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QCD SUM RULES: AN INTRODUCTION AND SOME APPLICATIONS*

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I present an introduction to the basic ideas of QCD sum rules, and discuss various applications.

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CONTENT

1. Introduction

These lectures are an elementary introduction to the basic ideas of the QCD sum rules invented by Shifman, Vainshtein and Zakharov (Ref. [1]). The numerous applications of these sum rules made in recent years have greatly improved our understanding of hadronic physics. I will not be able to give an extensive review of all applications but will only treat a few which I consider to be representative and most illuminating, in some detail.

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The objective of the QCD sum rules is to calculate resonance parameters (masses, couplings) in terms of the QCD Lagrangian parameters (α_s, the quark masses m_q) and a number of vacuum matrix elements which are introduced to parametrize nonperturbative effects.

In QCD the property of asymptotic freedom makes the use of perturbation theory possible to calculate hard processes, i.e., at short distances. On the other hand, bound states of quarks and gluons (resonances) arise because of large distance confinement effects, i.e., strong coupling effects, which cannot be treated in perturbation theory. The idea of Shifman, Vainshtein, and Zakharov is to approach the resonance region from the asymptotic freedom side and include power corrections due to nonperturbative effects. These corrections signal the breakdown of asymptotic freedom and are introduced via non-vanishing vacuum expectation values of higher dimensional operators like

$$\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle \neq 0, \qquad \langle 0|\bar{q}q|0\rangle \neq 0, \tag{1}$$

where $G^a_{\mu\nu}$ is the gluon field tensor, and q, $\bar{q}$ are (light) quark fields. At intermediate distances there is hopefully an overlap between the asymptotic freedom region on one hand and the long distance part on the other hand, where reliable calculations of resonance parameters can be made.

These power corrections are more important than higher order perturbative α_s corrections as can be seen from the following example due to SVZ (Ref. [1]). Consider the vacuum polarizations of the vector current $j^V_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$ and the axial vector current $j^A_\mu(x) = \bar{q}(x)\gamma_\mu\gamma_5 q(x)$. Typical diagrams are given in Fig. 1, with $\Gamma = \gamma_\mu$ or $\gamma_\mu\gamma_5$ respectively.

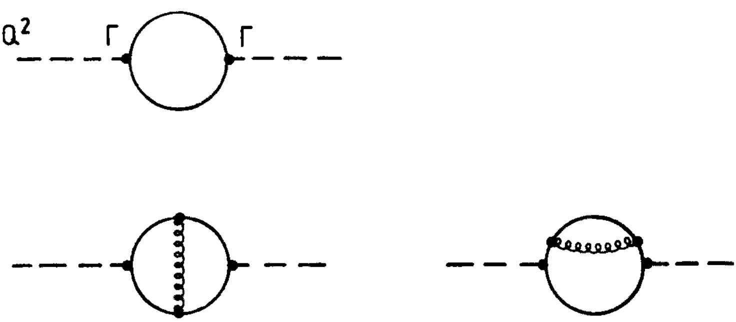


Fig. 1. Examples of diagrams that contribute to the vacuum polarization to first order in α_s. Curly lines depict gluons, solid lines quarks and dashed lines currents. Γ as in the first diagram is the quark-current vertex

In the chiral limit, i.e., for the quark mass $m_q = 0$, which should be a good approximation for light quarks, there is no difference between the two vacuum polarizations in every order of perturbation theory. The γ_5 in the quark-current vertex of the axial vector vacuum polarization can be pulled through to the other side encountering on its way an even number of γ matrices which gives rise to a factor $+1$. However, experimentally the mass spectra for the vector and axial vector channel look very different. In the vector channel there is the ϱ meson, while in the axial vector channel we have the A_1 meson and the pion. Therefore, it is the spontaneous breaking of the chiral symmetry which is responsible for the π–ϱ–A_1 splitting, i.e., $\langle 0|\bar{q}q|0\rangle \neq 0$ and the pion arises as the Goldstone boson of the broken symmetry. Since the operator $\bar{q}q$ has dimension three this can only give rise to a power correction to the perturbative result which behaves like a logarithm.

2. *Correlation functions*

Consider a current $j_\Gamma(x)$ where for mesons

$$j_\Gamma(x) = \bar{q}_i(x)\Gamma q_j(x), \quad \Gamma = 1, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \text{ etc.} \tag{2}$$

and for baryons

$$j_\Gamma(x) = \varepsilon_{abc}(q_i^a(x)\Gamma_1 q_j^b(x))\Gamma_2 q_k^c(x), \tag{3}$$

where the indices i, j, k denote the flavour of the quark, a, b, and c are colour indices and Γ determines the tensor structure. By choosing Γ and the appropriate combination of quark flavours $q_i(x)$ each current can be given definite quantum numbers (J, P, I). For instance,

$$j_\mu(x) = \tfrac{1}{2}(\bar{u}(x)\gamma_\mu u(x) - \bar{d}(x)\gamma_\mu d(x)) \tag{4}$$

has the quantum numbers $I = 1$, $J^{PC} = 1^{--}$, i.e., the same as ϱ, ϱ', ...

The vacuum polarization induced by such a current is given by the correlation function

$$T_{\mu\nu\ldots}\Pi^j(Q^2) = i\int d^4x e^{iqx}\langle 0|\mathrm{T}(j_\Gamma(x)\bar{j}_\Gamma(0))|0\rangle, \quad Q^2 = -q^2; \tag{5}$$

$\Pi^j(Q^2)$ is a scalar function of Q^2, $T_{\mu\nu}$ ... a tensor depending on the current in question, and T on the right-hand side denotes the T-ordered product.

In general, the function $\Pi^j(Q^2)$ obeys a dispersion relation:

$$\Pi^j(Q^2) = \frac{(q^2)^n}{\pi}\int \frac{\mathrm{Im}\,\Pi^j(s)ds}{s^n(s-q^2)} + \sum_{k=0}^{n-1} a_k(q^2)^k, \tag{6}$$

where the constants a_k are unknown subtraction constants. We note for future use that these can be removed by taking the appropriate number of derivatives with respect to Q^2. The dispersion relation (6) relates $\Pi^j(Q^2)$ to its imaginary part which in turn is related to a cross section, in particular, for the vector current $j_\Gamma(x) = j_\mu(x) = \bar{q}\gamma_\mu q$:

$$\mathrm{Im}\,\Pi^{\mathrm{V}}(s) = \frac{9}{64\pi^2\alpha^2}\, s\sigma(e^+e^- \to \text{hadrons}). \tag{7}$$

By selecting a particular flavour, e.g. charm in the case of $j_{\mathrm{V}}(x) = \bar{c}\gamma_\mu c$ only states with open and hidden charm and $J^{PC} = 1^{--}$ appear in $\mathrm{Im}\,\Pi^{\mathrm{V}}$, i.e., J/$\psi$, ψ', ψ'', ... and continuum states above threshold ($\mathrm{D\bar{D}}$ etc.). Similarly one can pick up states with other quantum numbers and/or quark content by choosing another current. At this point feeding hadronic states plus a continuum into $\mathrm{Im}\,\Pi^j(q^2)$ one obtains a representation of $\Pi^j(q^2)$ in terms of the parameters of the hadrons which correspond to the current $j_\Gamma(x)$. To parametrize $\mathrm{Im}\,\Pi^j(s)$ one normally uses a narrow resonance approximation writing the imaginary part as a sum over δ-functions, e.g. for the vector current of flavour q with charge e_q we have

$$\operatorname{Im} \Pi^{\mathrm{V}}(s) = \frac{\pi}{e_{\mathrm{q}}^2} \sum_{\mathrm{res}} \frac{m_{\mathrm{R}}^2}{g_{\mathrm{R}}^2} \delta(s-m_{\mathrm{R}}^2) + \frac{1}{4\pi}\left(1+\frac{\alpha_{\mathrm{s}}}{\pi}\right)\theta(s-s_0). \tag{8}$$

The θ-function on the right-hand side of (8) stands for the continuum. In the vector case g_{R} is related to the electronic width of the resonance, but for most other currents the coupling has no direct physical significance.

This procedure which as yet has nothing to do with QCD expresses $\Pi^j(Q^2)$ in terms of m_{R} and g_{R} and gives us the phenomenological side of the sume rule. In the next Sections we will discuss the construction of the theoretical QCD side of the sum rules. Apart from Ref. [1] this discussion relies heavily on Ref. [2].

3. The operator product expansion (OPE)

Let us turn again to the correlation function (5) and apply the operator product expansion (Ref. [3]) to the T-ordered product of currents, which is valid at short distances, i.e., $Q^2 = -q^2$ large:

$$i\int d^4x e^{iqx} \mathrm{T}(j_\Gamma(x)\bar{j}_\Gamma(0)) = C_I^\Gamma I + \sum_n C_n^\Gamma(q^2) O_n, \tag{9}$$

where I is the identity operator, C_I^Γ and C_n^Γ are the Wilson coefficients, and the O_n are local operators constructed from quark and gluon fields. The identity operator I has dimension $d = 0$ and C_I^Γ contains the ordinary perturbative contributions (like the diagrams of Fig. 1). The $C_n^\Gamma(q^2)$ are c-number functions of q^2, they depend on the Lorentz indices and the quantum numbers of $j_\Gamma(x)$ and O_n. The operators O_n are ordered by dimension and the $C_n^\Gamma(q^2)$ fall off by corresponding powers of q^2. Therefore, at short distances the operators with lowest dimensions dominate and give power corrections to the perturbative (logarithmic) contributions C_I^Γ.

The operator product expansion factorizes short and large distances. The short distance effects are contained in the coefficients $C_n^\Gamma(q^2)$ which can be calculated in perturbation theory by ordinary Feynman diagrammatic techniques, while all large distance effects are buried in the matrix elements of the operators O_n.

For $\Pi^j(Q^2)$ we require the vacuum expectation value of (9). Therefore, we only have to consider spin zero operators. The complete set of operators with spin zero and dimension $d \leqslant 6$ is:

$$\begin{aligned}
&I, && d = 0,\\
&O_m = m\bar{q}q, && d = 4,\\
&O_G = G_{\mu\nu}^a G_{\mu\nu}^a, && d = 4,\\
&O_\Gamma = \bar{q}\Gamma_1 q\bar{q}\Gamma_2 q, && d = 6,\\
&O_\sigma = m\bar{q}\sigma_{\mu\nu}\frac{\lambda^a}{2} q G_{\mu\nu}^a, && d = 6,\\
&O_f = f_{abc} G_{\mu\nu}^a G_{\nu\lambda}^b G_{\lambda\mu}^c, && d = 6,
\end{aligned} \tag{10}$$

where m is the quark mass, the λ^a are the usual Gell-Mann SU(3) matrices, $\mathrm{Tr}\,(\lambda^a\lambda^b) = 2\delta^{ab}$, and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

In the next Sections we will discuss which operators contribute to different quark systems and how to calculate the Wilson coefficients C_I^Γ and C_n^Γ. This results in an expression for the vacuum polarization $\Pi^j(Q^2)$ in the deep Euclidean region in terms of the fundamental parameters of QCD and the matrix elements $\langle 0|O_n|0\rangle$ where $|0\rangle$ is the physical, nonperturbative vacuum of QCD:

$$\Pi^j(Q^2) = C_I^\Gamma + C_m^\Gamma(Q^2)\,\langle 0|m\bar{q}q|0\rangle + C_G^\Gamma(Q^2)\,\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle + \dots \qquad (11)$$

Equating this expression with the physical representation discussed in Section 2 we have a relation between the parameters of the theory and hadron parameters.

4. *Further analysis of the operator product expansion*

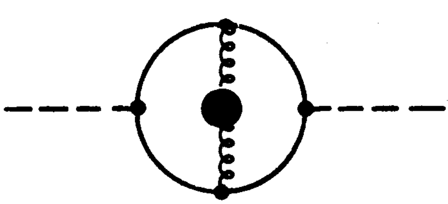
Consider the simple diagrams of Fig. 1 for heavy quarks with mass m at $Q^2 = 0$. In this case, the quarks (and gluons) are far-off mass shell ($p^2, k^2 \sim -m^2$). Consequently, they only travel a short distance ($\sim 1/2m$) and the free particle propagator will not be modified by nonperturbative effects, i.e., asymptotic freedom holds for heavy quarks (and gluons) even at $Q^2 = 0$!

