NON-PERTURBATIVE STUDIES IN THE CP^{n-1} MODEL*, **

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Two topics are discussed involving non-perturbative aspects of the quantum CP^{n-1} model in two dimensions. The first lecture deals with defining the topological charge and topological susceptibility through the lattice theory and the problems associated therewith. The second lecture discusses how to estimate the mass gap of the theory through a finite volume approach. The two lectures are tied together by considerations of universality.

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1. *Topology, the lattice and universality*

The CP^{n-1} model in two dimensions [1] is of interest because of the properties that it shares with four dimensional Yang-Mills gauge theory. In particular the model is asymptotically free and, because it also possesses gauge symmetry, the model has a non-trivial topological structure on the classical level. In these lectures I would like to discuss questions related to these two properties in the CP^{n-1} model which may help shed light also on the Yang-Mills theory. In particular, the first lecture will address the question of whether non-trivial topological effects can be shown to exist in the *quantized* CP^{n-1} model. In this lecture I will discuss the definition of topological charge on the lattice [9] as a way through which to establish that indeed non-trivial topological effects occur. In the second lecture I will illustrate a recently proposed method for calculating the low-lying mass spectrum of field theories which are asymptotically free [2]. In this method the property of asymptotic freedom is exploited to define the theory in a finite volume. Finally both lectures will be tied together by considerations of universality, that is, whether or not one finds the same answer for physical quantities when they are calculated through different lattice actions as one would find for the continuum theory.

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The work presented in the first lecture was done in collaboration with M. Lüscher [3] and that of the second with E. Floratos [4].

The fact that the clasical field space of the two-dimensional CP^{n-1} model, like that of the four-dimensional Yang-Mills theory, can be classified into different topological sectors according to the topological charge of each field is well known [5]. What is not so well understood is the role this topological structure plays on the quantum level. On the one hand, typical fields that contribute to the Feynman path integral are discontinuous [6], apparently indicating that topology might play no role whatsoever. On the other hand arguments have been made that topological effects can be seen in physical quantities in a non-trivial way [7]. Therefore, the question of whether one can define quantities associated with the topological charge in a precise way is a pertinent one. Here I address this problem through the lattice approach to the quantized CP^{n-1} model. First I will review the model in the continuum and define the topological charge for that model. Then I will give a heuristic approach as to how one might define the topological charge in the lattice theory and indicate what difficulties may arise. Next I will outline how these difficulties can be circumvented in order to insure that in the continuum limit of the *quantized* theory, non-trivial topological sectors survive. Then, as a check of whether these sectors occur with the correct probabilities, I'll present Monte Carlo calculations of the topological susceptibility for the CP^2 model to show that indeed the expected renormalization group behavior is obeyed. Finally I'll discuss the universality of the results obtained, a discussion which also entails Monte Carlo calculations of the correlation length of the system.

The CP^{n-1} model [1] is a generalization of the non-linear sigma model which has the important difference of admitting field configurations with non-trivial topological charge [5]. These non-trivial topological configurations arise because of an underlying gauge symmetry. The model is defined through the action

$$S = \frac{n}{2f_0}\int d^2x\overline{D_\mu Z}\cdot D_\mu Z, \tag{1}$$

where $Z \in \mathbf{C}^n$ with $|Z|^2 = 1$, $D_\mu = \partial_\mu - \bar{Z}\cdot\partial_\mu Z$ and f_0 is the bare coupling constant. This action is invariant under the local gauge transformation $Z(x) \to e^{i\Lambda(x)}Z(x)$, $\Lambda(x) \in [0, 2\pi]$ and therefore the physical degrees of freedom are the gauge equivalence classes

$$[Z(x)] = \{e^{i\Lambda(x)}Z(x)|\Lambda(x) \in [0, 2\pi], \forall\, x\}, \tag{2}$$

which are fields taking their values in the complex projective space CP^{n-1}. One can define a field

$$A_\mu = \bar{Z}\cdot\partial_\mu Z, \tag{3}$$

which transforms as an abelian gauge field under the above transformation:

$$A_\mu(x) \to A_\mu(x) + i\partial_\mu\Lambda(x). \tag{4}$$

The topological charge density can be defined in terms of this field

$$q(x) = \frac{1}{2\pi}\varepsilon_{\mu\nu}\partial_\mu A_\nu \tag{5}$$

giving the topological charge

$$Q = \int d^2x q(x). \tag{6}$$

The topological charge takes on an integer value for any given field configuration and its value is stable under smooth deformations of the field [5].

A useful formulation of the CP^{n-1} model which is manifestly gauge invariant is to express all fields in terms of the complex projection matrices $P \in CP^{n-1}$. These matrices are related to the Z fields by

$$P(x) = Z(x) \otimes \overline{Z(x)} \tag{7}$$

and have the properties

$$P^+ = P, \quad P^2 = P, \quad \mathrm{Tr}\, P = 1. \tag{8}$$

The action in terms of these fields becomes

$$S = \frac{n}{4f_0} \int d^2x \,\mathrm{Tr}\,[\partial_\mu P(x) \partial_\mu P(x)] \tag{9}$$

and the topological charge density is

$$q(x) = \frac{i}{2\pi} \varepsilon_{\mu\nu} \,\mathrm{Tr}\,[P(x) \partial_\mu P(x) \partial_\nu P(x)] \tag{10}$$

with $Q = \int d^2x q(x)$.

In order to get a better feel for the topological charge of a field, let us consider the simplest CP^{n-1} model, the case of $n = 2$. This model can be shown to be equivalent to the O(3) non-linear sigma model by the identification of the two-dimensional complex projection matrix fields $P(x)$ with the three component spin fields $\vec{S}$, $|\vec{S}|^2 = 1$ of the O(3) model through

$$P(x) = \tfrac{1}{2}(1 - \vec{S}(x) \cdot \vec{\sigma}) \tag{11}$$

where σ_1, σ_2 and σ_3 are the Pauli spin matrices. Then the action becomes the normal one for the O(3) model and the topological charge density becomes

$$q(x) = \frac{1}{4\pi} \varepsilon_{\mu\nu} \vec{S}(x) \cdot (\partial_\mu \vec{S}(x) \times \partial_\nu \vec{S}(x)). \tag{12}$$

The topological properties of the spin fields are easily seen if we map space time onto a two-dimensional sphere such that all points $|x| = \infty$ are identified. Then since each spin field takes its value also in the two-dimensional sphere S^2, we see that each field is now a mapping from S^2 into S^2. Because the topological density (12) is just the jacobian of the transformation from a two-dimensional flat surface to a two-sphere, the topological charge is just the number of times the sphere is covered in spin space as one integrates over space-time. Because the unit sphere has area 4π, the normalization factor $(4\pi)^{-1}$ in (12) serves to make Q always equal to an integer.

