corresponds to the $N = 1$ moment, and that the bulk of the jet radiation is in the range (1.7).

The singularities in (1.5) for $N = 1$ are of infra-red (IR) type since they arise in Feynman diagrams whenever the energy of gluons vanishes. The structure of these singularities in QCD is far more complex than in QED, and the question whether all singular terms in (1.5) are computable in perturbative QCD has been widely debated in the past years [8, 9] but only recently a perturbative technique has been developed [2–6] to compute all singular terms in (1.5).

In order to explain the interest of such a calculation let me recall some important results of this analysis.

a) The multiplicity. A direct consequence of the fact that $\gamma_N(\alpha)$ can be computed for $N = 1$ gives, due to (1.5), $\gamma_1(\alpha) = \text{const}\sqrt{\alpha}$ and consequently for the multiplicity

$$n(Q) \equiv D_{N=1}(Q) \sim \exp\sqrt{c \ln (Q^2/\Lambda^2)} \tag{1.8}$$

the constant c in (1.8) can actually be computed: $c = \dfrac{2C_A}{\pi b}$.

b) Depletion of soft radiation. The fragmentation function has a maximum for energy $E \sim \sqrt{Q\Lambda}$ and the radiation in the central region is inhibited. As it will appear from the calculation this is due to a genuine quantum effect peculiar of QCD: a destructive interference is taking place among soft gluons.

c) Branching structure. In spite of this interference the emission of radiation in a QCD jet can be described as a classical branching process for gluons and quarks. This implies that Monte Carlo simulation can be still constructed [10, 11] to obtain quantitative results.

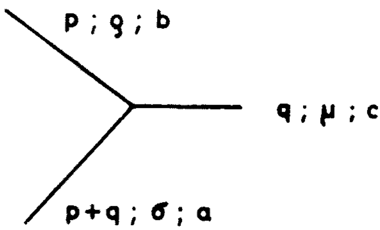
d) Screening of colour charges. The colour charges are screened by Sudakov type of form factors, i.e. are preconfined [12]. This property not only holds for the fast part of the radiation, as shown in Ref. [9, 12], but also for the soft part [10, 11]. As suggested [9, 12] this fact can be used to construct a reliable phenomenological model of hadronization [11].

In this lecture I describe in detail some aspect of such a recent analysis: it is mainly devoted to the description of the formalism for dealing with soft radiation and it is based on the formulation of the soft gluon problem given in the forthcoming Physics Reports article [6] written in collaboration with A. Bassetto and M. Ciafaloni.

The plan of the lecture is the following. In Section 2 it is reviewed the soft gluon technique which allows one to set up a recurrence procedure to compute the general n-gluon emission amplitude. This will be done by using approximations, partially known from QED, which allow one to calculate the leading IR and collinear singularities. In Section 3 a factorization formula is deduced which provides the physical basis for the understanding of the mentioned depletion of soft emission. In Section 4 the general emission probability is computed by using the results of Sect. 2. In Section 5 it is shown that, in spite of all interference, the emission process can be described as a branching process and it is deduced the probability density for the branching that has been used to construct the Monte Carlo simulation in Ref. [10, 11].

2. *Soft gluon emission amplitudes*

Since only gluons are relevant for the IR singularities I restrict the analysis to the pure gauge theory; quarks will be included only incidentally. In this Section I describe an iterative technique to express the n-gluon emission amplitude which is useful to prove factorization theorems (see Sect. 3) and to compute emission probabilities (see Sect. 4). For the moment virtual corrections are neglected and will be considered later. This technique is based on soft approximations, which allows one to reliably compute only the leading IR and collinear singularities. The corresponding technique in QED, the eikonal approximation, gives the Bloch-Nordsiek exponentiation leading to uncorrelated photon radiation. In QCD instead one expects strong correlations due to gluon self interaction. In spite of this some of the classical soft photon approximation of QED can be used also in QCD. In particular in QCD one can introduce the soft gluon vertex approximation. Consider the three gluon vertex



In the soft limit $p^2 \sim q^2 \sim 0$ and $E_q \ll E_p$ one has

$$ig f_{acb} \varepsilon^{\lambda}_{\sigma}(p+q) \Gamma_0^{\sigma\mu\varrho} \varepsilon^{\lambda'}_{\varrho}(q) \simeq g\langle a|T^c|b\rangle 2p^{\mu} \delta_{\lambda,\lambda'}, \tag{2.1}$$

where, similarly to the eikonal vertex appioximation in QED, the helicity λ is conserved and the neglected terms vanish as E_q/E_p for $E_q \to 0$. T^c are the SU(3) colour matrices in the adjoint representation

$$\langle a|T^c|b\rangle \equiv i f_{acb}; \quad \sum_c T^c T^c = C_A = 3. \tag{2.2}$$

Note that the indices of the matrices are the colours of the two hard gluons in the vertex.