In order to probe larger distances we will take derivatives with respect to Q^2 at $Q^2 = 0$. For these diagrams this gives integrals of the kind (all momenta are Euclidean)

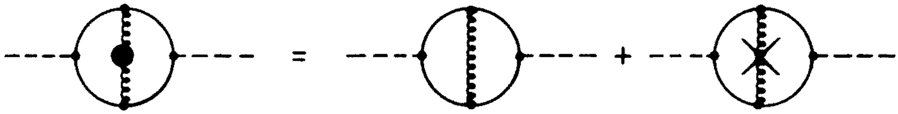
$$\int \frac{d^4p\,d^4k}{[(p+k)^2+m^2]^n},$$

which are dominated by the Euclidean momenta $p^2, k^2 \sim m^2/n$. For fixed m^2 and n becoming large p^2 and k^2 tend to zero. Even at $p^2 = 0$ the quarks are still far-off shell and their propagation is described by standard perturbation theory (no quark condensate). For $k^2 \to 0$ the gluons approach their mass shell and the gluon propagator will be strongly modified by nonperturbative effects, which will be expressed by a nonvanishing vacuum matrix element of the operator $G^a_{\mu\nu}G^a_{\mu\nu}$. Higher dimensional operators are suppressed by extra mass factors. Their contributions are not always negligible. We will come back to a recent calculation of six and eight dimensional gluon operators when we discuss applications to charmonium.

So, for gluons we have

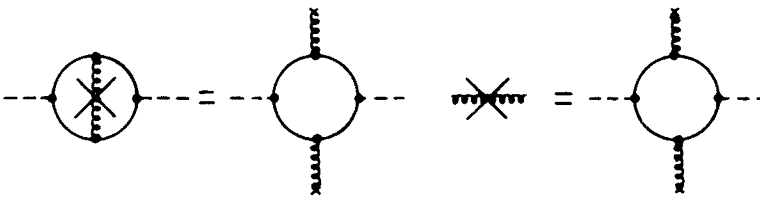


where the blob indicates that the gluon propagator has to be modified by large distance effects. We first separate the perturbative and soft pieces:



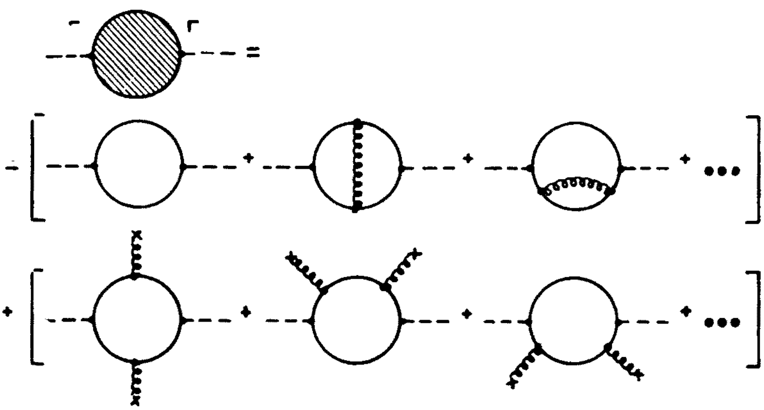
(12)

The first diagram on the right-hand side indicates the ordinary perturbative contribution to the Wilson coefficient C_I^{Γ}. Using the OPE the second diagram factorizes into a short and a large distance piece:



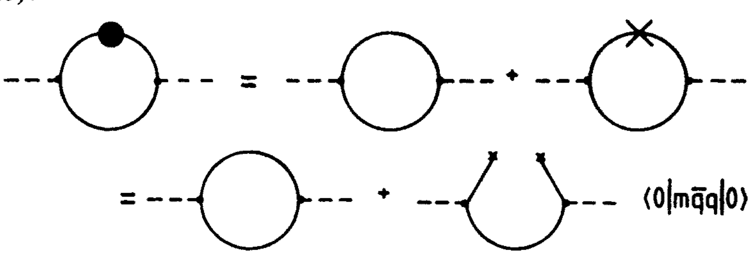
$\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle$ (13)

where the diagram in front of the matrix element $\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle$ contributes to the Wilson coefficient C_G^{Γ} and can be calculated in perturbation theory. A cross ($\times$) attached to a gluon line indicates that the gluon goes into the condensate. We will see later what the precise meaning of this diagram is. Diagrammatically we can now write for the polarization operator of heavy quark currents to first order in α_s:



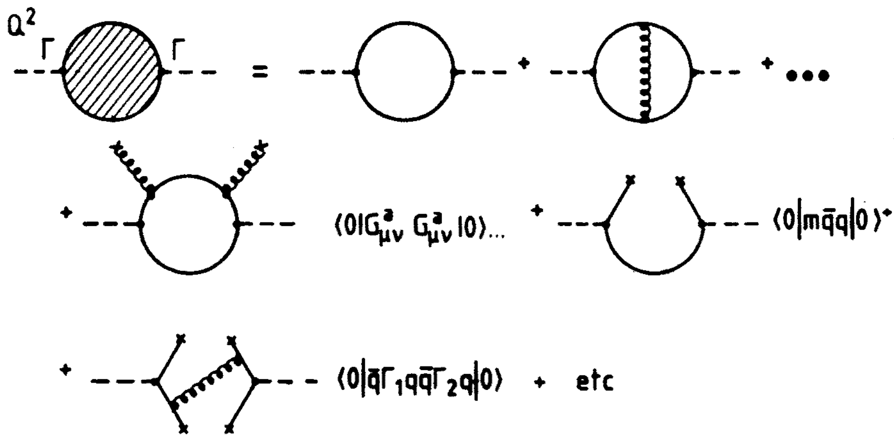
$\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle$ (14)

For light quarks we cannot work at $Q^2 = 0$ but have to take Q^2 large. In this case the light quark propagator can also be modified by nonperturbative effects. Apart from gluon operators quark condensate operators will contribute. We have analogous to (12) and (13):



$\langle 0|m\bar{q}q|0\rangle$ (15)

Again, a cross ($\times$) attached to a quark line indicates that the quark goes into the condensate. For the polarization operator of a light quark current to first order in α_s we can then write diagrammatically (including $d = 6$ operators, which now play an important role):



(16)

From this we can apparently deduce the following rule for obtaining the Wilson coefficients: cut quark and or gluon lines as appropriate for a particular operator. In the following Section we will discuss the calculation of the Wilson coefficients in more detail.

5. The calculation of the Wilson coefficients

The operator product expansion is an operator expansion. This means that the Wilson coefficients are independent of the states we sandwich with, and we can choose any state to single out a particular coefficient function. To calculate the gluon condensate coefficient $C_G^\Gamma(q^2)$ it is most convenient to take the matrix element of (9) between one-gluon states. Then

$$T_{\alpha\beta}((q+k)^2, q^2, k^2) = \langle k, \alpha| i \int d^4x e^{iqx} T(j_\Gamma(x)\bar{j}_\Gamma(0))|k, \beta\rangle \tag{17}$$

is the forward gluon scattering amplitude on a colour singlet current. This contains two invariant functions, but since we are only interested in the term which survives when sandwiched between states with vacuum quantum numbers we have only one:

$$\begin{aligned} T_{\alpha\beta}((q+k)^2, q^2, k^2) &= 2(k^2 g_{\alpha\beta} - k_\alpha k_\beta)C^\Gamma((q+k)^2, q^2, k^2)) \\ &= \langle k, \alpha| G^a_{\mu\nu}G^a_{\mu\nu}|k, \beta\rangle C^\Gamma((q+k)^2, q^2, k^2) \end{aligned} \tag{18}$$

and $C_G^\Gamma(q^2)$ is given by

$$C_G^\Gamma(q^2) = C^\Gamma((q+k)^2, q^2, k^2)|_{k_\mu \to 0}, \tag{19}$$

which implies the following interpretation for the calculation of the diagram in front of the matrix element in (13): expand each (quark) propagator in k^2, take the k^2 coefficient and let $k \to 0$.

As a second example let us calculate in detail the $\bar{d}d$ coefficient for the baryon current $\eta_N(x)$ with the quantum numbers of the proton

$$\eta_N(x) = \varepsilon_{abc}(u^{a^T}(x)C\gamma_\mu u^b(x))\gamma_5\gamma^\mu d^c(x), \tag{20}$$

where a, b, c are colour indices, T indicates the transposed and C is the charge conjugation operator. Keeping only the quark condensate terms in the OPE of the correlation function of two of these currents we have

$$i\int d^4x e^{ipx} \mathrm{T}\,(\eta_N(x)\bar{\eta}_N(0)) = \ldots + C_d(p^2)\bar{d}d + C_u(p^2)\bar{u}u + \ldots \tag{21}$$

To select C_d we sandwich by single d-quark states with momentum p_3:

$$\begin{aligned} C_d(p^2)\delta_{cc'}\bar{d}(p_3)d(p_3) &= i\int d^4x e^{ipx}\langle p_3^c|\mathrm{T}(\eta_N(x)\bar{\eta}_N(0))|p_3^{c'}\rangle \\ &= -i\int d^4x e^{i(p-p_3)x}\varepsilon_{abc}\varepsilon_{a'b'c'}\gamma_5\gamma^\mu d(p_3)\bar{d}(p_3)\gamma_5\gamma^\nu \\ &\quad \langle 0|T(u^{a^T}(x)C\gamma_\mu u^b(x)\bar{u}^{b'}(0)\gamma_\nu C\bar{u}^{a'^T}(0)|0\rangle \end{aligned}$$

Contracting the u-quark fields we get

$$\begin{aligned} C_d(p^2)\delta_{cc'}\bar{d}(p_3)d(p_3) &= -2i\int d^4x e^{i(p-p_3)x}\varepsilon_{abc}\varepsilon_{abc'}\gamma_5\gamma^\mu d(p_3)\bar{d}(p_3)\gamma_5\gamma^\nu \\ &\times\left(\frac{i}{(2\pi)^4}\right)^2\int d^4k d^4k' e^{-i(k-k')x}\mathrm{Tr}\,(C\gamma_\mu\gamma k\gamma_\nu C\gamma k'^{\mathrm{T}})\,\frac{1}{k^2}\cdot\frac{1}{k'^2}\,. \end{aligned}$$

Using $C\gamma k^T C = \gamma k$, $C^2 = -1$ and performing the integrations gives

$$C_d(p^2)\delta_{cc'}\,\bar{d}(p_3)d(p_3) = -\frac{1}{3\pi^2}\,\delta_{cc'}(p_\mu p_\nu - g_{\mu\nu}p^2)\ln\left(\frac{\Lambda^2}{-p^2}\right)\gamma_5\gamma^\mu d(p_3)\bar{d}(p_3)\gamma_5\gamma^\nu,$$

with $\bar{d}(p_3)d(p_3) = 1$ and $d(p_3)\bar{d}(p_3) = \gamma(p_3+m)/4m$ we finally obtain, letting p_3 tend to zero

$$C_d(p^2) = -\frac{1}{(2\pi)^2}\,p^2\ln\left(\frac{\Lambda^2}{-p^2}\right). \tag{22}$$

Diagrammatically this contribution can be represented as the bare loop vacuum polarization diagram for the current (20) with the d-quark line cut (Fig. 2).

These examples are rather simple. Especially for higher dimensional gluon operators the method presented above is too complicated. Recently, considerable progress has been

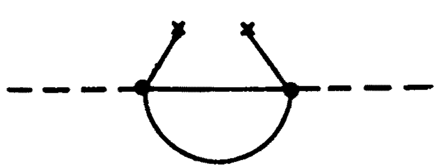


Fig. 2. Diagram that gives the coefficient $C_d(p^2)$ of the operator $\bar{d}d$ in the OPE of two baryon currents

made by working in the so-called fixed-point gauge, originally discovered by Fock and Schwinger (Ref. [4]) and rediscovered by a number of people (Ref. [5]). The gauge condition is

$$x_\mu A^\mu(x) = 0. \tag{23}$$

In this gauge $A_\mu(x)$ can be expressed directly in terms of the gluon field tensor $G_{\mu\nu}$ and its derivatives

$$\begin{aligned} A_\mu(x) &= \int_0^1 \alpha d\alpha G_{\varrho\mu}(\alpha x)x_\varrho \\ &= \tfrac{1}{2}\, x_\varrho G_{\varrho\mu}(0) + \tfrac{1}{3}\, x_\alpha x_\varrho D_\alpha G_{\varrho\mu}(0) + \dots . \end{aligned} \tag{24}$$

Gluonic power corrections can be calculated by considering the polarization operator in the external field of a gluon vacuum fluctuation $A^a_\mu(x)$. To calculate the $G^a_{\mu\nu}G^a_{\mu\nu}$ correction only the first term in the expansion (24) for A_μ has to be taken, for the $d = 6$ G^3 correction also the next two etc. The gauge invariant structures $\sim G^2$ etc. emerge automatically in this case and do not have to be constructed by hand.