Now let us look at a couple of examples. The first one is a case where $Q = 0$. This is just the trivial map $\vec{S}(x) = (1, 0, 0)$ for all x, and is pictorially represented by Fig. 1a. In this case no area is swept out in spin space as the space-time sphere is integrated over. The second example, for which $Q = -1$, is the identity map followed by a reflection. This map is illustrated in Fig. 1b. Clearly the sphere is covered once in spin space as the integration is done. This example also points out the importance of the orientation of the area swept out. The field in Fig. 1b has charge $Q = -1$ because of the reflection; the sphere in spin space is covered backwards relative to the space-time integration. But the identity map itself has $Q = +1$. Clearly these illustrations can be generalized so that Q takes on any positive or negative integer value. They also generalize to a host of other cases of boundary conditions.

Now of what interest are these classical configurations to the quantum theory? The answer is not fully known. Arguments have been made however that relate the mass of the

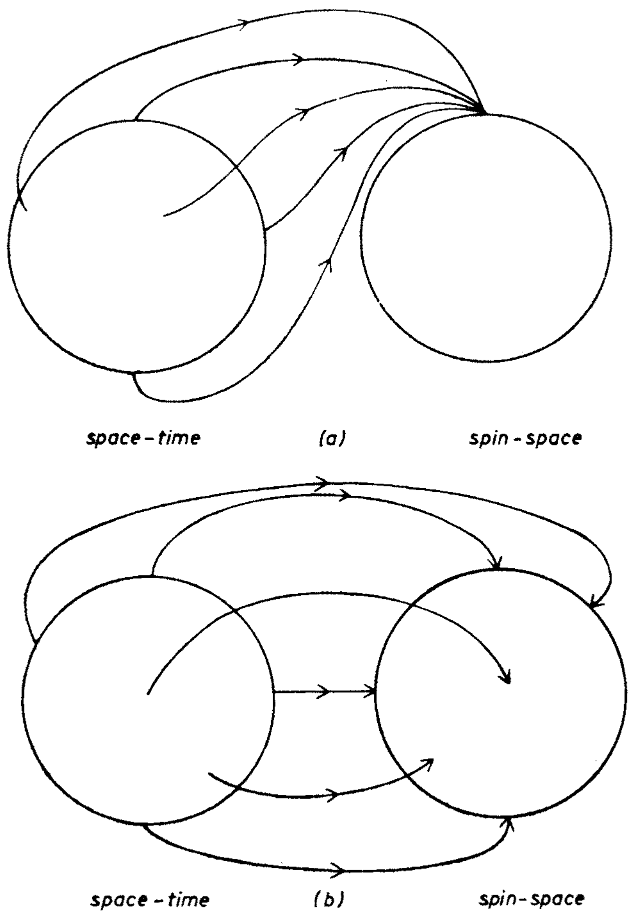

Fig. 1. Examples of mappings from a two dimensional space-time sphere into the field space of the O(3) σ-model (also a sphere) with (a) topological charge $Q = 0$ (the trivial mapping) and (b) topological charge $Q = -1$ (the identity mapping followed by a reflection)

η' meson to the topological susceptibility χ_t in the Yang-Mills theory [7] where χ_t is defined by

$$\chi_t = \int d^4x \langle q(x)q(0) \rangle \tag{13}$$

in which $q(x)$ is the topological charge density in that theory and $\langle \ldots \rangle$ means quantum mechanical averaging. Also the topological charge is argued to play a role in chiral symmetry breaking (for a review see Ref. [8]).

We can define an analogous quantity to (13) for the CP^{n-1} model:

$$\chi_t = \int d^2x \langle q(x)q(0) \rangle \tag{14}$$

with $q(x)$ given in (5). The problem is that in both cases (13) and (14), χ_t has never been given an adequate definition outside of an approximation scheme. Consider the case for the CP^{n-1} model. In perturbation theory χ_t is easily shown to be zero, by considering that it is the zero momentum projection of $\langle q(x)q(0) \rangle$ which is of dimension four and therefore vanishes with momentum (up to infrared divergences which are usually only logarithmic). For a more rigorous version of this argument see Ref. [9]. Alternatively one can consider small deformations of fields around the perturbative vacuum along with the topological properties of Q. On the other hand, for short distances $\langle q(x)q(0) \rangle \sim \dfrac{1}{|x|^4}$ and is therefore not integrable.

A way to get around these problems is to define χ_t as the continuum limit of a suitable quantity defined in the lattice CP^{n-1} model. Then the problems mentioned above would be automatically solved. Thus let us take the point of view that the quantity χ_t is *defined* through the continuum limit of a lattice theory. If we start with a lattice theory in a *finite* physical volume and also have a suitable definition of Q for lattice fields in this finite volume theory, then we can define

$$\chi_t^V = \langle Q^2 \rangle / V, \quad (V\text{: volume}) \tag{15}$$

and

$$\chi_t = \lim_{V \to \infty} \chi_t^V. \tag{16}$$

So the question now becomes how can we find a suitable definition of Q in a lattice theory which is topological in nature, vanishes to all orders of perturbation theory, and assigns a unique integer to any given lattice field.

To answer this question let us go back to our picture of the topological charge for the O(3) sigma model. In that case we saw that the topological charge is just a measure of the number of times the area of the sphere in spin space is swept out as one sweeps out the space-time volume. A local picture of this can be formulated. Suppose we pick three points in space time in a neighborhood of a particular point and connect these three points by geodesics to form a geodesic triangle $\triangle$. Then the contribution to the topological charge from this triangle would be the area in spin space swept out as one integrates over $\triangle$. Of course this area may not be a geodesic triangle in spin space but would be an area bounded by arbitrary curves joining the spin values from the three corners of the space-