In order to set the iterative technique and the notations let me relate M_3, the amplitude for three gluon emission, to the amplitude M_2 for two gluons. In the following I consider the emission out of a colour singlet source at rest. By using the eikonal vertex appoximation the emission amplitude for three gluons of momentum, colour and helicity p_i, b_i and λ_i, in the phase space region

$$E_3 \ll E_2 \simeq E_1 \simeq Q/2, \tag{2.3}$$

is approximately given by

$$\langle M_3|b_1 b_2 b_3\rangle \simeq g\left\{\langle M_2|b_1' b_2\rangle \langle b_1'|T^{b_3}|b_1\rangle \frac{p_1^{\mu_3}}{p_1 p_3}\right.$$

$$\left.+\langle M_2|b_1 b_2'\rangle \langle b_2'|T^{b_3}|b_2\rangle \frac{p_2^{\mu_3}}{p_2 p_3}\right\} \varepsilon^{\lambda_3}_{\mu_3}(p_3), \tag{2.4}$$

where the conservation of helicity in (2.1) gives

$$\langle M_2|b_1 b_2\rangle = M_2(p_1\lambda_1;\, p_2\lambda_2)\delta_{b_1 b_2}, \tag{2.5}$$

since the source is a colour singlet. Eq. (2.4) can be written in a simpler way by introducing the classical colour current for the soft gluon p_3:

$$J^{b_3\mu_3}(p_3) \equiv T_1^{b_3} \frac{p_1^{\mu_3}}{p_1 p_3} + T_2^{b_3} \frac{p_2^{\mu_3}}{p_2 p_3}, \tag{2.6}$$

where T_i^b is the colour matrix for the gluon i such that on a general colour state one has:

$$T_i^b |b_1 b_2 \dots b_n\rangle = |b_1 \dots b_i' \dots b_n\rangle \langle b_i'|T^b|b_i\rangle. \tag{2.7}$$

In fact (2.4) can be written as

$$\langle M_3|b_1 b_2 b_3\rangle \simeq g\langle M_2 J^{b_3\mu_3}(p_3)|b_1 b_2\rangle \varepsilon_{\mu_3}^{\lambda_3}(p_3), \tag{2.8}$$

where the two terms in (2.6) represent the emission of the softest gluon p_3 from p_1 and p_2, with the corresponding colour matrices. Note that since the source is a colour singlet one has

$$\langle M_2(T_1+T_2)^b |b_1 b_2\rangle = 0, \tag{2.9}$$

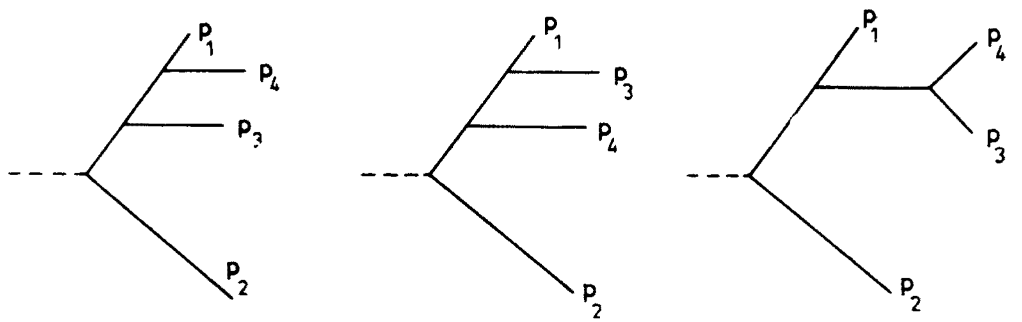
as can be checked from (2.5). This ensures the current conservation $p_3^\mu J_\mu^b(p_3) = 0$.

A similar expression holds if gluons 1 and 2 are substituted with quark-antiquark: In (2.6) one has to replace T_i with λ_i, the matrices in the fundamental representation.