Simple rules can now be formulated for the calculation of the coefficients (Ref. [6]). For light quarks it turns out to be most convenient to work in the x-representation. The quark propagator in an external vacuum fluctuation field can then be written as (with $\gamma x = x_\mu \gamma^\mu$)

$$S_F^{aa'}(x) = \frac{i}{2\pi^2}\cdot\frac{\gamma x}{x^4}\,\delta^{aa'} - \frac{ig}{32\pi^2}\,(t^c)^{aa'} G^c_{k\lambda}\,\frac{1}{x^2}\,[\gamma x \sigma_{k\lambda} + \sigma_{k\lambda}\gamma x] + \dots , \tag{25}$$

where the first term is the free quark propagator and the second term the propagator with one gluon line attached. (Averaged over the indices of the gluon fields a massless quark propagator with two gluon lines attached gives zero). For a cut quark line we have the simple rule

$$S_F^{aa'}(x)_{ij} = -\tfrac{1}{12}\,\delta^{aa'}\delta_{ij}\langle 0|\bar{q}q|0\rangle. \tag{26}$$

For heavy quarks the p-representation is more convenient and we have for the various terms in the propagator:

$$\frac{i}{\gamma p - m} \quad \text{the free propagator,} \tag{27a}$$

$$-\frac{i}{4}\, g t^a G^a_{k\lambda}\,\frac{1}{(p^2-m^2)^2}\,\{\sigma_{k\lambda}(\gamma p + m) + (\gamma p + m)\sigma_{k\lambda}\} \tag{27b}$$

with one gluon line attached and

$$\tfrac{1}{12}\, i g^2 G^a_{\varrho\sigma} G^a_{\varrho\sigma}\,\frac{m}{(p^2-m^2)^4}\,(p^2 + m\gamma p) \tag{27c}$$

with two gluon lines attached (and averaged over the indices). For higher order terms in this expansion see Ref. [7].

Let us consider the calculation of the polarization function for the nucleon current (20) to illustrate the simplicity of these rules

$$\Pi_N(q) = i \int d^4x e^{iqx} \langle 0|\mathrm{T}(\eta_N(x)\bar{\eta}_N(0))|0\rangle = i \int d^4x e^{iqx} \Pi_N(x) \tag{28}$$

and $\Pi_N(x)$ can be written as

$$\Pi_N(x) = 2\varepsilon_{abc}\varepsilon_{a'b'c'}\gamma_5\gamma^\mu S_d(x)^{cc'}\gamma_5\gamma^\nu \operatorname{Tr}(S_u^{\mathrm{T}}(x)^{aa'}C\gamma_\mu S_u(x)^{bb'}\gamma_\nu C), \tag{29}$$

where the u, d subscripts indicate which type of quark propagates. To calculate the various Wilson coefficients we only have to substitute the appropriate expressions for S_u and S_d. (We take the quarks to be massless).

For instance:

a) the bare loop, the zeroth order α_s contribution to the Wilson coefficient C_I. All propagators are free propagators, i.e.,

$$S_u = S_d = \frac{i}{2\pi^2}\frac{\gamma x}{x^4}$$

and it is straightforward to find

$$\Pi_N^{(0)}(x) = -\frac{24i}{\pi^6}\frac{\gamma x}{x^{10}}.$$

Substituting this in (28) we obtain

$$\Pi_N^{(0)}(q) = -\frac{1}{4(2\pi)^4}\gamma q q^4 \ln\left(\frac{-q^2}{\Lambda^2}\right) \tag{30a}$$

b) C_d: the coefficient of $\bar{d}d$ in the OPE. The d-quark line is cut, therefore for S_d we have to substitute expression (26), while the two u-quark propagators are free propagators. We find

$$\Pi_N^{(d)}(x) = \frac{2}{\pi^4}\frac{1}{x^6}\langle 0|\bar{d}d|0\rangle$$

and after substitution into (28)

$$\Pi_N^{(d)}(q) = \frac{q^2}{(2\pi)^2}\ln\left(\frac{-q^2}{\Lambda^2}\right)\langle 0|\bar{d}d|0\rangle, \tag{30b}$$

which agrees with (22).

c) C_u: the coefficient of $\bar{u}u$ in the OPE. Now one of the u-quark lines is cut, i.e., for one of the propagators inside the trace we have to substitute (26) and the trace will give zero, i.e., $C_u \equiv 0$.

d) C_G: the coefficient of $G^a_{\mu\nu}G^a_{\mu\nu}$. In this case two of the propagators in (29) are given by the second term of (25), the other one is a free propagator (a quark line with two gluons attached does not have to be considered). The calculation is still rather lengthy but straightforward with the result

$$\Pi_N^{(G)}(q) = -\frac{1}{32\pi^2}\gamma q \ln\left(\frac{-q^2}{\Lambda^2}\right)\langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle \tag{30c}$$

etc.

To conclude this Section let us give as an example the total result for the polarization operator for the vector current $j_V(x) = \bar{q}\gamma_\mu q$ of light quarks ($Q^2 = -q^2$)

$$i\int d^4x e^{iqx}\langle 0|T(j_\mu(x)j_\nu(0))|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi^V(Q^2)$$

$$= (q_\mu q_\nu - q^2 g_{\mu\nu})\left\{-\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{Q^2}{\mu^2} + \frac{2m}{Q^4}\langle 0|\bar{q}q|0\rangle\right.$$

$$+\frac{1}{12Q^4}\langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle - \frac{2\pi\alpha_s}{Q^6}\langle 0|\bar{q}\gamma_\alpha\gamma_5\lambda^a q\bar{q}\gamma_\alpha\gamma_5\lambda^a q|0\rangle$$

$$\left.-\frac{4\pi\alpha_s}{9Q^6}\langle 0|\bar{q}\gamma_\alpha\lambda^a q\bar{q}\gamma_\alpha\lambda^a q|0\rangle + \ldots\right\}. \tag{31}$$

For heavy quarks the Wilson coefficients are (complicated) functions of m and q^2 e.g. for the gluon condensate coefficient C^S_G for the scalar current $\bar{Q}Q$ with Q a heavy quark field:

$$C^S_G(q) = \frac{\alpha_s}{48\pi}\cdot\frac{1}{q^2}\left[\frac{3(3+v^2)(1-v^2)}{v^2}\frac{1}{2v}\ln\frac{1+v}{1-v} - \frac{9+4v^2+3v^4}{v^2(v^2-1)}\right] \tag{32}$$

with $v^2 = 1-4m^2/q^2$.

Expressions for other coefficients for light and heavy quark currents can be found in the literature. For a compilation see Ref. [8].

6. *Moments and the Borel transform*

So far, we have two expressions for the vacuum polarization operator: one in terms of physical resonance parameters as discussed in Section 2, and the other a theoretical expression which is a function of q^2, α_s, the quark masses and the vacuum expectation values of the operators O_n. The theoretical expression has been calculated for large $Q^2(= -q^2)$ where asymptotic freedom prevails and perturbation theory can be used. To probe large distances we have to take derivatives of $\Pi^j(Q^2)$ at some space-like Q_0^2 (for heavy quarks one can even choose $Q_0^2 = 0$). This leads to the moments $M_n^j(Q_0^2)$

$$M_n^j(Q_0^2) = \frac{1}{n!}\left(-\frac{d}{dQ^2}\right)^n \Pi^j(Q^2)\bigg|_{Q^2=Q_0^2} = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\text{Im}\,\Pi(s)ds}{(s+Q_0^2)^{n+1}}\,. \tag{33}$$

For n large enough all subtraction constants in (6) will have been eliminated. Inserting the representation (8) into (33) we can write

$$M_n^j(Q_0^2) = \frac{1}{e_q^2} \cdot \frac{m_R^2}{g_R^2} \frac{1}{(m_R^2+Q_0^2)^{n+1}} [1+\delta_n^j(Q_0^2)], \tag{34}$$

where m_R and g_R are the parameters of the lowest lying resonance and $\delta_n^j(Q_0^2)$ contains the contributions from higher resonances and the continuum. For high n $\delta_n^j(Q_0^2)$ will go to zero because of the factors

$$\frac{g_R^2}{g_{R'}^2} \left(\frac{m_R^2+Q_0^2}{m_{R'}^2+Q_0^2}\right)^{n+1}$$

it contains and because $g_{R'}, m_{R'}, > g_R, m_R$. So, from a certain n onwards $M_n^j(Q_0^2)$ will be practically equal to the contribution of the first resonance. This dominance will be even greater for ratios of moments

$$r_n^j(Q_0^2) = \frac{M_n^j(Q_0^2)}{M_{n-1}^j(Q_0^2)} = \frac{1}{m_R^2+Q_0^2} \cdot \frac{1+\delta_n^j(Q_0^2)}{1+\delta_{n-1}^j(Q_0^2)}, \tag{35}$$

which immediately gives the mass of the lowest resonance if we are at sufficiently high n where $\delta_n^j(Q_0^2) \simeq \delta_{n-1}^j(Q_0^2)$.

For instance, for charmonium a large number of resonances is known and $M_n^V(Q_0^2)$ can be calculated to high accuracy. One can easily verify that the J/ψ alone gives $\sim 50\%$ of $M_1^V(Q_0^2 = 0)$ and $\sim 90\%$ of $M_4^V(Q_0^2 = 0)$. For $Q_0^2 \neq 0$ $\delta_n^j(Q_0^2)$ will converge less fast to zero and for very large Q_0^2 it will be difficult to extract the parameters of the lowest lying resonance from (34) and (35). Large (space-like) Q_0^2 means moving away from the resonance region in the Q^2 plane up to a point from where it will be impossible to distinguish individual resonances. In principle this can be compensated by taking large n. Indeed, taking higher derivatives of $\Pi(q^2)$ means testing larger distances, i.e., moving towards the resonance region. The important observation of SVZ is that there is a region in Q_0^2 (starting at about $Q_0^2 = 0$ for heavy quarks) where asymptotic freedom holds.

The expressions above give the phenomenological side of the moments. Taking derivatives of the Wilson coefficients we can write for the theoretical moments (for heavy quarks in which case only the gluon condensate gives a nonperturbative contribution)

$$M_n^j(\xi) = A^j(n)\,[1+a_n(j;\xi)\alpha_s + b_n(j;\xi)\phi], \tag{36}$$

where $\xi = Q_0^2/4m^2$ and

$$\phi = \frac{4\pi^2}{9} \langle 0|\, \frac{\alpha_s}{\pi}\, G_{\mu\nu}^a G_{\mu\nu}^a |0\rangle/(4m^2)^2 \tag{37}$$

is the matrix element of the gluon condensate. $A^j(n)$ is the n^{th} derivative of the bare loop contribution; $A^j(n) \sim (m^2)^{-n}$. The dimensionless coefficients $a_n(j;\xi)$ and $b_n(j;\xi)$ are the moments of the α_s contribution of the Wilson coefficients C_I^Γ and C_G respectively normalized with respect to the bare loop. Explicit expressions for a_n and b_n can be found in Refs. [1] and [9] (for $\xi = 0$) and in Ref. [2] (for $\xi \neq 0$). The coefficients $|a_n|$ and $|b_n|$ grow with n (b_n like n^3 for S-states and like $3n^3$ for P-states) but decrease with ξ (or Q_0^2), which has to be chosen such that for a range of n values $|a_n\alpha_s|$, $|b_n\phi| \ll 1$ for first order perturbation theory to make sense. We will see later in applications to the charmonium spectrum that for the calculations to be reliable Q_0^2 has to be chosen unequal to zero. As an example we have plotted the coefficients a_n and b_n for the axial vector current as a function of n for various values of ξ in Figs. 3 and 4. For a certain value of ξ there will be a range of n values

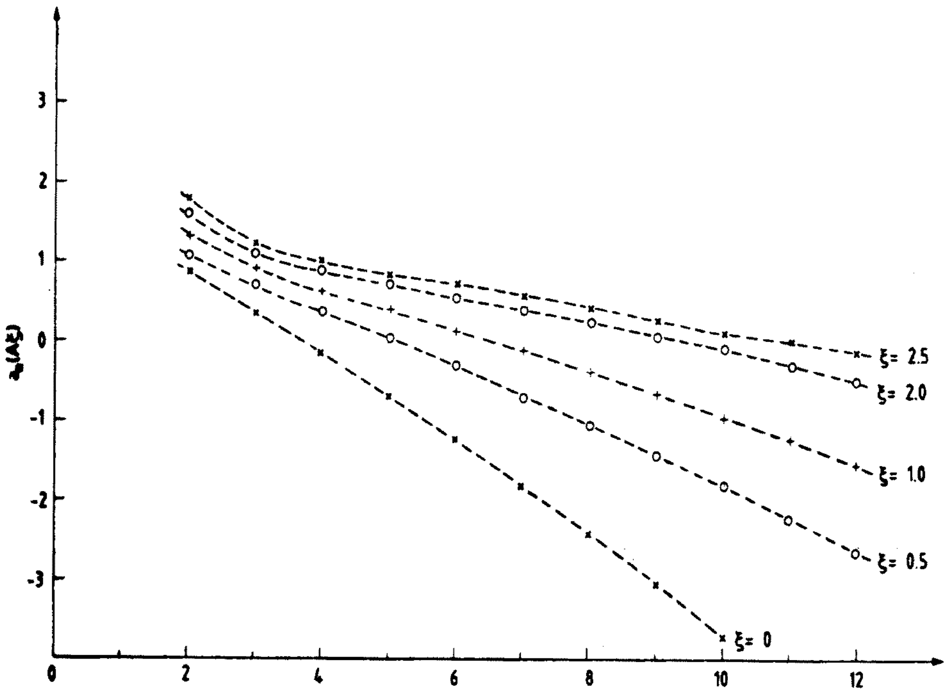


Fig. 3. The coefficient $a_n(A;\,\xi)$ for the axial vector current as a function of n for various values of ξ

for which the experimental side of the moment equation (34) (or the ratio (35)) is dominated by a single resonance, while the asymptotic freedom side (36) is still valid. For small n there occurs a breakdown since the effect of higher states in (34) should become noticeable and at high n the expansion (36) does not hold when $b_n\phi$ becomes too large compared to 1. The stability region in n will change with ξ and one should study the stability of the moments for growing ξ. Further on we will discuss applications of this method to the charmonium system.