-time triangle (see Fig. 2). Recall also that the orientation of the area is important. Thus if we cover space-time with a mesh of triangles and define the contribution of each triangle to the topological charge as

$$Q(\Delta) = \int_{\Delta} q(x) d^2x, \tag{17}$$

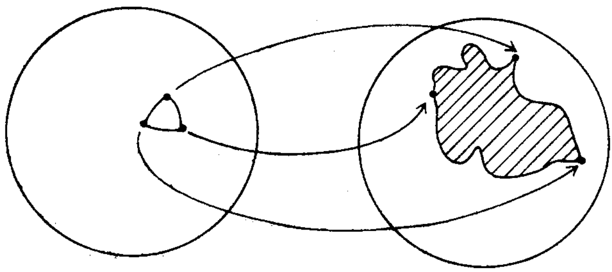

Fig. 2. Mapping of an arbitrary triangle in space-time into the spin manifold of the O(3) σ-model

then the topological charge is given by

$$Q = \sum_{\triangle} Q(\triangle). \tag{18}$$

Now a natural definition of Q on the lattice becomes apparent. If, for simplicity, we define our theory on a triangular lattice, the field takes values on the corners of the triangles of the lattice. The natural definition for $Q(\triangle)$ for any given triangular plaquette of the lattice is to connect the three values of fields at the corners of the triangle by *geodesics* in spin space, and define for the lattice theory [9]

$$Q(\triangle) = \frac{1}{4\pi} \sigma(\triangle) A(\triangle), \tag{19}$$

where $A(\triangle)$ is the area of the triangle in spin space after the three spin values are connected with geodesics and $\sigma(\triangle)$ is ± 1 depending on the orientation of the triangle (see Fig. 3). The choice of connecting the spin values on lattice sites with geodesics is of course arbitrary, but is most natural. Furthermore, the definition obviously goes to the correct classical continuum limit for smooth fields as it is topological in nature, and it is well defined and integer valued for all lattice fields except for a set of measure zero in the functional integral of the quantized theory. These fields, called *exceptional* fields are characterized in the O(3) model by fields containing at least one plaquette which takes its three spin values on a great circle in spin space. Thus $A(\triangle)$ for this plaquette is either 0 or 2π and the orientation is undefined. However if one moves one spin value slightly off the great circle $Q(\triangle)$ again becomes defined with the orientation depending on which direction the spin value is moved. Thus these configurations lie on the boundaries of topological charge sectors, as a smooth deformation of a spin which causes the field to go through an exceptional configuration changes the charge by one unit. This situation is in contrast to the classical continuum theory where the topological sectors are without boundary and no smooth deformation will bring a field from one sector into another. At first site one might think that exceptional configurations would present no problem because they form a set of measure zero in the functional integral of the quantized theory. However, the possibility still exists that configurations in a neighborhood around exceptional configurations can become too probable as the continuum limit is taken, effectively washing out the boundaries between topological

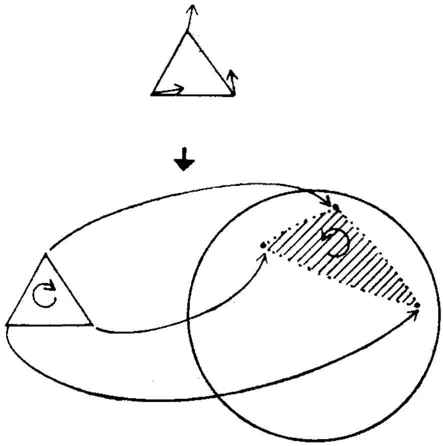

Fig. 3. Mapping of three lattice points in spacetime to three points in spin-space. The points in spin space are then connected with geodesic interpolating fields (...)

sectors in the quantum theory. As this phenomena may cause the expected renormalization group behavior of χ_t to be violated this situation must be treated with care. This will be discussed in some detail in what follows.

The definition of topological charge for the lattice theory outlined above can be easily extended to the general CP^{n-1} model by using geodesics in the CP^{n-1} manifold for interpolating fields between sites. The definition will be given below.

Now let us go back and define the lattice CP^{n-1} model. We start with a triangular two-dimensional lattice with lattice sites $x = n_1 e_1 + n_2 e_2$ where $e_1 = (1, 0)a$ and $e_2 = \frac{1}{2}(1, \sqrt{3})a$, a is the lattice spacing, and $n_1 = 1, \ldots, l_1$, $n_2 = 1, \ldots, l_2$. Thus the space time volume of the lattice is given by $V = \frac{\sqrt{3}}{2} L_1 L_2$ where $L_1 = l_1 a$ and $L_2 = l_2 a$. Then let

$$h(\triangle) = 3 - \mathrm{Tr}\,[P_1 P_2] - \mathrm{Tr}\,[P_2 P_3] - \mathrm{Tr}\,[P_3 P_1], \tag{20}$$

where P_1, P_2 and P_3 are the projection matrices in the CP^{n-1} manifold at the three corners of the plaquette $\triangle$. We define the theory through the lattice action [3]

$$S = \frac{\beta}{\sqrt{3}} \sum_{\triangle} [h(\triangle) + \gamma h(\triangle)^2], \tag{21}$$

where $\beta = \frac{n}{4f_0}$ and γ is a parameter yet to be determined. In the classical continuum limit the first term goes to the classical action and the second term vanishes. Let the boundary conditions be periodic. Finally $Q(\triangle)$ is defined through [9]

$$e^{2\pi i Q(\triangle)} = \frac{\mathrm{Tr}\,[P_3P_2P_1]}{|\mathrm{Tr}\,[P_3P_2P_1]|}, \qquad Q(\triangle) \in (-\tfrac{1}{2}, \tfrac{1}{2}) \tag{22}$$

and

$$Q = \sum_{\triangle} Q(\triangle). \tag{23}$$

Note that $Q(\triangle)$ is undefined if Tr $[P_3P_2P_1]$ is real and negative or zero. Fields with a plaquette where $Q(\triangle)$ is undefined are exceptional fields. By way of review, this definition of Q has the properties:

1) Q is defined and is an integer for (almost) every lattice configuration (not for exceptional configurations).

2) Q is topological in nature. That is, Q does not change under smooth deformations of the field as long as one does not go through an exceptional configuration.

3) Q is the sum of a local density $Q(\triangle)$.

4) In the classical continum limit $Q(\triangle)$ approaches the continuum topological charge density $q(x)$ given by (10).

Also note that property 2) leads to the vanishing of χ_t to all orders of perturbation theory.