The first essential differences with respect to QED appear in the case of M_4. In the strongly ordered region

$$E_4 \ll E_3 \ll E_2 \simeq E_1 \simeq Q/2, \tag{2.10}$$

consider the graphs



In the eikonal approximation for the vertices in (2.1) the first two graphs give

$$g^2 < M_2 \frac{1}{(p_3+p_4)p_1} \left\{ \frac{1}{p_1 p_4} T_1^{b_3} T_1^{b_4} + \frac{1}{p_1 p_3} T_1^{b_4} T_1^{b_3} \right\} |b_1 b_2\rangle p_1^{\mu_3} p_1^{\mu_4}. \tag{2.11}$$

The difference with QED is the non commutativity of colour matrices so that one can not use the eikonal identity

$$\frac{1}{(p_3+p_4)p_1}\left(\frac{1}{p_1 p_4} + \frac{1}{p_1 p_3}\right) = \frac{1}{p_1 p_4}\,\frac{1}{p_1 p_3}. \tag{2.12}$$

However in the strongly ordered region (2.10) the first term in (2.11) is the most singular collinear contribution in (2.11) and corresponds to the emission of the softest gluon p_4 from the external line p_1 (first graph). In the following only this contribution will be kept thus implying that only the leading collinear singularities are reliably computed. On the same basis one can approximate the propagator

$$\frac{1}{(p_3+p_4)p_1} \simeq \frac{1}{p_3 p_1}, \qquad E_4 \ll E_3, \tag{2.13}$$

thus the contribution to M_4 in (2.11) is approximated by

$$g^2 < M_2 \left\{ T_1^{b_3} \frac{p_1^{\mu_3}}{p_1 p_3} \, T_1^{b_4} \frac{p_1^{\mu_4}}{p_1 p_4} \right\} | b_1 b_2 \rangle. \tag{2.14}$$

Note that in QED Eq. (2.12) allows one to keep also non leading collinear contributions and gives a result analogous to (2.14). Thus in QED these non leading contributions are cancelled.

In the region (2.10) the contribution for the third graph can be expressed by repeated use of eikonal vertex approximation in (2.1): first p_3 is emitted by the hard gluon p_1; then the softest gluon p_4 is emitted by the external line p_3, and one obtains

$$g^2 < M_2 \left\{ T_1^{b} \frac{p_1^{\mu_3}}{p_1 p_3} \langle b | T^{b_4} | b_3 \rangle \frac{p_3^{\mu_4}}{p_3 p_4} \right\} | b_1 b_2 \rangle. \tag{2.15}$$

These examples show how the full amplitude M_4 can be obtained in the phase space (2.10): first the soft gluon p_3 is emitted from the hard gluons p_1 and p_2 thus giving the amplitude M_3. Then the softest gluon p_4 is radiated by the harder gluons p_1, p_2, and p_3. By using the previous expression for M_3 in fact one obtains the iterative form:

$$\langle M_4 | b_1 b_2 b_3 b_4 \rangle \simeq g \langle M_3 J^{b_4 \mu_4}(p_4) \, | b_1 b_2 b_3 \rangle \varepsilon_{\mu_4}^{\lambda_4}(p_4), \tag{2.16}$$

where $J^b(p_4)$ is the classical colour current for the emission of the softest gluon p_4 from the harder ones p_1, p_2, p_3:

$$J^{b_4 \mu_4}(p_4) \equiv \sum_{i=1}^{3} T_i^{b_4} \frac{p_i^{\mu_4}}{p_i p_4}. \tag{2.17}$$

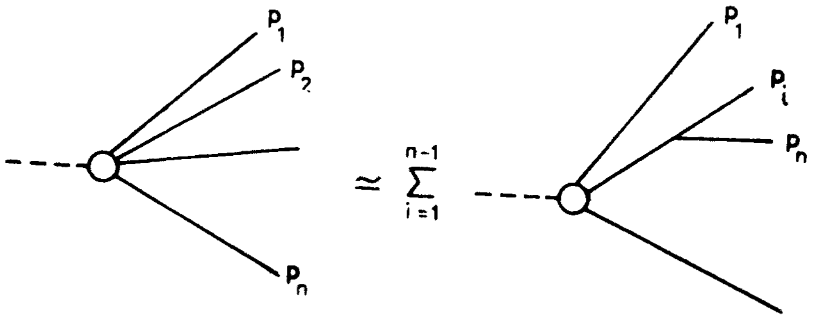
The iterative structure in (2.16) and (2.17) can be generalized and the amplitude M_n for the emission of n gluons $p_1 p_2 \ldots p_n$ can be related to M_{n-1}. In fact if p_n is the softest gluon, it is emitted by the harder ones via the classical colour current

$$J^{b_n \mu_n}(p_n) \equiv \sum_{i=1}^{n-1} T_i^{b_n} \frac{p_i^{\mu_n}}{p_i p_n}; \qquad E_n \ll E_{n-1}, \ldots, E_1; \tag{2.18}$$