Equation (36) is only correct for equal mass heavy quark systems. For light-heavy systems the quark condensate will also give a contribution to (36) which again must be small compared to 1 for the expansion to be valid. For light quark systems more higher dimensional operators come in. The moment method can in principle also be used in this case, choosing a large mass scale Q^2 where all corrections are small. If Q^2 tends to

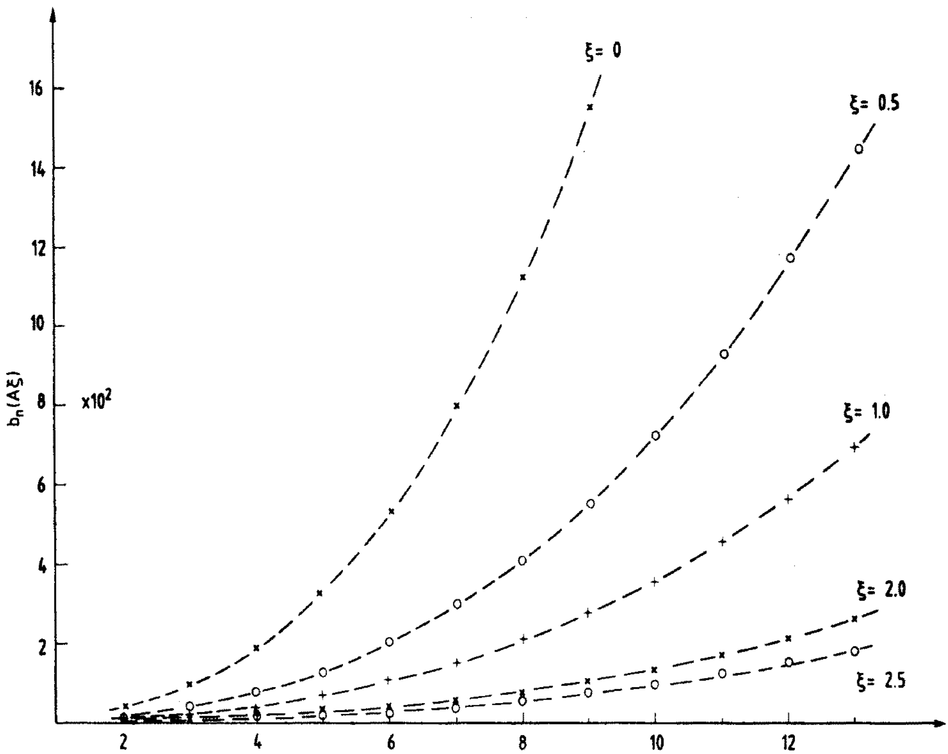


Fig. 4. The coefficient $b_n(A, \xi)$ for the axial vector current as a function of n for various values of ξ

infinity the number of derivatives which can reliably be calculated is also arbitrarily large and one can consider the limit

$$Q^2 \to \infty, n \to \infty, Q^2/n \equiv M^2 \text{ fixed.} \tag{38}$$

In this way a new variable M^2 is introduced instead of Q^2. It corresponds to introducing the Borel transform of $\Pi^j(Q^2)$:

$$\mathrm{L}_M \Pi^j(Q^2) = \lim_{\substack{Q^2, n \to \infty \\ Q^2/n \equiv M^2}} \frac{1}{(n-1)!} (Q^2)^n \left(- \frac{d}{dQ^2} \right)^n \Pi^j(Q^2), \tag{39}$$

which results in a Borel improvement of the series (9) since an operator of dimension d is suppressed by a factor $1/(\frac{1}{2} d - 1)!$. Applying L_M to (6) we get

$$\mathrm{L}_M \Pi^j(Q^2) = \frac{1}{\pi M^2} \int ds e^{-s/M^2} \operatorname{Im} \Pi^j(s), \tag{40}$$

where the weight function in the dispersion integral has been replaced by an exponential one. In one of the next Sections we will discuss a few applications of this method.

7. The operator matrix elements

Before we can turn to applications of the sum rules we need the vacuum expectation values of the various operators O_n.

For heavy quark systems we only have $\langle 0|\alpha_s/\pi G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle$. This matrix element has been determined phenomenologically from the charmonium spectrum (see next Section), resulting in the value

$$\langle 0|\frac{\alpha_s}{\pi} G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle \simeq (340\,\mathrm{MeV})^4. \tag{41}$$

It is very encouraging that recent determinations from lattice simulations agree with this value (Ref. [10]).

The quark condensate $\langle 0|m\bar{q}q|0\rangle$ is known from current algebra

$$(m_u+m_d)\langle 0|\bar{u}u+\bar{d}d|0\rangle = -m_\pi^2 f_\pi^2, \tag{42}$$

where f_π is the $\pi \to \mu\nu$ decay constant, $f_\pi = 133$ MeV. Using isospin invariance and SU(3) symmetry, and $m_u+m_d \simeq 15$ MeV, we find

$$\begin{aligned}\langle 0|\bar{q}q|0\rangle &\simeq -(250\,\mathrm{MeV})^3, \qquad q = u, d, s;\\ \langle 0|m\bar{q}q|0\rangle &\simeq -(100\,\mathrm{MeV})^4, \qquad m = m_u, m_d.\end{aligned} \tag{43}$$

We note that $\langle 0|m\bar{q}q|0\rangle \ll \langle 0|\alpha_s/\pi G^2|0\rangle$.

For the four-fermion operators we follow SVZ in assuming dominance of the vacuum intermediate state, which makes it possible to express them in terms of $\langle 0|\bar{q}q|0\rangle$. For instance,

$$\begin{aligned}\langle 0|\bar{q}\gamma_\mu\lambda^a q\bar{q}\gamma_\mu\lambda^a q|0\rangle &= -\tfrac{16}{9}\langle 0|\bar{q}q|0\rangle^2,\\ \langle 0|\bar{q}\gamma_5\lambda^a q\bar{q}\gamma_5\lambda^a q|0\rangle &= -\tfrac{4}{9}\langle 0|\bar{q}q|0\rangle^2.\end{aligned} \tag{44}$$

In the applications that we will discuss other operators do not play an important role.

Some of the operators listed above are not renormalization group invariant and will have anomalous dimensions. In the following we have neglected these but they can be included in a straightforward way.

8. Charmonium

As a first application we will consider the moment method described in Section 6 for the charmonium spectrum. The first important result in charmonium spectroscopy using QCD sum rules was obtained by Shifman, Vainshtein and Zakharov (Ref. [11]) when they predicted the 0^{-+} resonance η_c at 3.0 GeV (at the same time ruling out the X(2.83) as a possible candidate). Later a full treatment of all charmonium levels (including the P-states) has been given in Ref. [2].

As shown in Section 6 the mass of the lowest lying resonance follows directly from the ratio of moments (35) and (36) giving the moment equation

$$m_R^2+Q_0^2 = \frac{A^j(n-1)\,[1+a_{n-1}(j;\xi)\alpha_s+b_{n-1}(j;\xi)\phi]}{A^j(n)\,[1+a_n(j;\xi)\alpha_s+b_n(j;\xi)\phi]}. \tag{45}$$

(Here we have neglected the contributions from higher resonances and the continuum. A continuum contribution can be included on the theoretical side of (45), but as can be seen from Fig. 5 for the vector channel it does not play a role in determining the mass.) Since the values of $b_n(j;\xi)$ for P-waves are on average a factor three larger than the $b_n(j;\xi)$ coefficients for S-waves, ξ has to be chosen larger for P-waves than for S-waves to fulfil the condition $|b_n\phi| \ll 1$. In Fig. 6 we show the stability region for the ratio of the pseudoscalar to the vector current for various values of ξ. It is clear from this picture that $\xi = 0$ provides a rather small window for determining resonance parameters. We will come back to this at the end of the Section when we discuss recent results by Nikolaev and Radyushkin on the calculation of contributions from $d = 6$ and $d = 8$ gluon condensate operators. On the other hand, if we choose ξ very large the contribution from higher resonances to the sum rule will be so large that we cannot extract the parameters of the lowest lying resonance in a reliable way. We have no criterion to decide which ξ to choose

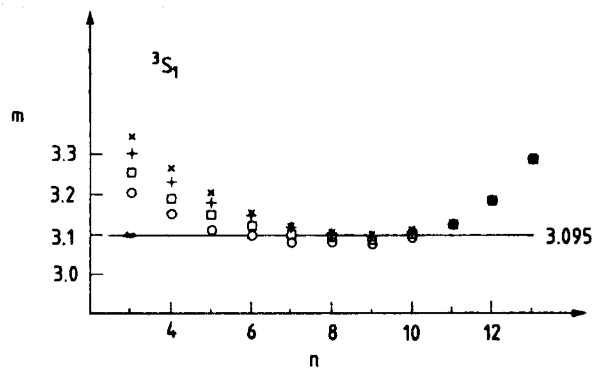


Fig. 5. Results for the vector state of charmonium from the moments (45) for various choices of the starting point s_0 of the continuum contribution ($\times : \sqrt{s_0} = 4.4$ GeV; $+ : \sqrt{s_0} = 4.2$ GeV; $\square : \sqrt{s_0} = 4.0$ GeV and $\bigcirc : \sqrt{s_0} = 3.8$ GeV)

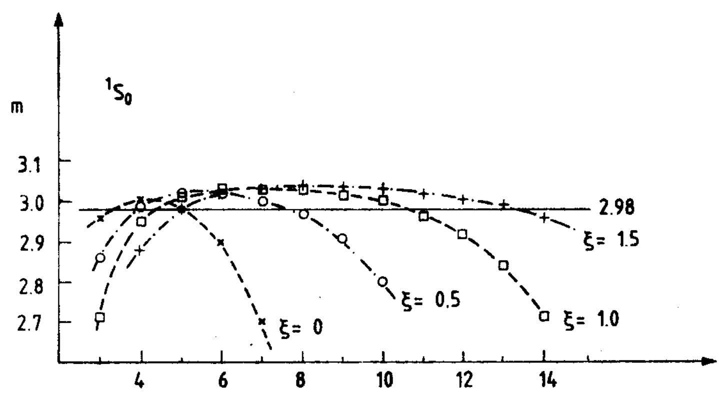


Fig. 6. Stability region in n for the pseudoscalar S-state of charmonium for various values of ξ. For all ξ values the parameters are as in (49)

as for most channels there are no higher resonances known, and we cannot calculate its contribution to the moments. For each channel we have determined ξ empirically, but apart from the vector channel we have no guarantee that the contribution from higher resonances goes quickly enough to zero with n. It seems reasonable however to assume that the spectra in other channels are similar to the vector channel.

One should also note that α_s (and m_c) being running parameters are ξ dependent since the region which dominates the dispersion integral in (33) will be different for different Q_0^2. Moreover, α_s also depends on n as noted by SVZ. At present we are not in a position to be able to give quantitative estimates of this dependence. Instead, at a certain value of ξ we choose α_s constant at $\alpha_s(Q_0^2+4m_c^2)$, with its value calculated from the asymptotic freedom formula

$$\alpha_s(Q_0^2+4m_c^2) = \frac{\alpha_s(4m_c^2)}{1+\frac{25}{12\pi}\alpha_s(4m_c^2)\ln\frac{Q_0^2+4m_c^2}{4m_c^2}} \tag{46}$$

with $\alpha_s(4m_c^2) \simeq 0.3$.