Now the rest of the program will be as follows. We first examine what the probability of near exceptional configurations is. When we compare this with what the renormalization group predicts about the scaling behavior of χ_t as the continuum limit is taken, we find that if χ_t has a chance of scaling according to the renormalization group then exceptional configurations (and a neighborhood around them) occur with zero probability in the quantum continuum limit. That is to say that walls of probability are built up between the topological sectors dynamically as the continuum limit is taken. Thus the occurrence of non-trivial topological sectors is implied. As we shall see the renormalization group behavior enforces a condition on what value the action may take in the $Q = 1$ sector and therefore a condition on γ in (21). Another way to say this is that in order for χ_t to (possibly) scale according to the renormalization group if χ_t is given by our definition above, the action must be suitably ferromagnetic which means that a certain notion of continuity of the fields at a scale *of the order of a lattice spacing* must be enforced. Let us see how this works. Before engaging in the above program let us argue the following. If $\beta\varepsilon$ is defined to be the lowest value the action can take in the $Q = 1$ sector for an infinite volume theory then the inequality [10]

$$\lim_{\beta\to\infty} \left[-\frac{1}{\beta} \ln (a^2\chi_t) \right] \leqslant \varepsilon \tag{24}$$

is obeyed. Although this inequality is not yet rigorous, it can be made plausible as follows (see Ref. [10]). First we consider a lattice of size l^2. Then define χ_t on this lattice. If l is small enough χ_t^{l2} will behave according to the semiclassical approximation $\chi_t^{l2}(\beta) \sim e^{-\beta\varepsilon(l)}$. Then one can argue that for $L \gg l$, $\chi_t^{L2} \gg \chi_t^{l2}$ so finally in the limit, $\chi_t \gg \chi_t^{l2}$ and the inequality

$$\lim\left[-\frac{1}{\beta}\ln(a^2\chi_t)\right] \leqslant \varepsilon(l) \tag{25}$$

follows. The best bound is achieved as $l \to \infty$.

The next thing to note is that as $l \to \infty$ we can pick a field arbitrarily close to an instanton solution of the continuum theory so

$$\varepsilon \leqslant 4\pi. \tag{26}$$

Third we examine the predictions of the renormalization group. For this I will give a brief introduction to the use of the renormalization group in this context (see [11] for a review).

In order to introduce the subject let us first consider the continuum CP^{n-1} model in dimensional regularization. If we calculate in d dimensions the coupling constant must have dimension $2-d$ in order to keep the action dimensionless. Thus we may introduce a length scale μ so that our renormalized coupling constant f remains dimensionless. That is, f is defined through

$$f_0 = \mu^{2-d} f Z(f, d), \tag{27}$$

where Z has the form

$$Z(f, d) = 1 + \frac{Z_1(f)}{2-d} + O((2-d)^{-2}). \tag{28}$$

Clearly since μ is arbitrary, a change in μ can be compensated by a change in f to represent the same physics. This is expressed by the equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(f)\frac{\partial}{\partial f}\right](\text{Physical Quantity}) = 0, \tag{29}$$

where

$$\beta(f) = \lim_{d\to 2} \mu \left.\frac{\partial f}{\partial \mu}\right|_{f_0, d} = f^2 \frac{\partial}{\partial f} Z_1(f). \tag{30}$$

Since there is only one scale in the theory in terms of which physical quantities may be defined, we may replace the scale μ by a scale Λ_{MS} defined to satisfy (29) [12]

$$\Lambda_{MS} = \mu(b_0 f)^{-b_1/b_0^2} e^{-1/b_0 f} \lambda(f), \tag{31}$$

where

$$\lambda(f) = 1 + O(f) \tag{32}$$

and b_0 and b_1 are the coefficients of the β-function

$$\beta(f) = f^2[-b_0 - b_1 f - \ldots]. \tag{33}$$

For the CP^{n-1} model $b_0 = \frac{1}{\pi}$ and $b_1 = \frac{2}{n\pi^2}$ [13]. Now that we have a scale that automatically satisfies the renormalization group equation (29), any physical quantity may be written as a constant times this scale. For example the topological susceptibility is

$$\chi_t = C_\chi \Lambda_{\overline{MS}}^2 \tag{34}$$

and the correlation length is

$$\xi = C_\xi \Lambda_{\overline{MS}}^{-1}, \tag{35}$$

where [14]

$$\Lambda_{\overline{MS}} = e^{1/2(\log 4\pi + \Gamma'(1))} \Lambda_{MS}. \tag{36}$$

In perturbation theory the function $\lambda(f)$ may also be expressed in terms of the β-function:

$$\lambda(f) = e^{-\int_0^f dx \left[\beta(x)^{-1} + \frac{1}{b_0 x^2} - \frac{b_1}{b_0{}^2 x}\right]}. \tag{37}$$

We can analyze the situation with a lattice regulator a as well giving the renormalization group equation and the definition for the physical scale from the lattice theory

$$\left[a \frac{\partial}{\partial a} - \beta(f_0) \frac{\partial}{\partial f_0}\right] \Lambda_L = 0 \tag{38}$$

from which follows

$$\Lambda_L = \frac{1}{a} \left(\frac{\pi}{f_0}\right)^{2/n} e^{-\pi/f_0} \lambda_L(f_0) \tag{39}$$

for the CP^{n-1} model where

$$\lambda_L(f_0) = 1 + O(f_0). \tag{40}$$

We may thus also express χ_t and ξ in terms of Λ_L:

$$\chi_t = C'_\chi \Lambda_L^2, \tag{41}$$

$$\xi = C'_\xi \Lambda_L^{-1}. \tag{42}$$

Keep in mind that Λ_L itself is not a lattice quantity but is the physical scale *defined* by the lattice theory in the *continuum limit*. That is, we take the continuum limit by letting $a \to 0$, $f_0 \to 0$ but keeping Λ_L fixed. Thus from (34), (35), (41) and (42)

$$C_\chi \Lambda_{\overline{MS}}^2 = C'_\chi \Lambda_L^2 \tag{43}$$

and

$$C_\xi \Lambda_{\overline{MS}}^{-1} = C'_\xi \Lambda_L^{-1}. \tag{44}$$

One final remark is that $\Lambda_{\overline{MS}}$ and Λ_L must be related simply by a constant factor. This factor can be calculated exactly from a one loop calculation in perturbation theory [15] (for a review see Ref. [16]) and for the case of the action given in (21) can be shown to be [3]

$$\frac{\Lambda_L}{\Lambda_{\overline{MS}}} = \frac{1}{2\sqrt{3}} \exp\left\{\tfrac{1}{2}(\ln \pi + \Gamma'(1)) - \frac{\pi}{\sqrt{3}} + \gamma 2\pi\sqrt{3}\left(2 - \frac{1}{n}\right)\right\}. \tag{45}$$