and one obtains the reduction formula

$$\langle M_n | b_1 b_2 \dots b_n \rangle \simeq g \langle M_{n-1} J^{b_n \mu_n}(p_n) | b_1 b_2 \dots b_{n-1} \rangle \varepsilon_{\mu_n}^{\lambda_n}(p_n), \tag{2.19}$$

which corresponds to the diagrams



This reduction formula is obtained under the following approximations. *i*) Eikonal vertex formula (2.1). This implies that only leading IR singularities are under control. *ii*) The softest gluon is emitted only by external lines and the propagators are approximated as in (2.13). This implies that only leading collinear singularities are under control. The approximations in *ii*) are not necessary in QED where one can use (2.12) and its generalizations.

The iterative structure in (2.18) and (2.19) can be used to obtain M_n in the strongly ordered region of phase space, e.g.

$$E_n \ll E_{n-1} \ll \dots \ll E_3 \ll E_2 \simeq E_1 \simeq Q/2. \tag{2.20}$$

In fact in (2.19) the amplitude M_{n-1} for the emission of gluons $p_1 \dots p_{n-1}$ can be related to M_{n-2} by reducing the softest p_{n-1} and so on.

Notice finally that for the conservation of charge one has (cfr. 2.9)

$$\langle M_{n-1}(T_1 + \dots + T_{n-1})^b | b_1 b_2 \dots b_{n-1} \rangle = 0 \tag{2.21}$$

thus the emission current in (2.18) is conserved. This implies that instead of (2.19) one may use the current

$$J(p_n) = \sum_{i=1}^{n-1} T_i \left(\frac{p_i}{p_i p_n} - \frac{\eta}{\eta p_n} \right), \tag{2.22}$$

with η a fixed vector. This form is useful to show the gauge independence of physical results and to prove general factorization theorems.

3. *Factorization theorem*

The reduction formula for the softest gluon allows one to derive various factorization properties according to the angular configuration of the harder gluons. Here I review a particular case which provides the physical basis for understanding the mentioned inhibition for soft gluon emission. Consider the emission from the colour singlet source of $n+1$ gluons $p, p_1, \dots p_n$ where in the c.m. the softest gluon p is emitted with large angles with respect to the others. The gluons $p_1 \dots p_n$ are in two opposite narrow jets J_1 and J_2 and then the soft current for the emission of p can be approximated by

$$J^{b\mu}(p) = \sum_{i=1}^{n} T_i^b \frac{p_i^\mu}{p_i p} \simeq T_{J_1}^b \left(\frac{q_1}{q_1 p} - \frac{q_2}{q_2 p} \right). \tag{3.1}$$

Here q_1 (q_2) is the momentum of a gluon in the jet J_1 (J_2), and charge conservation in (2.22) implies,

$$T_{J_1} \equiv \sum_{i \in J_1} T_i = - \sum_{i \in J_2} T_i \equiv - T_{J_2}; \tag{3.2}$$

T_{J_1} and T_{J_2} are the total charges of gluons in J_1 or J_2. Within the approximations used in Sect. 2 each jet J_1 or J_2 originates from a single gluon then T_{J_1} and T_{J_2} are simply the charges of the adjoint representation: $T_{J_1}^2 = T_{J_2}^2 = C_A$. Using now (2.19) one finds that the emission of the gluon p factorizes:

$$\langle M_{n+1} M_{n+1} \rangle \simeq g^2 \langle M_n J_\mu^b(p) J^{\mu b}(p) M_n \rangle \simeq g^2 C_A \frac{2 q_1 q_2}{(q_1 p)(q_2 p)} \langle M_n M_n \rangle. \tag{3.3}$$

Similarly to QED one finds then that the softest gluon emitted at large angle can only probe the total charge of the system, while the charges of individual gluons are not resolved.

The important consequence of this fact is that the inclusive emission of p in this kinematical configuration is inhibited. In fact due to the Lee Nauenberg Kinoshita theorem when Eq. (3.3) is summed over n, all IR and collinear singularities in $|M_n|^2$ cancel against virtual corrections. The emission of p in this configuration is then essentially given by the Born contribution, a far non leading term. On the other hand the soft radiation requires a large angular phase space which is then inhibited. As recalled in the Introduction this physical effect is due to interference of soft gluons. In fact the factorization property (3.3) is obtained by taking into account all possible emission graphs of the softest gluon p. This class of graphs can not be reduced, in any gauge, to ladder type of Feynman graphs.

