TESTING QCD WITH CURRENT ALGEBRA*, **

By H. Leutwyler***

CERN — Geneva

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Spontaneously broken chiral symmetry fixes the low energy structure of QCD to a large extent. I show how to determine the Green's functions to first nonleading order in a simultaneous expansion in powers of the momenta and of the u- and d-quark masses. In particular, I discuss the corrections of order M_π^2 to the low energy theorems for $\pi\pi$ scattering.

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1. Motivation

In the framework of the standard model the gauge theory of $SU(3)_c \times U(1)_{e.m.}$ is a very accurate effective low energy theory. The Lagrangian

$$\mathcal{L}_{eff} = \mathcal{L}_{QCD} + \mathcal{L}_{QED} \tag{1.1}$$

should describe all low energy processes (energies $E \ll M_W$) up to small corrections of order $e^2(E/M_W)^2$.

The QED part of the effective low energy theory is responsible for the structure of atoms, molecules or solids — it affects the structure of the proton or of the pion very little. In the following I will discuss the low energy properties of the Green's functions associated with the quark currents; in this context, QED only generates small perturbations which are adequately described by the first few terms of a perturbative expansion on powers of e. I neglect these perturbations altogether, i.e. work to lowest order in e:

$$\mathcal{L}_{eff} = \mathcal{L}_{QCD}. \tag{1.2}$$

Apart from the coupling constant g (or, equivalently, the renormalization group invariant scale Λ) the QCD Lagrangian contains the vacuum angle θ and the mass matrices associated with the quarks of equal charge as basic parameters. The general quark mass term is of the form $\overline{U}_R M_U U_L + \overline{D}_R M_D D_L + \text{h.c.}$, where U and D collect the quark fields of charge

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*** On leave of absence from Universität Bern, permanent address: Institut für Theoretische Physik, Sidlerstrasse 5, 3012 Bern, Switzerland.

$\frac{2}{3}$ and $-\frac{1}{3}$ respectively. With suitable unitary transformations of U_R and U_L one may bring the matrix M_U to diagonal form with positive eigenvalues $m_u, m_c, \ldots$; likewise M_D may be brought to positive diagonal form [1]. If the determinant of the original quark mass matrix, $\det M = \det M_U \cdot \det M_D$ is not real and positive, this change of basis however involves a chiral U(1) transformation which only leaves the theory invariant, if one simultaneously transforms the vacuum angle in such a way that the sum

$$\bar{\theta} = \theta + \arg \det M \tag{1.3}$$

remains the same. (Note that the experimental information about the eigenvalues of the quark mass matrix [2] excludes $\det M = 0$ by many standard deviations. The phase of the determinant is therefore a meaningful number.)

If $\bar{\theta}$ is not a multiple of π the theory does not conserve CP. From the experimental upper bound on the electric dipole moment of the neutron one concludes [3] that $\bar{\theta}$ must be very close to a multiple of π. It is clear from (1.3) that $\bar{\theta}$ is defined only modulo 2π. We are therefore faced with two distinct possibilities $\bar{\theta} \simeq 0$ or $\bar{\theta} \simeq \pi$. I have recently argued [4] that the spectrum of QCD does not distinguish between these two possibilities and that we do not know which of the two is realized in nature. (The issue is of relevance for models which predict quark mass relations; if it were known that $\bar{\theta}$ is close to zero rather than close to π, models which predict $\theta \simeq 0$, $\det M > 0$ would be ruled out.) To simplify the discussion I disregard the small CP violating effects and idealize the two cases to $\bar{\theta} = 0$, $\bar{\theta} = \pi$. Instead of pinning down the chiral U(1) transformations by the requirement that all quark masses be positive, one may use this degree of freedom to eliminate the vacuum angle in the Lagrangian of QCD

$$\theta = 0. \tag{1.4}$$

In this convention $\bar{\theta}$ is the phase of the quark mass determinant. If $\bar{\theta} = 0$ the determinant is positive; in this case the quark field basis may be chosen such that all quark masses are positive. If $\bar{\theta} = \pi$ the determinant is negative, in this case it is convenient to take the three light quark masses m_u, m_d, m_s as negative, whereas the heavy quark masses m_c $m_b, \ldots$ are taken positive. With these conventions the matrices M_U and M_D are real, the quark mass term in the Lagrangian is independent of γ_5 and may be written as $\bar{q}Mq = m_u\bar{u}u + m_d\bar{d}d + \ldots$.