Now let us return to our problem. From (39), (41) and the definition of β we can easily show

$$\lim_{\beta \to \infty} \left[-\frac{1}{\beta} \ln (a^2 \chi_t)\right] = \frac{8\pi}{n}. \tag{46}$$

Thus *if* χ_t is a physical quantity *then* Eq. (46) holds. Combining this with (24) we have the following result: if χ_t defined as above through our definition of the topological charge Q is to be a physical quantity then we must have an action such that

$$\varepsilon \geqslant \frac{8\pi}{n}. \tag{47}$$

In other words, if for some reason $\varepsilon < \frac{8\pi}{n}$ then the definition of Q we have chosen will not result in the proper renormalization group scaling for χ_t. In this case small scale fluctuations on the order of the scale of a lattice spacing will dominate as the continuum limit is taken. To prevent this from happening we may tune the parameter γ to insure that $\varepsilon > \frac{8\pi}{n}$. This is the enforcement of some notion of continuity on the scale of the lattice spacing I spoke of earlier. Keep in mind that this in no way forces fields to be continuous on physical scales.

Finally we return to the program outlined before. In answer to the question of what probability near exceptional fields have to occur in the *quantum* theory, one can show that if W is this probability then [3]

$$W \leqslant C\beta^p e^{\beta\left(\frac{8\pi}{n} - \varepsilon\right)} \tag{48}$$

where C is a constant and p is some power. That is to say that if $\varepsilon > \frac{8\pi}{n}$ then the probability that nearly exceptional configurations occur goes to zero in the continuum limit and walls of probability are dynamically generated between topological sectors. Thus condition (47) which is necessary if χ_t is to have a chance to scale properly is exactly the one needed to insure the division of field space into topological sectors.

Now let us summarize. We have shown that if the lowest value of the action in the charge $Q = 1$ sector is greater than or equal to $\beta \frac{8\pi}{n}$ then in the quantized continuum limit the field space will be dynamically chopped into definite topological sectors. Further we have made it plausible that if χ_t satisfies the renormalization group equation then the above inequality is implied. On the other hand, the inequality was not shown to be sufficient to make χ_t scale according to the renormalization group. In order to check this, Monte Carlo calculations have been performed [3].

For the Monte Carlo calculations the case $n = 3$ has been studied. The reason is that for the CP^1 or O(3) model ($n = 2$) (26) and (47) imply that ε must equal 4π. In this case the fluctuations on the scale of a lattice spacing will be competitive and power scaling violations are likely to occur. These would be very difficult to see in a Monte Carlo calculation. In the CP^2 model on the other hand ε may range between $8\pi/3$ and 4π. By numerically relaxing the system in the $Q = 1$ sector one can determine that for $\gamma = 0.4$, $\varepsilon \approx 8.7$ and for $\gamma = 0.6$, $\varepsilon \approx 10$ for the action given by (21). These two values of γ have been chosen for Monte Carlo runs.

The Monte Carlo results for χ_t for these two values of γ are shown in Figs. 4 and 5 respectively. As can be seen by the figures if one assumes that λ_L in (40) is equal to 1 the scaling behavior predicted by (39) is rather closely followed. This indicates that not only is the field space chopped into topological sectors in the quantum theory but also that these sectors occur with the right probability as expected if χ_t is to be a physical quantity. In one sense this completes the story of the non-triviality of the topological charge in the quantum theory.

On the other hand if we recall equation (43) we should be able to obtain a prediction for C_χ from our Monte Carlo data. In fact the prediction should be the same no matter which value of C'_χ we use, i.e. for $\gamma = 0.4$ or $\gamma = 0.6$. This is the assumption of universality, that no matter what lattice action we start with, we should obtain the same continuum

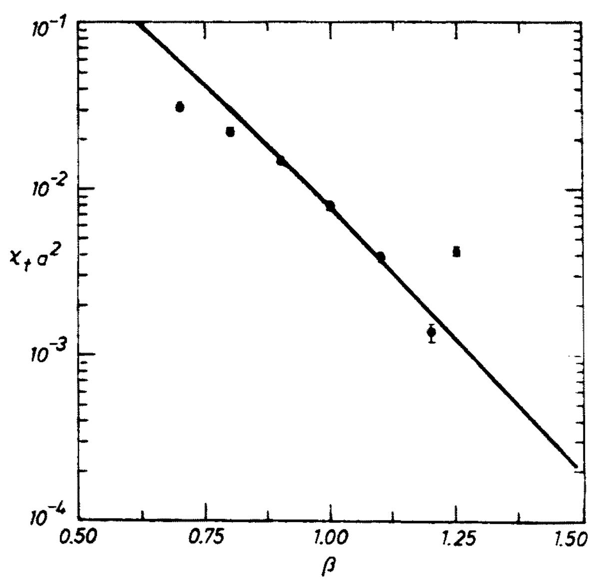

Fig. 4. The topological susceptibility in lattice spacing units versus β for $\gamma = 0.4$. The solid line represents the expected scaling behavior with the overall normalization fit to the data

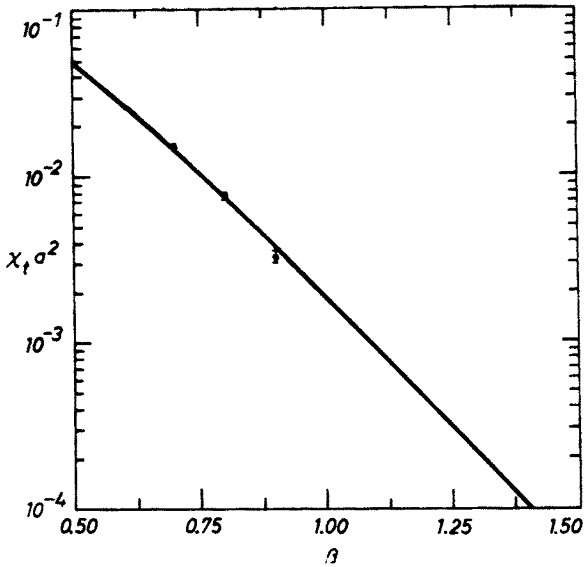

Fig. 5. Same as Figure 4 but for $\gamma = 0.6$

limit. But if we use the numbers we obtain from Monte Carlo along with the Λ-parameter ratio (45) the following values of C_χ are predicted [3].

$$C_\chi = 5.43 \times 10^3 \quad (\gamma = 0.4), \tag{49a}$$

$$C_\chi = 1.88 \times 10^6 \quad (\gamma = 0.6). \tag{49b}$$

Obviously something is wrong. So we must go back and look at our assumptions. One assumption that may be suspect is that in (40) λ_L is set to 1. λ_L is a power series in β^{-1} with the approximation that $\chi_L = 1$ being good if β is very large. On the contrary all the Monte Carlo calculations were performed with $\beta \cong 1$. Thus if λ_L is a slowly varying function, in the small range of β that was studied it is possible that λ_L acts simply like a constant which does not substantially affect the exponential behavior of the scaling but serves to throw off the predictions for C_χ.