4. *Emission probabilities*

The results of Sec. 2 are used here to compute the n-gluon emission probability in the soft limit. Virtual corrections will be discussed in the next Section. In the strongly ordered region of phase space

$$E_n \ll E_{n-1} \ll \dots \ll E_3 \ll E_2 \simeq E_1 \simeq Q/2, \tag{4.1}$$

the softest gluon p_n can be reduced by (2.19) and gives for the probability

$$\langle M_n M_n \rangle \simeq -g^2 \langle M_{n-1} J^2(p_n) M_{n-1} \rangle; \tag{4.2}$$

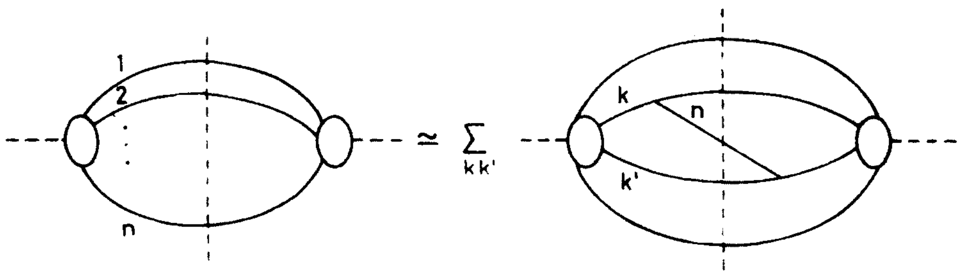
where

$$J^2(p_n) = \sum_{h \neq k'=1}^{n-1} T_k^b T_{k'}^b \frac{(p_k p_{k'})}{(p_k p_n)(p_n p_{k'})} = \sum_{k,k'} T_k^b T_{k'}^b \frac{1}{E_n^2} \frac{\xi_{kk'}}{\xi_{kn}\xi_{nk'}}, \tag{4.3}$$

and

$$\xi_{kk'} \equiv \frac{p_k p_{k'}}{E_k E_{k'}} = 1 - \cos\theta_{kk'}. \tag{4.4}$$

Here the form (2.18) for the current has been used, and the sum over the polarization is substituted by $-g_{\mu\nu}$ since the currents are conserved. This corresponds to use the Feynman gauge where the Feynman diagrams corresponding to (4.2) are given by



Similarly one can work in the axial gauge ($A \cdot \eta = 0$) and it is easy to show that this still corresponds to (4.2) where the emission current is given by (2.22). The charge conservation in (2.21) implies then that the reduction formula (4.2) is gauge invariant.

Eq. (4.3) shows explicitly both the IR ($E_n = 0$) and collinear ($\xi_{kn}\xi_{nk'} = 0$) singularities coming from the softest gluon p_n. The dependence on the other momenta can be obtained in a similar way by successive reduction of $p_{n-1}, \ldots, p_3$ gluons. This calculation however can be greatly simplified by the following observations.

1) Note first that the current for the emission of the softest gluon p_n scale as $1/E_n$. This fact determines the dependence of $|M_n|^2$ on the various gluon energies. In fact the reduction of each gluon p_i ($i = n, n-1, \ldots, 3$) brings a factor $1/E_i^2$ and this gives

$$|M_n|^2 \simeq \frac{F_n(\Omega_1, \ldots, \Omega_n)}{E_n^2 \ldots E_3^2} = \left(\frac{Q}{2}\right)^4 \frac{F_n(\Omega_1, \ldots, \Omega_n)}{E_n^2 \ldots E_1^2}, \tag{4.5}$$

where F_n depends on the angular directions only.

2) In spite of the strongly ordered phase space considered (cfr. 4.1) the result is symmetric in $E_1, \ldots, E_n$. As a consequence of Bose statistic also the angular dependence in F_n must be symmetric.

3) According to (4.2) and (4.3) the dependence of $F_n(\Omega_1, \ldots, \Omega_n)$ on the direction of p_n is in the form $(\xi_{kn}\xi_{nk'})^{-1}$ $k \neq k'$. In fact the momentum p_n does not enter in successive reduction of $p_{n-1}, \ldots, p_3$. Due to the symmetry of F_n the dependence of any other direction

must be of the same type. Conditions 2) and 3) determine the form of F_n:

$$F_n(\Omega_1, \ldots, \Omega_n) = h_n \sum_{\text{perm.}} \frac{1}{\xi_{i_1 i_2} \cdots \xi_{i_n i_1}} \tag{4.6}$$

with ξ_{ij} given in (4.4).