Similarly m_c depends on Q_0^2. At $Q_0^2 \neq 0$ the quark mass is chosen to be renormalized at $m_c(p^2 = -(\xi+1)m_c^2)$ which (in the Landau gauge) is expressed in terms of $m_c = m_c(p^2 = -m_c^2)$ by the following formula

$$\frac{m_c(\xi)}{m_c} = 1-\frac{\alpha_s}{\pi}\left\{\frac{2+\xi}{1+\xi}\ln(2+\xi)-2\ln 2\right\}. \tag{47}$$

Our parameters are ϕ (as defined in (37)), α_s and m_c. The values chosen for ξ are $\xi = 1.0$ for S-waves and $\xi = 2.5$ for P-waves. For these values the stability region for the moment equations (45) is sufficiently large to make reliable estimates for the masses of the states. Using Eq. (46) the α_s values for S- and P-waves have been calculated.

Figures 7 and 8 show the masses as determined from the ratios (45). These moments are very sensitive to the quark mass. Even a slight change in the quark mass would spoil the beautiful agreement with the experimental mass value for the vector case (Fig. 7a). We believe that this calculation provides the best evaluation of the charmed current quark mass at this value of Q_0^2. Values at other Q_0^2 can then be obtained from fromula (47). In particular, the value of the quark mass to be used at $\xi = 2.5$ for the P-waves is now fixed, and we can also calculate the quark mass at the Euclidean point $p^2 = -m_c^2$, i.e., $Q^2 = 0$:

$$m_c(p^2 = -m_c^2) \simeq 1.28 \text{ GeV}. \tag{48}$$

The following set of parameters has been obtained

$$\begin{array}{ll} \text{S-waves } (\xi = 1.0) & \text{P-waves } (\xi = 2.5) \\ m_c = 1.25 \text{ GeV}, & m_c = 1.22 \text{ GeV}, \\ \phi = 0.14\times 10^{-2}, & \phi = 0.14\times 10^{-2}, \\ \alpha_s = 0.27 & \alpha_s = 0.25, \\ \sqrt{s_0} = 4.0 \text{ GeV}, & \sqrt{s_0} = 4.8 \text{ GeV}. \end{array} \tag{49}$$

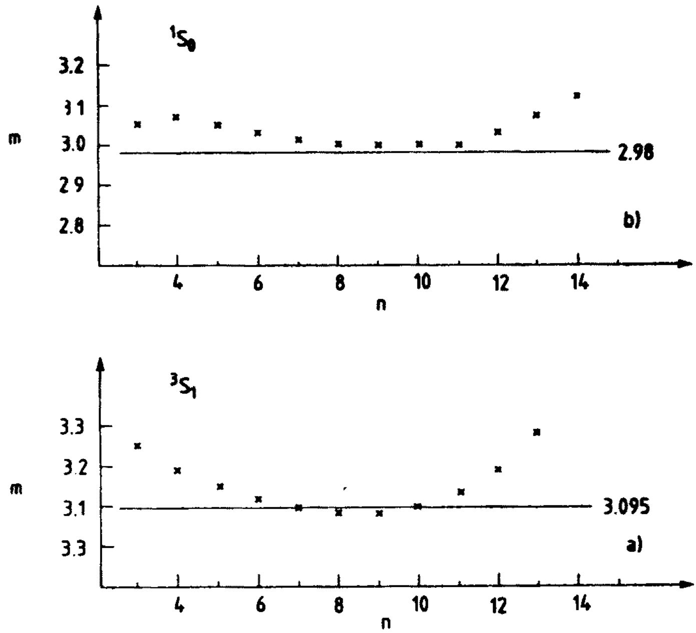


Fig. 7. Results for (a) the vector and (b) the pseudoscalar states of charmonium from the moments (45). All parameters as in (49). For comparison the experimental mass values have also been indicated

We stiess that only ϕ and $m_c(\xi = 1.0)$ are free parameteis. The values for α_s follow from Eq. (46) with $\alpha_s(4m_c^2) \simeq 0.3$ and Eq. (47) has been used to calculate m_c at $\xi = 2.5$. The parameter set (49) leads to the following numerical results for the masses

$$
\begin{aligned}
\eta_c(^1S_0)&:\ m = 3.01 \pm 0.02\ \text{GeV}, && (\text{exp. } 2.98),\\
J/\psi(^3S_1)&:\ m = 3.10 \pm 0.01, && (3.095),\\
\chi(^3P_0)&:\ m = 3.40 \pm 0.01, && (3.41),\\
\chi(^3P_1)&:\ m = 3.50 \pm 0.01, && (3.51),\\
\chi(^3P_2)&:\ m = 3.56 \pm 0.01, && (3.56),\\
\chi(^1P_1)&:\ m = 3.51 \pm 0.01, && (?)
\end{aligned}
\tag{50}
$$

The agreement with known masses is extremely good.

Having determined the masses, we can use the moments (34) directly to calculate the couplings g_R. Contamination by higher resonances will be larger than for the ratios (35) and we do not expect to obtain as accurate results as for the masses. Only foı the vector channel g_R is related to the physical width $\Gamma(e^+e^-)$. Neglecting all higher resonances we find

$$\Gamma_{J/\psi}(e^+e^-) = 5.3\ \text{keV}, \tag{51}$$

which compares quite favourably with the experimental number $\Gamma_{J/\psi}(e^+e^-) = (4.7 \pm 0.6)$ keV. We estimate that the error in (51) is 10–20% due to higher resonances.

Recently Nikolaev and Radyushkin (Refs. [12] and [13]) have reported calculations of the Wilson coefficients of six- and eight-dimensional gluon operators. The coefficients of the six dimensional operators $\langle G^3 \rangle$ turn out to be small and their contribution to the moments can safely be neglected. However, the coefficient functions of the eight dimensional operators $\langle G^4 \rangle$ appear to be much larger than expected (at $Q^2 = 0$). Employing a few models for the vacuum to calculate their contributions to the ratios r_n they find

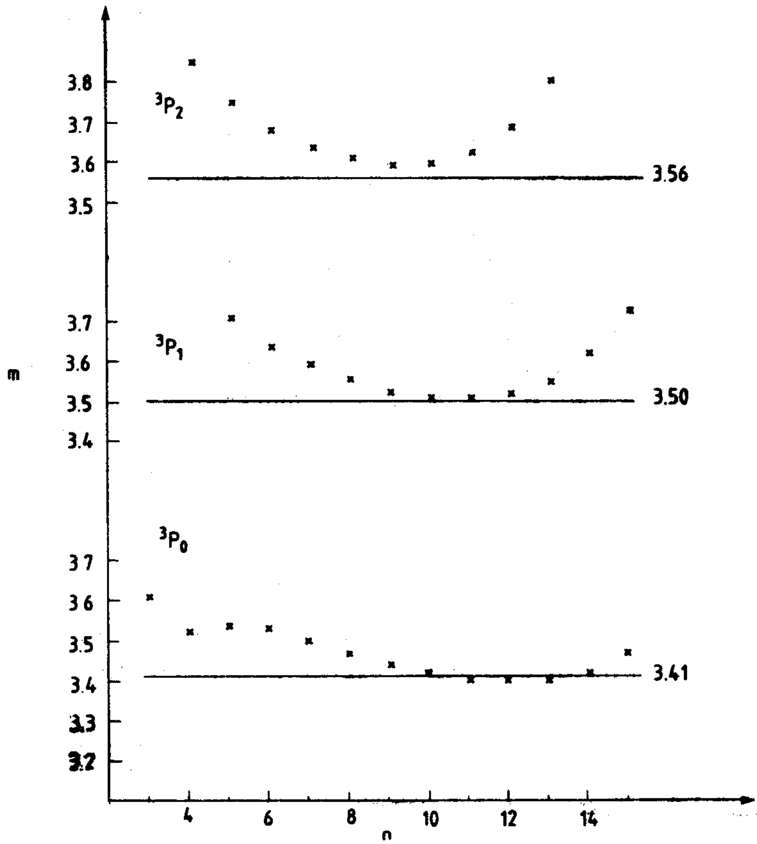


Fig. 8. Results for the P-states of charmonium from the moments (45). All parameters as in (49). For comparison the experimental mass values have also been indicated

those to be large ($\sim$ 50% of the $\langle G^2 \rangle$ contribution for r_6). These vacuum models and in particular the assumption that a similar factorization procedure as for the six-dimensional quark condensate operators (see Section 7) can be used to express the matrix elements $\langle G^4 \rangle$ in terms of $\langle G^2 \rangle^2$ has recently been criticized by Novikov et al. (Ref. [14]). They show that contrary to the quark condensate operators such a factorization does not give a reliable estimate for the matrix element.

However, this still does not tell us that the $\langle G^4 \rangle$ contribution is indeed small. We have already seen (see Fig. 6) that $Q^2 = 0$ provides a very small window for determining resonance parameters. The moment $n = 6$ is already outside the fiducial region. If we move to $Q^2 \neq 0$ the values of a_n and b_n will be suppressed (see Figs. 3 and 4). The point is that contributions from higher dimensional operators will be even more suppressed. At $Q_0^2 = 4m_c^2$ (i.e. $\xi = 1$) the suppression of contributions from $d = 8$ operators will be about four times larger than from $d = 4$ operators, which could easily solve the problems raised by Nikolaev and Radyushkin.

Let us conclude this Section with a few remarks on the upsilon system. Here the method used for charmonium fails (see Ref. [2]). Due to the higher quark mass the non-perturbative effects are now essentially negligible:

$$\phi_b = \phi_c \left(\frac{m_c}{m_b}\right)^4 \sim \phi_c/100$$

with ϕ defined by (37). On the phenomenological side this is reflected in the fact that the resonances in the upsilon system are relatively much closer together than in charmonium and one would have to go to very high n to ensure dominance of the lowest lying state (and at the same time a non-negligible nonperturbative contribution). At high n higher order perturbative contributions have to be taken into account (the k^{th} order α_s correction grows essentially like $(\sqrt{n}\alpha_s)^k$ with n the moment number). A program of this type has been performed by Voloshin (Ref. [15]) using nonrelativistic Borel transformed sum rules. His result for the P-levels, however, is lower than the recent data from CUSB and CLEO (Ref. [16]).

9. *Light quark meson systems*

As explained in Section 6, for light quark systems we use Borel transformed sum rules (Eqs. (39) and (40)). In this section we will discuss the ϱ and A_1 mesons in some detail. For other examples see Refs. [1, 17] and [18].

The polarization operator for the vector current is given by Eq. (31). For the ϱ meson the current is the $I = 1$ combination $j_\mu(x) = \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)$. From (31) one easily finds for the polarization operator for this current

$$\Pi(Q^2) = -\frac{1}{8\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{Q^2}{\mu^2} + \frac{1}{2Q^4}\langle 0|m_u\bar{u}u + m_d\bar{d}d|0\rangle$$
$$+\frac{1}{24Q^4}\langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle - \frac{\pi\alpha_s}{2Q^6}\langle 0|(\bar{u}\gamma_\mu\gamma_5\lambda^a u - \bar{d}\gamma_\mu\gamma_5\lambda^a d)^2|0\rangle$$
$$-\frac{\pi\alpha_s}{9Q^6}\langle 0|(\bar{u}\gamma_\mu\lambda^a u + \bar{d}\gamma_\mu\lambda^a d)\sum_{q=u,d,s}\bar{q}\gamma_\mu\lambda^a q|0\rangle. \tag{52}$$

Substituting the approximations for the various operators discussed in Section 7, and performing the Borel transform of (52) we arrive at the sum rule

$$\int e^{-s/M^2} \operatorname{Im} \Pi(s) ds = \frac{1}{8\pi} M^2 \left[1 + \frac{\alpha_s(M)}{\pi} + \frac{8\pi^2}{M^4} \langle 0|m\bar{q}q|0\rangle + \frac{\pi^2}{3M^4} \langle 0| \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle - \frac{448}{81} \cdot \frac{\pi^3 \alpha_s}{M^6} \langle 0|\bar{q}q|0\rangle^2 \right]. \tag{53}$$

For α_s we choose α_s ($M \simeq m_\varrho$) $\simeq 0.6$ ($\Lambda \simeq 200$ MeV). Substituting the numbers given in Section 7 for the various matrix elements we get (M^2 in GeV^2)

$$\int e^{-s/M^2} \operatorname{Im} \Pi(s) ds = \frac{1}{8\pi} M^2 \left[1 + \frac{\alpha_s}{\pi} + \frac{0.04}{M^4} - \frac{0.03}{M^6} \right]. \tag{54}$$

A second sum rule can be derived by differentiating (54) with respect to M^2:

$$\int e^{-s/M^2} \operatorname{Im} \Pi(s) s ds = \frac{1}{8\pi} M^4 \left[1 + \frac{\alpha_s}{\pi} - \frac{0.04}{M^4} + \frac{0.06}{M^6} \right]. \tag{55}$$

In these two expressions the power corrections are relatively small compared to the unit term even at $M^2 \simeq m_\varrho^2 \simeq 0.6\ \mathrm{GeV}^2$, and at such M^2 the integral over $\operatorname{Im} \Pi(s)$ is dominated by the ϱ meson.