In order to explore this conjecture the correlation length as defined through the exponential fall off of the connected invariant two-point function has also been measured [3]. The results are predictions for C_ξ as follows

$$C_\xi = 4.12 \times 10^{-3} \quad (\gamma = 0.4), \tag{50a}$$

$$C_\xi = 2.06 \times 10^{-4} \quad (\gamma = 0.6). \tag{50b}$$

If we take the results for both C_χ and C_ξ and combine them to form the universal quantity

$$\chi_t \xi^2 = C_\chi C_\xi^2 \tag{51}$$

which does not depend on λ_L, then the results are [3]

$$\chi_t \xi^2 = 0.092 \quad (\gamma = 0.4), \tag{52a}$$

$$\chi_t \xi^2 = 0.080 \quad (\gamma = 0.6). \tag{52b}$$

These results are in fairly close agreement with one another considering how strongly the individual predictions for C_χ and C_ξ deviated between the two different values for γ. This is fairly good evidence that the problem did lie in the failure to take into account the power corrections in the scaling law (39) which are present when we are not working close enough to the continuum. Perhaps the large deviations found here for physical predictions in terms of the Λ-parameter are rather extreme, but it seems naive to think that on present day lattice sizes one would not easily make a mistake of a factor of at least two or more. These factors are simply unknown. On the other hand, the calculation of dimensionless quantities as, for example, mass ratios or the quantity $\chi_t \xi^2$, seems in good shape. More will be said on this in the next lecture.

Now we are ready for conclusions. The main conclusion is that

1) Topology survives in a non-trivial way in an example of a *quantized* field theory. Thus it seems that topological quantities can be treated on the lattice provided sufficient care is taken.

We also have the less positive statement that is more a warning:

2) The relation of the physical quantities to the Λ-parameter cannot be reliably extracted from lattice Monte Carlo calculations on lattice sizes accessible presently. However, dimensionless universal quantities which are independent of the Λ-scale should be more reliably extracted.

In the next lecture a finite volume approach will be used towards calculating the same quantities as above in the *continuum* theory. Thus we will have a further test of our notion of universality.

2. Finite volume, the mass gap and universality revisited

In this lecture I will illustrate a method for calculating the low-lying mass spectrum in asymptotically free theories which has been recently proposed [2]. The application of the method is obvious: to calculate glueball masses in Yang-Mills theory. Although the method has not been as successful in this program as hoped for [17], the method applied to two dimensional theories seems to work extremely well. Here I will illustrate the method as applied to the CP^{n-1} model which was introduced in the first lecture. First I will outline the strategy and then remind you of the definition of the CP^{n-1} model. After that I will indicate the steps of the calculation in only slightly more detail, and present results. The results can then be compared with Monte Carlo calculations and provide a test as to the accuracy of the latter.

The strategy of the method is as follows. The first observation is that because perturbation theory for an asymptotically free theory is good at high energies or alternatively at small distances, perturbation theory in a finite volume makes sense. The next observation is that if one puts a theory in which the fields take their values in a compact manifold into a finite space time volume, the spectrum of the Hamiltonian becomes discrete. In fact for a box of size L the energy levels will be of the form

$$E = \frac{1}{L} g(\mu L, f), \tag{53}$$

where g is a function of the available dimensionless quantities as indicated. Now we know that in perturbation theory the energy levels are expanded in terms of the coupling constant:

$$E = E_0 + E_1 f + E_2 f^2 + \dots . \tag{54}$$

Since to zero order in perturbation theory the lowest lying states (the constant fields) are degenerate with the vacuum, the mass gap of the theory (the difference between the first excited state and the vacuum) will be of the form

$$m_0 = E_1 f + E_2 f^2 + \dots . \tag{55}$$

Now how do we calculate the low-lying mass spectrum? I will illustrate one possible way; to calculate a suitable Euclidean correlation function leading to an effective action from which an effective Hamiltonian can be derived. Alternatively one could regularize with say a lattice cutoff and set up a standard Hamiltonian formalism [2]. The important point is that the effective Hamiltonian will act on the unperturbed eigenvector space of the zero order Hamiltonian, i.e. the space of constant fields, and yet give the same eigenvalues as the exact Hamiltonian order by order in perturbation theory. The construction of such an effective Hamiltonian for systems with a finite number of degrees of freedom is well known to nuclear physicists through the work of Bloch [18].

Once the effective Hamiltonian is found we must solve for its eigenvalues, and finally form the quantity $m_0/\Lambda_{\overline{MS}}$. This latter quantity can be expressed in terms of dimensionless quantities, and in particular as a function of the quantity

$$z = m_0 L. \tag{56}$$

We may attempt to extrapolate the result to large z, a limit in which the correct answer must simply be the universal number C_ξ^{-1} of the last lecture (since $\xi = m_0^{-1}$). Thus when we are finally finished with the calculation we will have a result to compare with our Monte Carlo calculations of Lecture I.

This was the whirlwind tour of the method. Now let us go back and do it in slightly more detail. First let me remind you of the CP^{n-1} model. The model is defined through the Euclidean action given in (1), but this time we will take the fields to live on a cylinder $[0, T] \times S^1$ where $[0, T]$ is an interval of the real line that serves as the time dimension, and S^1, a one dimensional sphere (a circle) of length L which is the space dimension. The latter indicates that fields are given periodic boundary conditions in space with period L. We choose to give the fields Dirichlet boundary conditions in time.