Finally the constant h_n can be obtained by computing the contribution to $|M_n|^2$ for the fundamental permutation in (4.6) i.e. the term $(\xi_{12} \ldots \xi_{n1})^{-1}$. Consider, in the phase space region (4.1), the successive reduction of $p_n \ldots p_3$. The wanted contribution to $|M_n|^2$ can be obtained by the following steps.

1) For the reduction of p_n in (4.2) one keeps only the terms with the factor $(\xi_{n-1,n}\xi_{n,1})^{-1}$. This corresponds in (4.3) to the terms with $(k, k') = (1, n)$ or $(n, 1)$ (i.e. p_n is emitted by p_{n-1} and p_1):

$$|M_n|^2 = -g^2 \frac{2}{E_n^2} \frac{\xi_{n-1,1}}{\xi_{n-1,n}\xi_{n,1}} \langle M_{n-1} T_l^b T_{n-1}^b M_{n-1} \rangle + \ldots . \tag{4.7}$$

2) For the successive emission current of the soft gluon p_{n-1} one keeps only the contributions for the emission from p_1 and p_{n-2}

$$J^{b_{n-1}}(p_{n-1}) = \frac{p_1}{p_1 p_{n-1}} T_1^{b_{n-1}} + \frac{p_{n-2}}{p_{n-2} p_{n-1}} T_{n-2}^{b_{n-1}} + \ldots . \tag{4.8}$$

In fact the two terms give the momentum factor $\xi_{1,n-2}/(\xi_{1,n-1}\xi_{n-1,n-2})$. The colour matrix operation is made by using the identity

$$-(T_1^a T_1^b T_{n-2}^{a'} + T_{n-2}^a T_1^b T_1^{a'}) \langle a|T^b|a' \rangle = C_A T_1^c T_{n-2}^c, \tag{4.9}$$

and one finds

$$|M_n|^2 \simeq -g^4 C_A \frac{2}{E_n^2 E_{n-1}^2} \frac{\xi_{n-2,1}}{\xi_{n-2,n-1}\xi_{n-1,n}\xi_{n,1}} \langle M_{n-2} T_1^c T_{n-2}^c M_{n-2} \rangle + \ldots . \tag{4.10}$$

Note that (4.10) has the same colour structure as (4.7). Therefore the reduction of p_{n-2} goes through similar steps. After the reduction of $p_n \ldots p_3$ one finally finds

$$|M_n|^2 \simeq -\frac{1}{C_A}(g^2 C_A)^{n-2} \frac{2}{E_n^2 \ldots E_3^2} \frac{\xi_{2,1}}{\xi_{2,3} \ldots \xi_{n,1}} \langle M_2 T_1^b T_2^b M_2 \rangle + \ldots$$

$$= |M_2|^2 (g^2 C_A)^{n-2} \frac{2}{E_n^2 \ldots E_3^2} \left\{ \frac{\xi_{2,1}}{\xi_{2,3} \ldots \xi_{n,1}} + \text{permutations of } (3, 4, \ldots n) \right\}; \tag{4.11}$$

where the charge conservation $(T_1 + T_2)^2 = 0$ is taken into account to give $-T_1 T_2 = C_A$. If instead of the fast gluon $p_1 p_2$ one has a quark antiquark pair, T_1 and T_2 must be replaced by λ_1 and λ_2 so that $-\lambda_1\lambda_2 = C_F = \frac{4}{3}$. Notice that introducing $Q^2 = 2p_1 p_2 = 2E_1 E_2 \xi_{12}$, the final result in (4.11) can be written in the very symmetric form

$$|M_n|^2 = |M_2|^2 \tfrac{1}{2} Q^4 (g^2 C_A)^{n-2} \sum_{\text{perm.}}' (\xi_{i_1 i_2} \cdots \xi_{i_n i_1})^{-1}, \tag{4.12}$$

where the sum is over unequivalent terms. This result is obtained in the strongly ordered phase space region (4.1) but due to its symmetry it holds in any other ordered region.