We now saturate as before for the charmonium system $\operatorname{Im} \Pi(s)$ by one resonance in a narrow resonance approximation, plus a continuum with threshold s_0 in the form of a θ function:

$$\operatorname{Im} \Pi(s) = \frac{\pi m_\varrho^2}{g_\varrho^2} \delta(s - m_\varrho^2) + \frac{1}{8\pi} \left(1 + \frac{\alpha_s}{\pi} \right) \theta(s - s_0). \tag{56}$$

Substituting (56) into the left-hand side of (54) and (55), transferring the continuum contribution to the right-hand side of these equations and taking the ratio we find the following expression for the mass of the resonance

$$m_\varrho^2 = M^2 \frac{\left(1 + \frac{\alpha_s}{\pi}\right)\left[1 - \left(1 + \frac{s_0}{M^2}\right) e^{-s_0/M^2}\right] - \frac{0.04}{M^4} + \frac{0.06}{M^6}}{\left(1 + \frac{\alpha_s}{\pi}\right)[1 - e^{-s_0/M^2}] + \frac{0.04}{M^4} - \frac{0.03}{M^6}}. \tag{57}$$

The value of s_0 chosen, $s_0 = 1.5\ \mathrm{GeV}^2$, is suggested by the experimental data on R in e^+e^- annihilation. For this value of s_0 a stable mass prediction is obtained for a range of M^2. For very small M^2 the power corrections blow up and for large M^2 (57) is dominated by s_0. In Fig. 9 we have plotted the mass prediction as a function of M^2 for a few values of s_0. Contrary to the charmonium system where the results were insensitive to the threshold choice, it can be seen from Fig. 9 that the results are sensitive to the actual choice of s_0, but the stability criterion fixes s_0 quite accurately.

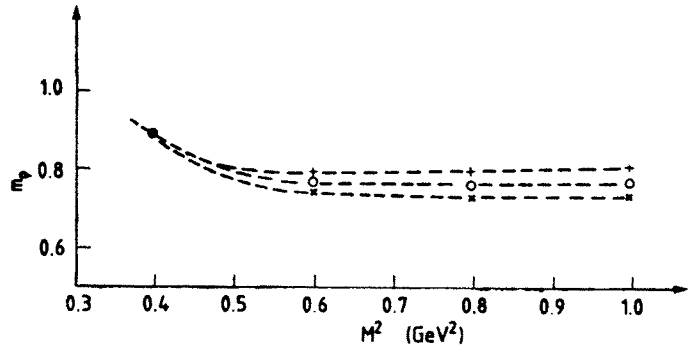


Fig. 9. The theoretical prediction for the ρ-meson mass from Eq. (57) for a few values of the continuum threshold $s_0(+ : s_0 = 1.75\,(\mathrm{GeV})^2,\ \bigcirc : s_0 = 1.50\,(\mathrm{GeV})^2,\ \times : s_0 = 1.25\,(\mathrm{GeV})^2)$. The strong coupling constant $\alpha_s = 0.6$

To consider the situation a little more in detail we follow SVZ and write for the prediction of the mass

$$m_\varrho^2 = M^2 f_{\mathrm{cont}}(M^2) f_{\mathrm{th\,corr}}(M^2), \tag{58}$$

where $f_{\mathrm{th\,corr}}(M^2)$ is given by (57) without the continuum contributions ($s_0 = \infty$) and f_{cont} is the ratio of (57) and $f_{\mathrm{th\,corr}}$. Both these functions have been plotted in Fig. 10. Without

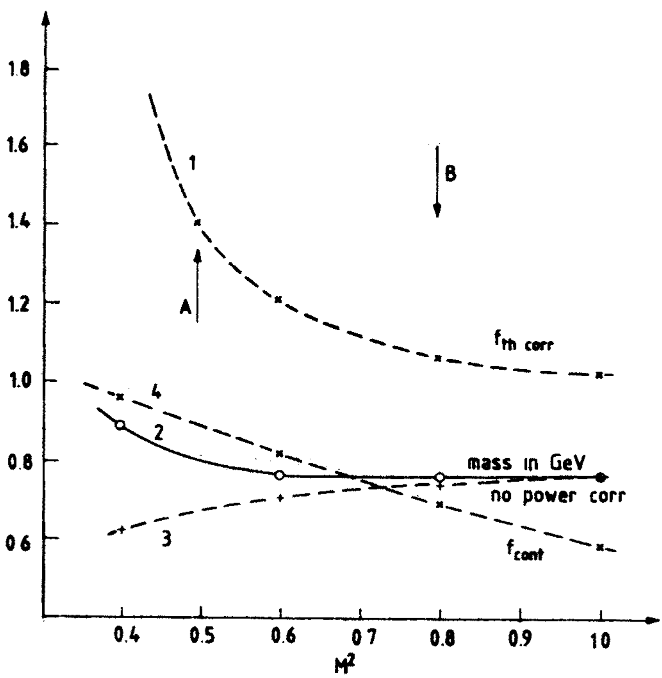


Fig. 10. The ρ-meson mass with and without power corrections for $s_0 = 1.5\,(\mathrm{GeV})^2$. Also shown are the functions f_{cont} and $f_{\mathrm{th\,corr}}$ defined in the text. The region between the arrows A and B is considered to be reliable for determining the resonance parameters

power corrections $f_{\text{th corr}}$ would be equal to 1 and without continuum ($s_0 = \infty$) f_{cont}would be equal to 1. The deviations from these values give a measure of the importance of the power corrections and the continuum. The arrows A and B indicate the region in M^2 which is reliable from a theoretical point of view and still sensitive to the resonance contribution. For these reasons the sum rules are not considered when the continuum contribution and/or the power corrections exceed 30%. The assumption is that unaccounted power corrections are of the order of the square of the power corrections that are kept. This keeps the accuracy of the calculations at the 10% level.

Curve 3 in Fig. 10 gives the result of the sum rule without power corrections while curve 2 is the actual mass prediction. No stability is obtained without power corrections which contribute substantially to get a stable mass prediction.

The sum rule (54) can be used directly to determine the ϱ-meson coupling. The final results for the ϱ-meson parameters are

$$\frac{g_\varrho^2}{4\pi} \simeq 2.42, \qquad m_\varrho \simeq 770 \text{ MeV}, \tag{59}$$

which compares very favourably with the experimental values 2.36 ± 0.18 and 776 ± 3 MeV.

Similarly one can also consider the $I = 0$ currents $j_\mu(x) = \frac{1}{2}(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)$ and $j_\mu(x) = \frac{1}{3}\bar{s}\gamma_\mu s$, which correspond to the ω and ϕ mesons respectively.

For the A_1 meson we have to consider the isospin $I = 1$ ($J^{CP} = 1^{++}$) axial current $j_\mu(x) = \frac{1}{2}(\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d)$. For the polarization operator we have

$$\begin{aligned} \Pi_1(Q^2) = {} & \frac{1}{8\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)Q^2 \ln\frac{Q^2}{\mu^2} + \frac{1}{2Q^2}\langle 0|m_u\bar{u}u + m_d\bar{d}d|0\rangle \\ & - \frac{1}{24Q^2}\langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle + \frac{\pi\alpha_s}{2Q^4}\langle 0|(\bar{u}\gamma_\mu\lambda^a u - \bar{d}\gamma_\mu\lambda^a d)^2|0\rangle \\ & + \frac{\pi\alpha_s}{9Q^6}\langle 0|(\bar{u}\gamma_\mu\lambda^a u + \bar{d}\gamma_\mu\lambda^a d)\sum_{q=u,d,s}\bar{q}\gamma_\mu\lambda^a q|0\rangle. \end{aligned} \tag{60}$$

Using the values for the operator matrix elements and performing the Borel transform we find the sum rule

$$\int e^{-s/M^2}\,\text{Im}\,\Pi_1(s)ds = \frac{1}{8\pi}M^4\left[1+\frac{\alpha_s}{\pi} - \frac{0.046}{M^4} - \frac{0.10}{M^6}\right]. \tag{61}$$

Again a second sum rule is found by differentiating both sides of (61) with respect to $1/M^2$. Finally we get

$$m_R^2 = M^2\,\frac{2\left(1+\frac{\alpha_s}{\pi}\right)\left[1-\left(1+\frac{s_0}{M^2}+\frac{s_0^2}{2M^4}\right)e^{-s_0/M^2}\right] + \frac{0.10}{M^6}}{\left(1+\frac{\alpha_s}{\pi}\right)\left[1-\left(1+\frac{s_0}{M^2}\right)e^{-s_0/M^2}\right] - \frac{0.046}{M^4} - \frac{0.10}{M^6}} \tag{62}$$

and we can use this equation to find the mass of the A_1 meson.

However, we can derive an alternative sum rule by considering the divergence of the axial vector current $j_\mu(x) = \bar{q}\gamma_\mu\gamma_5 q$. This divergence has a pseudoscalar and an axial component. The resulting polarization function has been given by SVZ in Ref. [1] and leads to the sum rule

$$\int e^{-s/M^2}\,\mathrm{Im}\,\Pi_2(s)ds = \frac{1}{8\pi}M^2\left[1+\frac{\alpha_s}{\pi}+\frac{0.046}{M^4}+\frac{0.05}{M^6}\right]. \qquad (63)$$

Actually, the sum rule (61) is just the derivative of (63) because up to the order we work $\mathrm{Im}\,\Pi_1 = s\,\mathrm{Im}\,\Pi_2$ in the chiral limit. This implies that if we saturate $\mathrm{Im}\,\Pi_1$ and $\mathrm{Im}\,\Pi_2$ by resonances, plus a continuum, the continuum threshold is the same in both cases (or different by m_π^2 which is negligible). We have

$$\mathrm{Im}\,\Pi_2 = \frac{\pi}{2}f_\pi^2\delta(s)+\pi m_{A_1}^2 f_{A_1}^{-2}\delta(s-m_{A_1}^2)+\frac{1}{8\pi}\left(1+\frac{\alpha_s}{\pi}\right)\theta(s-s_0), \qquad (64a)$$

$$\mathrm{Im}\,\Pi_1 = \pi m_{A_1}^4 f_{A_1}^{-2}\delta(s-m_{A_1}^2)+\frac{s}{8\pi}\left(1+\frac{\alpha_s}{\pi}\right)\theta(s-s_0), \qquad (64b)$$

where the constants f_{A_1} and f_π are defined in the usual way

$$\langle 0|\bar{u}\gamma_\mu\gamma_5 d|\pi\rangle = if_\pi P_\mu \quad \text{and} \quad \langle 0|\bar{u}\gamma_\mu\gamma_5 d|A_1\rangle = \sqrt{2}f_{A_1}^{-1}m_{A_1}^2\varepsilon_\mu.$$

Substituting (64a) into (64b), transferring the continuum and pion pole contributions to the right-hand side and taking the ratio with the second sum rule we find

$$m_R^2 = M^2\frac{\left(1+\frac{\alpha_s}{\pi}\right)\left[1-\left(1+\frac{s_0}{M^2}\right)e^{-s_0/M^2}\right]-\frac{0.046}{M^4}-\frac{0.10}{M^6}}{\left(1+\frac{\alpha_s}{\pi}\right)[1-e^{-s_0/M^2}]+\frac{0.046}{M^4}+\frac{0.05}{M^6}-\frac{4\pi f_\pi^2}{M^2}}, \qquad (65)$$

where the pion coupling constant $f_\pi = 133$ MeV. The behaviour of the two sum rules (62) and (65) as a function of s_0 is different and it turns out that they only result in the same resonance mass for $s_0 \simeq 1.75\ (\mathrm{GeV})^2$. The result for the A_1 meson mass is shown in Fig. 11 from which we find

$$m_{A_1} = 1.15\pm 0.04\ \mathrm{GeV} \qquad (66)$$

in good agreement with the experimental value usually quoted.

We can also use (61) and (63) directly for determining the A_1 coupling constant. Again the two sum rules agree very nicely and give (Fig. 12)

$$\frac{4\pi}{f_{A_1}^2} \simeq 0.15-0.18.$$

This completes our discussion of light quark meson systems. Following the same procedure all $L = 0$ and $L = 1$ mesons have been discussed in Refs. [1, 17] and [18].