Now in the infinite volume theory all physical quantities may be given in terms of $\Lambda_{\overline{MS}}$ defined in (36). In particular as indicated previously

$$m_0 = C_\xi^{-1}\Lambda_{\overline{MS}}. \tag{57}$$

But for small volumes we expect the mass gap to be proportional to L^{-1}. Let us therefore define m_0 as a function of L, using (56):

$$m_0(L) = \frac{1}{L}\, z(\mu L, f), \tag{58}$$

where μ is the scale introduced in (31). Now z must be a renormalization group invariant because m_0 is. Thus since $m_0(\infty)$ is given by (57), z and L in (57) must cooperate in this limit so that $m_0(L)/\Lambda_{\overline{MS}}$ goes to a constant. Inverting (58), i.e. solving for f, and then substituting back into the expression for $m_0(L)/\Lambda_{\overline{MS}}$ one obtains

$$\frac{m_0(L)}{\Lambda_{\overline{MS}}} = C_0(z). \tag{59}$$

The right hand side can only depend on z because of renormalization group invariance. That is, since it no longer depends on f by virtue of the substitution in favor of z, it cannot depend on μ. And since C_0 is dimensionless it can no longer depend on L explicitly. Finally from (57) we know that the exact $L = \infty$ limit is

$$C_0(\infty) = C_\xi^{-1}. \tag{60}$$

The important thing to realize now is through (58) and (55), the perturbative expansion of m_0 generates the small z expansion of $C_0(z)$. Thus the strategy is to calculate $C_0(z)$ for small z through perturbation theory and then extrapolate to large z.

Now let us push the program through for the CP^{n-1} model and calculate $C_0(z)$ to one loop order in perturbation theory. The amplitude that I wish to calculate is the matrix element between a constant field η at time $t = 0$ and another constant field ζ at time $t = T$;

$$\mathscr{A}(\zeta, \eta) = \langle \eta, t = 0 | \zeta, t = T\rangle = \langle \eta | e^{-HT} | \zeta \rangle, \tag{61}$$

where in the second line the fields are taken at the same time. In reality we only want to look at states of the form (55) which to zeroth order are degenerate with the vacuum. Thus after expanding $\mathscr{A}$ as a power series in f we can throw away any terms that are exponentially small for large T. These are contributions of higher energy states. I will indicate this projection onto the low-lying states by the inclusion of a projection operator P_0. So what we actually calculate is

$$\mathscr{A}_0(\zeta, \eta) = \langle \eta | e^{-HT} P_0 | \zeta \rangle. \tag{62}$$

The method of calculation is to represent $\mathscr{A}_0$ as a path integral, fix the gauge (say let z_n be real and positive for all x), regularize with dimensional regularization, and calculate the amplitude to first order in perturbation theory. All this is standard with the only exception that the coordinate space representation of a propagator in a finite volume is a sum over momentum components rather than an integral. The calculation amounts to calculating the quadratic fluctuations around the classical action

$$S_{\text{classical}} = \frac{n}{2f_0} \frac{L}{T} \theta^2 \tag{63}$$

where θ is the geodesic distance in the CP^{n-1} manifold between the two constant fields at times 0 and T, i.e. $\zeta \cdot \eta = \cos\theta$. If we define the effective action through

$$e^{-S_{\text{eff}}} \sim \mathscr{A}_0(\zeta, \eta) \tag{64}$$

and follow the procedure outlined above we find [4]

$$S_{\text{eff}} = S_{\text{classical}} + S_{\text{quadratic}}, \tag{65}$$

where

$$S_{\text{quadratic}} = n\theta^2 \left[\tfrac{1}{24} \frac{L^2}{T^2} - \frac{1}{4\pi} \left(\ln \frac{\mu^2 L^2}{4\pi} - \Gamma'(1) \right) \frac{L}{T} - \tfrac{1}{6} \right]. \tag{66}$$

Note that S_{eff} is not a function of η or ζ explicitly but only of the geodesic distance between points. This is as it should be because all points look identical on the CP^{n-1} manifold.

Now $\mathscr{A}_0(\zeta, \eta)$ is a singular object for $T \to 0$ so we must be careful about this limit. In order to extract the effective Hamiltonian we may smear $\mathscr{A}_0$ with a function over the CP^{n-1} manifold. That is, define

$$\begin{aligned} \mathscr{A}_0(\psi; \eta) &= \int d\mu(\zeta) \mathscr{A}_0(\zeta, \eta) \psi(\zeta) \\ &\sim (1 - H_{\text{eff}} T + \tfrac{1}{2} H_{\text{eff}}^2 T^2 - \ldots) P_0 \psi(\eta), \end{aligned} \tag{67}$$

where $d\mu$ is the invariant measure over the CP^{n-1} manifold. Alternatively we could recognize that $\mathscr{A}_0(\zeta, \eta)$ is proportional to the heat kernel of an operator which is a constant times the Laplace-Beltrami operator $\Delta_{CP^{n-1}}$ on the CP^{n-1} manifold, to the order we are interested in. Either way the effective Hamiltonian is found to be [4]

$$H_{\text{eff}} = -\frac{1}{L} \left(\frac{f}{2n} + \frac{f^2}{n} \frac{1}{4\pi} \left(\ln \frac{\mu^2 L^2}{4\pi} - \Gamma'(1) \right) \right) \Delta_{CP^{n-1}}. \tag{68}$$

The spectrum of the operator $\Delta_{CP^{n-1}}$ can be calculated by standard techniques [19] and can be shown to be $\{-4k(k+n-1) \,|k = 0, 1, 2, \ldots\}$. Our result for the mass gap ($k = 1$) is then

$$m_0 = \frac{1}{L} \left(2f + f^2 \frac{1}{\pi} \left(\ln \frac{\mu^2 L^2}{4\pi} - \Gamma'(1) \right) \right). \tag{69}$$

Note that m_0 is a renormalization group invariant as expected. Finally, carrying out the algebra outlined previously one finds

$$\frac{m_0(L)}{\Lambda_{\overline{MS}}} = \tfrac{1}{2} e^{-\Gamma'(1)} \left(\frac{z}{2\pi} \right)^{\frac{2}{n}+1} e^{\frac{2\pi}{z}} (1 + O(z)). \tag{70}$$

This is the one loop approximation to the function $C_0(z)$ in (59) and is plotted in Fig. 6 for $n = 2$ and $n = \infty$, as a function of z. In order to get an approximate value for $C_0(\infty) = C_\xi^{-1}$ let us evaluate (70) at its minimum to obtain

$$\frac{m_0(\infty)}{\Lambda_{\overline{\mathrm{MS}}}} \cong \frac{1}{2} e^{-\Gamma'(1)} \left(\frac{ne}{n+2}\right)^{\frac{n+2}{n}}. \tag{71}$$