Let me now come to discuss the structure of the singularities in $|M_n|^2$. In Eq. (4.11) there are $n-2$ IR singularities as $E_i \to 0$ $i = 3, \ldots, n$. Moreover there is a maximum number of $n-2$ collinear singularities since at most $n-2$ variables ξ_{ij} are allowed to vanish. In fact consider for instance the contribution from the fundamental permutation in the phase space region (4.1). Since p_1 and p_2 are opposite in the cm, the maximum collinearity comes for $p_3 \ldots p_k$ parallel to p_2, and $p_n \ldots p_{k+1}$ parallel to p_1, where $k = 2, \ldots, n$. For this configuration $\xi_{12} \simeq \xi_{k,k+1} \simeq 2$ while the remaining $\xi_{i,i+1}$ are small, thus

$$\frac{\xi_{1,2}}{\xi_{2,3} \cdots \xi_{n,1}} \simeq \frac{1}{\xi_{2,3} \cdots \xi_{k-1,k}} \cdot \frac{1}{\xi_{k+1,k+2} \cdots \xi_{n,1}} . \tag{4.13}$$

This proves that for the leading collinear contributions, the only ones that are reliably computed within the used approximations, the emission probabilities factorize in a forward and a backward jet distributions.

5. *Generating function and branching process*

In order to discuss the structure of the emission process it is useful to introduce the generating function. According to the factorization in (4.13) one may limit the discussion to a single jet. Apart for virtual corrections the differential distribution for n-gluons in a jet of total energy $E = Q/2$ is given by (cfr. 4.11)

$$dw^{(n)} = \prod_1^n \frac{d^3 p_i}{2E_i} 2\pi\delta\left(\sum_1^n x_i - 1\right)\left\{\left(\frac{\alpha C_A}{\pi}\right)^{n-1} \prod_1^n \frac{1}{E_i^2} \sum_{\text{perm.}} (\xi_{i_1 i_2} \cdots \xi_{i_{n-1} i_n})^{-1}\right\}, \tag{5.1}$$

where $x_i = E_i/E$. Similarly to QED, the IR and collinear divergences can be cut off by attributing a virtual mass Q_0 to the emitted gluons so that

$$\xi_{ij} = \frac{p_i p_j}{E_i E_j},$$

$$1 \sim \xi > \xi_{ij} > \tfrac{1}{2} \frac{Q_0^2}{E^2}\left(\frac{1}{x_i^2} + \frac{1}{x_j^2}\right) \equiv \bar{\xi}_{ij}. \tag{5.2}$$

The upper limit ξ is set to ensure that the radiation is not emitted in the direction opposite to the jet. Since ξ is related to the jet total aperture, the jet invariant mass Q_{Jet} is related to ξ by

$$Q_{\text{Jet}}^2 \sim 2E^2 \xi. \tag{5.3}$$

The generating function for the jet distribution in (5.1) is defined by

$$G_\xi(E, u) \equiv \sum_{n=1}^{\infty} \frac{1}{n!} u^n \int dw^{(n)} = \sum_{n=1}^{\infty} u^n \left(\frac{\alpha C_A}{\pi}\right)^{n-1} \int \prod_1^n \frac{dx_i}{x_i}$$

$$\times \frac{d^2\Omega_i}{4\pi} \delta\left(\sum_1^n x_i - 1\right) T_\xi(p_1 p_2 \dots p_n), \tag{5.4}$$

where T_ξ contains the angular dependence

$$T_\xi(p_1 p_2 \dots p_n) \equiv \prod_1^{n-1} \frac{1}{\xi_{i,i+1}} \theta(\xi - \xi_{i,i+1})\theta(\xi_{i,i+1} - \xi_{i,i+1}). \tag{5.5}$$

From this expression one derives an evolution equation which allows one to take into account also the virtual corrections. As suggested in Ref. [5], instead of considering as usual the evolution in the jet mass Q_{Jet}, because of (5.3), one may consider the evolution in ξ. In fact by using the identity

$$\xi \frac{\partial}{\partial \xi} T_\xi(p_1 \dots p_n) = \sum_k T_\xi(p_1 \dots p_k) T_\xi(p_{k+1}, \dots, p_n), \tag{5.6}$$

one finds the evolution equation

$$\xi \frac{\partial}{\partial \xi} G_\xi(E, u) = \tfrac{1}{2} \int_\varepsilon^{1-\varepsilon} dz \frac{\alpha}{\pi} \frac{C_A}{z(1-z)} [G_\xi(zE, u) G_\xi((1-z)E, u) - G_\xi(E, u)], \tag{5.7}$$

where $\varepsilon = Q_0/(E\sqrt{\xi})$ and the virtual corrections have been included by the usual procedure of replacing $1/(1-z)$ with the distribution $1/(1-z)_+$, which gives in (5.7) the additional term $-G_\xi(E, u)$ in the integral. This ensures that for $u = 1$ $G_\xi(E, 1) = 1$ is a solution. The Lee Nauenberg Kinoshita theorem is then satisfied since $G_\xi(E, 1)$ is independent of the cut off Q_0. Moreover the function $G_\xi(E, 1)$ is just the total emission probability which is then correctly normalized. Note that the z-distribution in (5.7) corresponds to the soft limit of the Altarelli-Parisi gluon distribution

$$P(z) = 2C_A \left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right) \simeq \frac{2C_A}{z(1-z)}, \tag{5.8}$$

where $P(z)$ at the leading collinear log level includes also IR non singular contributions.