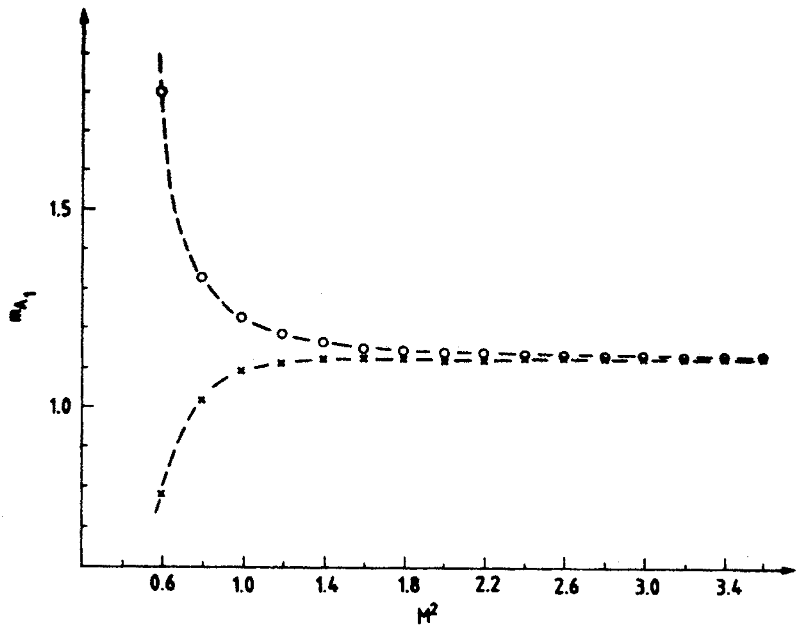


Fig. 11. The mass of the A_1 meson from the two sum rules Eq. (62) (○) and Eq. (65) (×). The continuum threshold $s_0 = 1.75\ (\mathrm{GeV})^2$

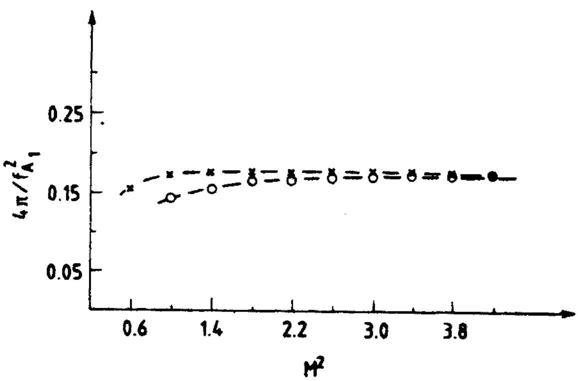


Fig. 12. The A_1 coupling constant $4\pi/f_{A_1}^2$ from the two sum rules Eq. (61) (○) and Eq. (63) (×) defined as $\langle 0|\bar{u}\gamma_\mu\gamma_5 d|A_1\rangle = \sqrt{2}\, f_{A_1}^{-1} m_{A_1}^2 \varepsilon_\mu$

10. Baryons

Two-point functions of baryons have been discussed by several authors (Refs. [6, 19–23]). As an example we will discuss the sum rules for the nucleon in some detail, since we need the results in the next Section when we discuss three-point functions. Extensions to all other octet resonances and to the decouplet resonances have been made in the papers referred to above.

For the nucleon we choose the current (20) which has the right quantum numbers. Then

$$\Pi(q) = i \int d^4x e^{iqx} \langle 0 | T\{\eta_N(x)\bar{\eta}_N(0)\} | 0 \rangle = \Pi_1(q^2) + \gamma q \Pi_2(q^2). \tag{67}$$

So there are two invariant functions. Counting dimensions, one easily finds that $\Pi_1(q^2)$ has an odd number of dimensions while $\Pi_2(q^2)$ is even because of the factor γq. This implies that in performing the operator product expansion the Wilson coefficients of even dimensional operators (I, $G^a_{\mu\nu}G^a_{\mu\nu}$, $\bar{q}\Gamma q\bar{q}\Gamma q$) in $\Pi_1(q^2)$ will be proportional to the small quark mass m_q while the operator $\bar{q}q$ will appear without m_q and gives the dominant contribution to $\Pi_1(q^2)$, the more so since the contribution of the five-dimensional operator $\bar{q}\sigma_{\mu\nu}G_{\mu\nu}q$ turns out to be zero for octet baryons (Ref. [6]). To $\Pi_2(q^2)$ all operators contribute but since

$$|\langle 0 | m\bar{q}q | 0 \rangle| \ll \langle 0 | \frac{\alpha_s}{\pi} G^a_{\mu\nu}G^a_{\mu\nu} | 0 \rangle$$

we can neglect the $\bar{q}q$ piece. Therefore the diagrams of Fig. 13a contribute to $\Pi_2(q^2)$ and those of Fig. 13b to $\Pi_1(q^2)$. The calculation of some of the Wilson coefficients for the polarization operator induced by the current (20) has been discussed in detail in Section 5.

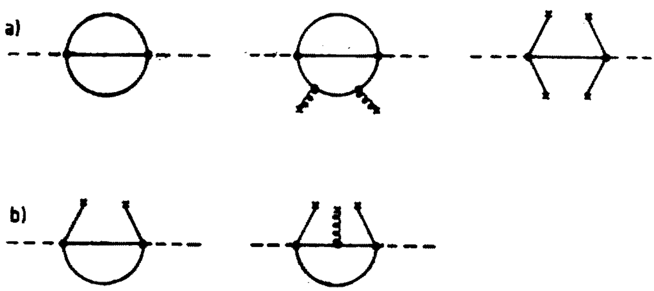


Fig. 13. Diagrams to be calculated for the Wilson coefficients of the operators in the OPE of two baryon currents; (a) for $\Pi_2(q^2)$ and (b) for $\Pi_1(q^2)$ as defined by (67)

The choice of the current (20) is not unique. A second possibility (of the same dimension) is the one with γ_μ replaced by $\sigma_{\mu\nu}$. However, it has been argued by Ioffe (Ref. [19]) that chiral symmetry breaking terms are strongly suppressed in the polarization operator of this current and consequently the resonance should not couple strongly to this current.

On the phenomenological side we take only the nucleon state into account with coupling λ_N to the current, therefore

$$\Pi(q) = \lambda_N^2 \frac{\gamma q + M_N}{q^2 - M_N^2} + \text{continuum}, \tag{68}$$

where the coefficient of γq gives $\Pi_2(q^2)$ and the M_N piece $\Pi_1(q^2)$. Collecting the various contributions (see Section 5) to the two invariant functions and performing the Borel transform we arrive at the two sum rules:

for $\Pi_1(q^2)$

$$2aM^4 = 2(2\pi)^4\lambda_N^2 M_N e^{-M_N^2/M^2} \tag{69a}$$

and for $\Pi_2(q^2)$

$$M^6 + bM^2 + \tfrac{4}{3}a^2 = 2(2\pi)^4\lambda_N^2 e^{-M_N^2/M^2}, \tag{69b}$$

where

$$a = -(2\pi)^2\langle 0|\bar{q}q|0\rangle \simeq 0.6$$

and

$$b = \pi^2\langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle \simeq 0.13.$$

Taking the ratio of these two equations (including a continuum contribution to the leading terms of (69a) and (69b)) we get

$$M_N(M^2) = \frac{2aM^4\left[1-\exp(-s_0/M^2)\left(\frac{s_0}{M^2}+1\right)\right]}{M^6\left[1-\exp(-s_0/M^2)\left(\frac{s_0^2}{2M^2}+\frac{s_0}{M^2}+1\right)\right]+\frac{4}{3}a^2+bM^2}. \tag{70}$$

In first approximation (neglecting the continuum and the power corrections in (69b)) we get with $M^2 = M_N^2$

$$M_N = \{-2(2\pi)^2\langle 0|\bar{q}q|0\rangle\}^{1/3} \simeq 1 \text{ GeV}, \tag{71}$$

which for a first approximation is amazingly good. This result can be improved by considering the full expression (70). Recently (Ref. [23]) the contributions of higher dimensional operators than the ones considered above have also been calculated. They turn out to be non-negligible in some cases. I feel however that progress in describing the fine structure in the octet and decouplet will be limited as long as the first order α_s corrections to the bare loop and the quark condensate piece have not been calculated.

11. Three-point functions

In this Section, I will discuss an extension of the sum rule formalism to three-point functions. Several applications have so far been made (Refs. [24–27]), but I will consider only some of the results for coupling constants of hadrons to Goldstone bosons (Ref. [24]), in particular the pion-nucleon coupling constant $g_{\pi NN}$ and $g_{\omega\varrho\pi}$.

Let us consider the three-point function of two baryon currents and one meson current

$$A(p, p', q) = \int d^4x \int d^4y e^{ip'x-iqy}\langle 0|T(\eta_B(x)J_m(y)\bar{\eta}_B(0))|0\rangle, \tag{72}$$

where $p' = p+q$. We are interested in the couplings of pions to mesons and baryons, i.e., for the meson current $J_m(x)$ in (72) we choose the pseudoscalar current (for π^0)

$$J_P(x) = \bar{u}(x)i\gamma_5 u(x) - \bar{d}(x)i\gamma_5 d(x). \tag{73}$$

We also assume in analogy to the two-point function case discussed in detail before that we can apply the operator product expansion to the T-ordered product of currents in (72)

$$\int d^4x \int d^4y e^{ip'x - iqy} \mathrm{T}(\eta_B(x) J_m(y) \bar{\eta}_B(0)) = \sum_k C_k(p, p', q) O_k, \tag{74}$$

where the $C_k(p, p', q)$ fall off by powers of p^2, q^2, and p'^2 according to the dimension of O_k.

Phenomenologically the baryon-pion coupling can be represented by the diagram in Fig. 14, λ_B is the coupling of the baryon to its current and g_P is the coupling of the meson to its current. For the neutral pion we have

$$g_P = \frac{f_\pi}{\sqrt{2}} \cdot \frac{m_\pi^2}{m_q} \tag{75}$$

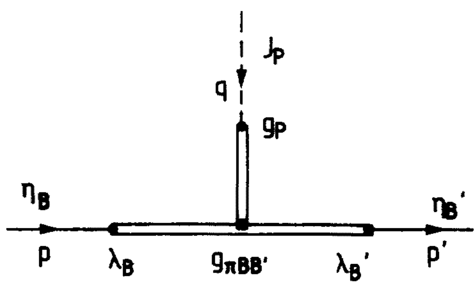


Fig. 14. Diagrammatic representation of the phenomenological side of the sum rule for vertex functions; η_B and $\eta_{B'}$ are baryon currents, J_P is the pseudoscalar meson current. λ_B and $\lambda_{B'}$ are the couplings of the lowest-lying baryons to the currents, g_P is the coupling of the pseudoscalar meson to the current, and $g_{\pi BB'}$ is the three-point coupling

with $f_\pi = 133$ MeV the pion decay constant and m_q the mass of the quark. We will assume that each channel is saturated by a single resonance (in principle of course contributions of higher resonances can be taken into account analogous to the two-point function case). For the transition $B \to \pi B$ with B a $J = 1/2$ baryon we then get for the phenomenological side of $A(p, p', q)$ (with $p^2 = p'^2$):

$$A(p, p', q) = \lambda_B^2 \frac{M_B}{(p^2 - M_B^2)^2} (\gamma q i\gamma_5) g_{\pi BB} \frac{1}{q^2 - m_\pi^2} \frac{f_\pi}{\sqrt{2}} \frac{m_\pi^2}{m_q} \tag{76}$$

which results from the effective Lagrangian $L = g\bar{B}i\gamma_5\tau \cdot \pi B$ and is just the product of the two fermion propagators, the pion propagator and the various couplings.

The first important observation is that (neglecting the pion mass) there will always be a $1/q^2$ pole on the phenomenological side. To get rid of all possible subtraction constants we take the Borel transform with respect to $Q^2 = -p^2$. There is a slight problem in this procedure since by taking $p^2 = p'^2$ there could in principle be subtraction terms which do not vanish under simultaneous borelization. However, they are not present in the cases we are considering (see also Refs. [14] and [28] for a discussion of this procedure). We will give an alternative derivation of $g_{\pi NN}$ using two-point functions wnich leads to an equivalent sum rule showing that the three-point function method is correct. The Borel transformed expression (76) gives

$$\lambda_B^2 \frac{e^{-M_B^2/M^2}}{M^4} M_B(\gamma q i\gamma_5) g_{\pi BB} \frac{1}{q^2 - m_\pi^2} \frac{f_\pi}{\sqrt{2}} \frac{m_\pi^2}{m_q}, \tag{77}$$

where M^2 is the mass scale connected with Q^2 via the Borel transformation.

To determine the pion-baryon coupling constant theoretically we will identify the leading terms in the operator product expansion (74) which have a $1/q^2$ term with Eq. (76) (and with (77) after borelization), i.e., we will determine the Wilson coefficients which have a $1/q^2$ term for the lowest dimensional operators in (7).