Now let us ask the question how reliable is this result as an estimate of $C_0(\infty)$. First we can compare with the large n solution. The solution for $n = \infty$ is known exactly for arbitrary z [4]. The exact result is $m_0(\infty)/\Lambda_{\overline{\mathrm{MS}}} = 2$. Our formula (71) when evaluated at

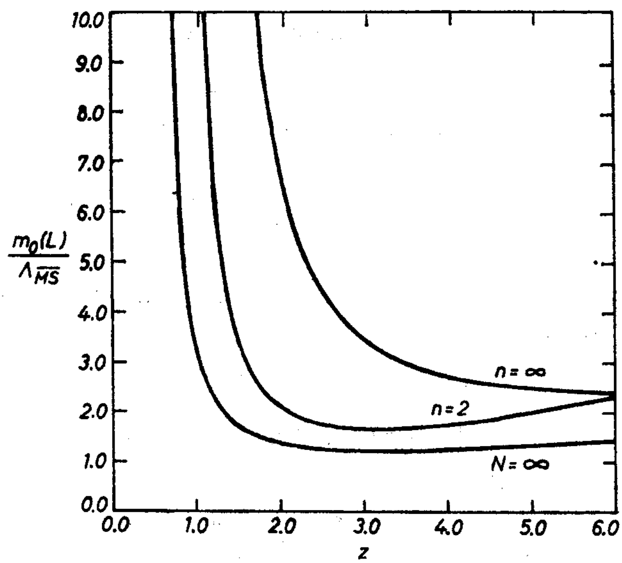

Fig. 6. One loop calculation of $M_0(L)/\Lambda_{\overline{\mathrm{MS}}}$ versus z for the CP^{n-1} model $n = 2$ and ∞, and the O(N) model $N = \infty$

$n = \infty$ gives $m_0(\infty)/\Lambda_{\overline{\mathrm{MS}}} \cong 2.42$ which is about 20% too high. Next we can gain insight from the O(N) model. In that case the exact result for $N = \infty$ is $m_0(\infty)/\Lambda_{\overline{\mathrm{MS}}} = 1$ and the approximation from a one loop finite volume calculation is $m_0(\infty)/\Lambda_{\overline{\mathrm{MS}}} \cong 1.21$, also about 20% too high [2]. The factor of two between these results and those of the CP^{n-1} model is no accident, but is exact order by order in the z expansion. The way we can use this latter result is to note that the O(3) model and the CP^1 model are the same, with $m_0(\infty)/\Lambda_{\overline{\mathrm{MS}}} \cong 1.65$ to one loop order. Because of this, $m_0(\infty)$ in units of $\Lambda_{\overline{\mathrm{MS}}}$ for the CP^{n-1} is an ascending series from 1.65 to 2.42 as a function of n, and for the O(N) model $m_0(\infty)/\Lambda_{\overline{\mathrm{MS}}}$ descends from 1.65 down to 1.21 as a function N, in the one loop approximation. Thus the system is rather rigid with all values of $m_0(\infty)/\Lambda_{\overline{\mathrm{MS}}}$ sandwiched between the large n and N values which are only about 20% off the correct values. Thus one would expect that all the values are probably within about 20% of being correct. This is rather remarkable since the other ways we have to estimate the mass gap in units of $\Lambda_{\overline{\rm MS}}$ of a system (i.e. lattice strong coupling or Monte Carlo) cannot claim such accuracy (see Lecture I). This will become more obvious when we compare the results directly.

Finally one may ask how fast will the z expansion approach the correct answer for large z. In the large n result an expansion around large z shows that deviations from the exact result are exponentially small [2]. In general, one can show this is true in lattice spin systems and lattice gauge theory to all orders in the strong coupling expansion [20].

Now let us look at Monte Carlo calculations of the quantity $m_0/\Lambda_{\overline{\rm MS}}$ for comparison. As stated above the prediction for the O(3) model is $m_0(\infty)/\Lambda_{\overline{\rm MS}} \cong 1.65$. Before this calculation was performed [2] the consensus from Monte Carlo (MC) and strong coupling (SC) was that the number should be somewhere between 3.2 and 5.7 (see [21]). Recently, however, MC calculations have been performed using Symanzik's "one loop improved" action [22] from which a value of 1.3 was obtained [23]. This shows that when extracting the relation of a physical quantity with the Λ-parameter from a MC calculation one can easily be off by a factor of 2 simply be changing the action. Of course the situation discussed in Lecture I concerning the CP^2 model was much more extreme with factors of 20 or more [3]. But now let us ask about the results for a quantity such as $\chi_t\xi^2$ discussed in Lecture I where all Λ dependence cancels. Recall that χ_t is the topological susceptibility and $\xi = 1/m_0$ is the correlation length.

An estimate for a lower bound for the quantity $\chi_t\Lambda_{\overline{\rm MS}}^{-2}$ was recently estimated for the CP^2 model by finite volume techniques to be about 0.81 [24]. The reason that it is a lower bound is that χ_t is expected to rise monotonically with volume and the estimate was taken in a volume small enough to make a small volume calculation almost certainly reliable. We then combine this result with equation (71) evaluated at $n = 3$ which gives the estimate for the CP^2 model $m_0\Lambda_{\overline{\rm MS}}^{-1} \cong 2.0$. So finally we obtain from the finite volume calculations

$$\chi_t\xi^2 \geqslant 0.2. \tag{72}$$

Assuming the bound for χ_t to be reliable this result is probably also because the expected systematic error in ξ would only serve to make the bound more severe. Now recall from Lecture I (52) our results for this quantity were 0.092 and 0.08 for two different lattice actions used in MC calculations. From the data [3] one could argue that points too far in the strong coupling region have contributed heavily to these results and in fact the value extracted from the MC calculations should be about 0.12 for $\chi_t\xi^2$. Given the wide range of values obtained for C_χ and C_ξ individually the different results for $\chi_t\xi^2$ are surprisingly close. However even the value 0.12 is several standard deviations outside of the bound derived from small volume calculations. We can still point to uncertainties such as the fit involved in obtaining ξ from MC calculations as being possible places to look for errors, but the situation is not completely clear to me. However, I think that we can safely say that at least in this context the finite volume results are more reliable than those from Monte Carlo and therefore provide a valuable check on the Monte Carlo calculations. I should say though that this is not necessarily the case in the calculation of glueball masses in Yang-Mills theory in four dimensions [17].

Finally I will close by saying that the finite volume approach appears to be a promising method with yet many undiscovered possibilities when it comes to analyzing asymptotically free theories.

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