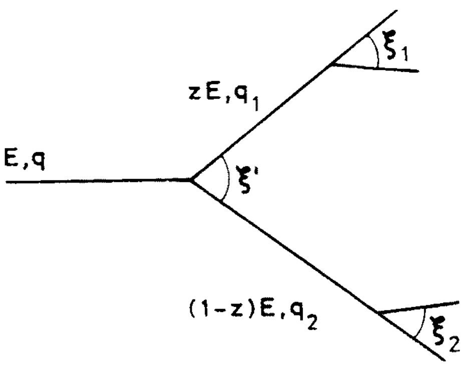
It is surprising that, in spite of all interference graphs here included, Eq. (5.7) allows one to interpret the jet emission as a tree process. In particular in the emission the opening angles of successive branches along the tree are ordered. Such a structure implies then that a Monte Carlo simulation for this jet emission can be constructed [10, 11]. In order to describe the branching, the role of interference and the structure of Monte Carlo simulation it is convenient to transform the evolution equation in (5.7) into the integral form by introducing the virtual form factor $\Delta(\xi, E)$ defined by

$$\Delta(\xi, E) \equiv \exp\left[-\tfrac{1}{2}\int_{\xi_0}^{\xi}\frac{d\xi'}{\xi'}\int_{\varepsilon'}^{1-\varepsilon'} dz P(z)\frac{\alpha}{2\pi}\right], \tag{5.9}$$

where $\varepsilon' = Q_0/(E\sqrt{\xi'})$; $\varepsilon' = 1/2$ for $\xi' = \xi_0$. Eq. (5.7) takes then the form

$$G_\xi(E, u) = u\Delta(\xi, E)+\tfrac{1}{2}\int_{\xi_0}^{\xi}\frac{d\xi'}{\xi'}\int_{\varepsilon'}^{1-\varepsilon'} dz P(z)\frac{\alpha}{2\pi}\frac{\Delta(\xi, E)}{\Delta(\xi', E)}G_{\xi'}(zE, u)G_{\xi'}((1-z)E, u), \tag{5.10}$$

where $\Delta(\xi, E)$ represents the probability for no branching while the second term describes the fragmentation according to the graph



$$q^2 \simeq 2E^2 z(1-z)\xi' \tag{5.11}$$

In the phase space region

$$\xi' = \frac{q_1 q_2}{z(1-z)E^2} < \xi. \tag{5.12}$$

In the successive branching for q_1 and q_2 the iteration of this angular ordering gives

$$\xi_1, \xi_2 < \xi' < \xi, \tag{5.13}$$

so that along the tree the branching angles are ordered. According to Eq. (5.10) the branching probability for (5.11) is given by

$$dP = \tfrac{1}{2}\frac{d\xi'}{\xi'} dz P(z)\frac{\alpha}{2\pi}\frac{\Delta(\xi, E)}{\Delta(\xi', E)}\theta(\xi-\xi')\theta(\xi'-\xi_1)\theta(\xi'-\xi_2), \tag{5.14}$$

which provides the basis for the Monte Carlo simulation of Ref. [10, 11].

To understand the role of interference observe that, when soft gluons are involved ($z \to 0$, or $(1-z) \to 0$), the phase space region of angular ordering in (5.13) does not exhaust the full kinematical phase space of the tree graph in (5.11). The meaning of this limitation is that outside the region (5.13) tree and crossed Feynman diagrams cancel each other, while within the region (5.13) only tree graphs are leading.

Note that the mentioned inhibition of the soft radiation is a direct consequence of this angular ordering: the emission of soft radiation requires a large angular phase space, but, after a few branches the allowed decaying angle is shrunk to small values.

When no soft gluons are involved the region (5.13) corresponds to the fully allowed phase space for the branching in (5.11) and one recovers the previous results of the jet calculus [9, 13].

Many other important physical consequences can be obtained from these results on the analysis of soft gluons. These have been examined by analytic asymptotic methods [2–6] and by Monte Carlo methods [10, 11]. These numerical simulations have been extended [11] to include a hadronization model which relies on preconfinement [12] of colour charges. I refer for these results to the original papers and to the talk by B. Webber to this School.

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