Counting dimensions it is easily verified that $A(p, p', q)$ has an even number of dimensions; therefore, taking into account the γq factor the Wilson coefficients of all even dimensional operators will be proportional to the small quark mass m_q, while the operator $\bar{q}q$ does not have this m_q and its contribution will be greatly enhanced compared to the other operators. This implies that up to dimension four we only have to take quark condensate contributions into account and that we can neglect all perturbative and gluon condensate contributions. Therefore,

$$A(p, p', q) = C_u(p, p', q) \langle 0|\bar{u}u|0\rangle + C_d(p, p', q) \langle 0|\bar{d}d|0\rangle. \tag{78}$$

The diagrams which we actually have to calculate to obtain C_u and C_d have one quark line cut and are given in Fig. 15. It can easily be seen that only the diagrams of Figs. 15a and b have a $1/q^2$ term.

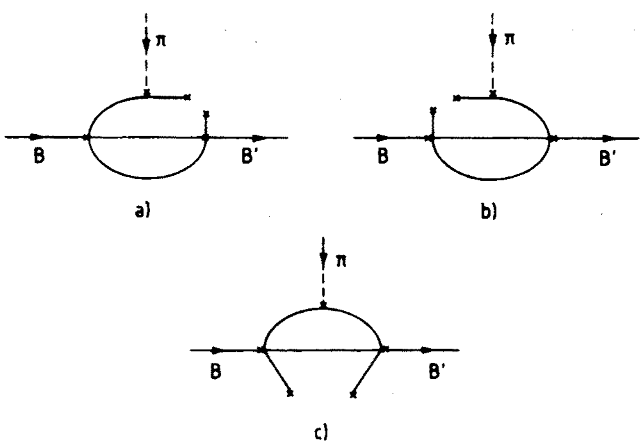


Fig. 15. Diagrams which contribute to the Wilson coefficient of the operator $\bar{q}q$. Only diagrams (a) and (b) have a $1/q^2$ term with q the pion momentum

Let us now apply this to the calculation of the pion-nucleon coupling constant $g_{\pi NN}$, i.e., in (72) we have the current (20) for η_B and the π^0 current (73) for J_m. The calculation of the diagrams is straightforward and gives the results

$$C_u = -\frac{1}{(2\pi)^2}\frac{10}{3}\frac{\gamma q}{q^2}(i\gamma_5)p^2\ln\frac{\Lambda^2}{-p^2}, \tag{79a}$$

$$C_d = -\frac{1}{(2\pi)^2}\frac{2}{3}\frac{\gamma q}{q^2}(i\gamma_5)p^2\ln\frac{\Lambda^2}{-p^2}. \tag{79b}$$

Here we have already taken $p'^2 = p^2$ and only written down the $1/q^2$ terms.

Substituting (79) into (78), taking $\langle\bar{u}u\rangle \equiv \langle\bar{d}d\rangle \equiv \langle\bar{q}q\rangle$, performing the Borel transform, and identifying with the phenomenological side (77) we get

$$\lambda_{\mathrm{N}}^2\frac{e^{-M^2{}_{\mathrm{N}}/M^2}}{M^4}M_{\mathrm{N}}g_{\pi\mathrm{NN}}\frac{f_\pi}{\sqrt{2}}m_\pi^2 = \frac{1}{(2\pi)^2}M^2(-4m_{\mathrm{q}}\langle 0|\bar{q}q|0\rangle). \tag{80}$$

Now we use the PCAC relation (42) and the two-point function result for the nucleon current Eq. (69b) to eliminate λ_{N}^2. We get

$$g_{\pi\mathrm{NN}} = 2(2\pi)^2\sqrt{2}\frac{f_\pi}{M_{\mathrm{N}}}\left(1+\frac{4}{3}\frac{a^2}{M^2}+\frac{b}{M^4}\right)^{-1}. \tag{81}$$

A remarkable feature of this relation is that it is almost independent of M^2. In first approximation, i.e., without the power corrections $g_{\pi\mathrm{NN}} \simeq 16$. Taking $b \simeq 0.13$, $a \simeq 0.6$ and $M^2 \simeq M_{\mathrm{N}}^2$ we obtain the prediction

$$g_{\pi\mathrm{NN}} \simeq 12.5 \tag{82}$$

with an estimated error of about 20%, in very good agreement with the experimental value $g_{\pi\mathrm{NN}}^{\mathrm{exp}} \simeq 13.5$.

We could also use relation (69a) to eliminate λ_{N}^2 which gives an estimate of the quark mass

$$m_{\mathrm{q}}(M^2) = \frac{f_\pi m_\pi^2 g_{\pi\mathrm{NN}}}{4M^2\sqrt{2}} = \frac{(2\pi)^2 f_\pi^2 m_\pi^2}{2M^2 M_{\mathrm{N}}} \simeq (6-8)\ \mathrm{MeV}. \tag{83}$$

We also note that relation (81) has a completely different structure than the Goldberger--Treiman relation

$$g_{\pi\mathrm{NN}} = \frac{\sqrt{2}\,M_{\mathrm{N}}}{f_\pi}. \tag{84}$$

Let us now give an alternative derivation of $g_{\pi\mathrm{NN}}$ using two-point functions. For this we consider the quark condensate terms in the OPE of the T-ordered product of two nucleon currents

$$i\int d^4x e^{ipx}\mathrm{T}(\eta_{\mathrm{N}}(x)\bar{\eta}_{\mathrm{N}}(0)) = \ldots + \sum_{q,\Gamma} C_q^\Gamma(p)\bar{q}\Gamma q + \ldots \tag{85}$$

We find

$$\sum_{\Gamma} C_d^{\Gamma}(p)\bar{d}\Gamma d = -\frac{1}{(2\pi)^2} p^2 \ln\frac{\Lambda^2}{-p^2}\Bigg[\bar{d}d + i\gamma_5\bar{d}i\gamma_5 d$$

$$+\frac{1}{3p^2}(\gamma p\gamma^{\tau}\gamma p + 2p^2\gamma^{\tau})\{\bar{d}\gamma_{\tau}d + \gamma_5\bar{d}\gamma_{\tau}\gamma_5 d\} + \frac{1}{6p^2}\gamma p\sigma^{\alpha\beta}\gamma p\bar{d}\sigma_{\alpha\beta}d\Bigg] \tag{86}$$

and a similar expression for $\sum_{\Gamma} C_u^{\Gamma}(p)\bar{u}\Gamma u$ which, however, does not contain a $\bar{u}\gamma_5 u$ term. Sandwiching between the vacuum and a one pion state we select precisely the $\bar{q}\gamma_5 q$ terms since the pion does not project on any of the other structures.

For the phenomenological side we get (with $\eta_{\rm N}(x) = \lambda_{\rm N}N(x)$ and the effective Lagrangian $L = g_{\pi\rm NN}\bar{N}i\gamma_5(\tau\cdot\pi)N$)

$$\langle 0|i\int d^4x e^{ipx}\mathrm{T}(\eta_{\rm N}(x)\bar{\eta}_{\rm N}(0))|\pi^0\rangle$$

$$= i\int d^4x e^{ipx} i\int d^4u|\lambda_{\rm N}|^2\langle 0|\mathrm{T}(N(x)L(u)\bar{N}(0))|\pi^0\rangle$$

$$= |\lambda_{\rm N}|^2\frac{(-g_{\pi\rm NN})}{p^2 - M_{\rm N}^2}(i\gamma_5). \tag{87}$$

For the theoretical side we use

$$\tfrac{1}{2}(\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d) = \frac{1}{\sqrt{2}}\frac{f_\pi m_\pi^2}{m_{\rm u}+m_{\rm d}}\pi^0. \tag{88}$$

Borel transforming and equating the two sides we get

$$g_{\pi\rm NN}|\lambda_{\rm N}|^2 e^{-M_{\rm N}^2/M^2} = \frac{1}{(2\pi)^2}M^4\frac{1}{\sqrt{2}}\cdot\frac{f_\pi m_\pi^2}{2m_{\rm q}}. \tag{89}$$

Now we use Eq. (69a) to eliminate $\lambda_{\rm N}^2$ and we obtain the Goldberger-Treiman relation (84).

The most important point of this exercise is that the Wilson coefficient of the $\bar{q}\gamma_5 q$ term in (86) is identical to the one obtained in the three-point function case. This shows that the procedure followed there is correct.

As a final example we will calculate the coupling constant $g_{\omega\varrho\pi}$ using three-point functions. In this case all currents in (72) are mesonic currents, two vector currents and one pseudoscalar current. The phenomenological side now reads

$$A(p, p', q) = \frac{g_{\omega\varrho\pi}\varepsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta}{(q^2 - m_\pi^2)(p^2 - m_\varrho^2)(p'^2 - m_\omega^2)}\cdot\frac{\sqrt{2}f_\pi m_\pi^2}{2m_{\rm q}}\cdot\frac{m_\omega^2 m_\varrho^2}{f_\omega f_\varrho}. \tag{90}$$

Counting dimensions one can easily verify that also in this case the quark condensate contributions dominate. The possible diagrams are the same as in Fig. 15 with one quark line less. The calculation of these diagrams is extremely simple since they do not contain any loops. Selecting only the $1/q^2$ pieces, the result for the invariant amplitude is

$$\frac{\langle 0|\bar{q}q|0\rangle}{q^2}\left(\frac{1}{p^2} + \frac{1}{p'^2}\right). \tag{91}$$

Identifying this with the $1/q^2$ terms in (90), taking $p^2 = p'^2 = -Q^2$ and applying the Borel transform with respect to Q^2 as before we get (with $m_\omega^2 = m_\varrho^2, f_\omega = f_\varrho$)

$$g_{\omega\varrho\pi} \frac{e^{-m_\varrho^2/M^2}}{M^4} \frac{\sqrt{2} f_\pi m_\pi^2}{2m_q} \frac{m_\varrho^4}{f_\varrho^2} = -2 \frac{\langle 0|\bar{q}q|0\rangle}{M^2} . \tag{92}$$

To eliminate f_ϱ we use the two-point function result (53):

$$\frac{12\pi^2 m_\varrho^2}{f_\varrho^2} e^{-m_\varrho^2/M^2} = \tfrac{3}{2} M^2 \left[1 + \frac{\alpha_s}{\pi} + \text{higher corrections}\right]. \tag{93}$$

For explicit expressions of the higher corrections see Section 9. At $M^2 \simeq m_\varrho^2$ the sum of all corrections amounts to about 10%. Substituting (93) into (92), using PCAC, and rearranging the terms we get

$$g_{\omega\varrho\pi} = \sqrt{2}\,(2\pi)^2 \frac{f_\pi}{m_\varrho^2} \simeq 13\ \text{GeV}^{-1}, \tag{94}$$

which is very close to the current algebra result

$$g_{\omega\varrho\pi} = \frac{2}{f_\pi} = 15\ \text{GeV}^{-1}.$$

Again we note that our result (94) is independent of M^2.

This independence of M^2 resulting in extremely simple formulae is the great advantage of our method compared to other three-point function calculations. The calculation of Ref. [27], for instance, results in an extremely complicated expression for $g_{\omega\varrho\pi}$ which is not only a function of M^2 but also of $Q^2 = -q^2$. Both variables can vary in a certain range controlled by the magnitude of the power corrections, resulting in a range of values for $g_{\omega\varrho\pi}$.

12. *Conclusion*

I have given an introduction to the basic ideas underlying the QCD sum rules invented by Shifman, Vainshtein and Zakharov (Ref. [1]), and discussed some of the applications made in recent years ranging from heavy quark two-point functions to light quark meson-baryon couplings.

It is important to note that different nonperturbative operators control the various quark systems. The charmonium and upsilon systems have only gluon condensate contributions, and from this we obtain an accurate determination of the fundamental gluon condensate parameter ϕ (for a determination of $\langle \alpha_s/\pi G^2\rangle$ involving only light quarks see Ref. [14]). In the open charm and beauty systems (not discussed here but see Refs. [29–31]) the chiral symmetry breaking starts to play a dominant role, which is also the case in light quark baryons (Section 10) and in couplings of hadrons to Goldstone bosons (Section 11), while for light quark mesons (Section 9) the quark condensate terms are small since they are multiplied by a very small mass.

Although some states are difficult to calculate due to technical problems an impressive body of results has been accumulated, which shows that it is possible to calculate resonance masses and couplings to a high degree of accuracy in QCD in terms of the Lagrangian parameters and a number of vacuum expectation matrix elements which parametrize nonperturbative effects due to the complicated vacuum structure of QCD. The method has its limitations, of course, in particular it does not appear to be possible to determine the properties of radial excitations. In fact by using moments or Borel transformed sum rules we ensure dominance of the lowest lying resonance. It is never possible to eliminate completely the contribution from higher resonances and/or the continuum. This introduces errors which are typically $\sim 10\%$ in the determination of masses (with the exception of charmonium where much greater accuracy is obtained) and $\sim 20\%$ in the coupling constants which we have discussed.

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