The running masses of the heavy quarks c and b are known rather precisely [2]:

$$m_c = 1.35 \pm 0.05 \text{ GeV},$$
$$m_b = 5.6 \pm 0.4 \text{ GeV} \tag{1.5}$$

(running masses in $\overline{\text{MS}}$ scheme at scale $\mu = 1$ GeV). The ratios of the light quark masses are also known very accurately [2]:

$$\frac{m_d}{m_u} = 1.76 \pm 0.13,$$

$$\frac{m_s}{m_d} = 19.6 \pm 1.6,$$

$$\frac{m_s - \hat{m}}{m_d - m_u} = 43.5 \pm 2.2; \quad \hat{m} = \tfrac{1}{2}(m_u + m_d) \tag{1.6}$$

but their absolute size is known only to about 30%:

$$|m_u| = 5.1 \pm 1.5 \text{ MeV},$$
$$|m_d| = 8.9 \pm 2.6 \text{ MeV},$$
$$|m_s| = 175 \pm 55 \text{ MeV} \tag{1.7}$$

(running masses at $\mu = 1$ GeV).

The spectrum of the theory is approximately SU(3)-symmetric (flavour symmetry $u \leftrightarrow d \leftrightarrow s$): the mass of the Σ^+_{uus} e.g. is not very different from the mass of the proton ($M_{\Sigma^+} = 1189$ MeV, $M_p = 938$ MeV). The mass difference $m_s - m_d$ (which is responsible for the breaking of the symmetry) must therefore amount to a small perturbation. Since m_s is 20 times larger than m_d this implies that m_s itself must be small (m_s must be small in comparison to the typical energy of a quark in the proton or in the Σ^+; if this were not the case there would be no reason for the wave functions of the two bound states to be approximately the same and hence no reason for the bound state masses to be approximately equal).

The mass of the charmed quark on the other hand is not small. If one gradually increases the mass of the strange quark until it reaches the value m_c the mass of the bound state *uds* gradually increases from 1116 MeV (the mass of Λ_{uds}) to 2282 MeV (the mass of Λ^+_{udc}): the mass difference $m_c - m_s$ is responsible for half of the mass of the *udc* bound state, it is not a small perturbation.

Since the light quark masses are small, it is appropriate to treat the corresponding mass term in the Lagrangian as a perturbation

$$\mathscr{L}_{\text{QCD}} = \mathscr{L}^0_{\text{QCD}} - m_u \bar{u}u - m_d \bar{d}d - m_s \bar{s}s \tag{1.8}$$

and to expand the bound state masses, Green's functions etc. in powers of m_u, m_d, m_s (note that the masses of the heavy quarks are retained in $\mathscr{L}^0_{\text{QCD}}$). This procedure is called chiral perturbation theory [5].

The unperturbed system, $\mathscr{L}^0_{\text{QCD}}$, is exactly symmetric with respect to independent rotations of the right- and left-handed components of the quark fields $u(x)$, $d(x)$, $s(x)$. The vector and axial currents

$$V^i_\mu = \bar{q}\,\frac{\lambda^i}{2}\,\gamma_\mu q; \qquad A^i_\mu = \bar{q}\,\frac{\lambda^i}{2}\,\gamma_\mu \gamma_5 q; \qquad i = 1, \ldots, 8$$

are exactly conserved. One assumes that the ground state of the theory breaks this SU(3) $\times$ SU(3) symmetry down to SU(3): the vacuum is not symmetric with respect to the chiral transformations generated by the axial charges:

$$Q_{A^i}|0\rangle \neq 0.$$

The eight states $Q_{A^i}|0\rangle$ describe massless, pseudoscalar particles, the Goldstone bosons of the spontaneously broken symmetry. Since the eight lightest hadrons $\pi^\pm$, π^0, $K^\pm$, K^0, $\bar{K}^0$, η are indeed pseudoscalar, it is natural to identify them with these Goldstone bosons.

Why are they not exactly massless? The group SU(3)×SU(3) is an exact symmetry only if the quark masses m_u, m_d, m_s are turned off. In the real world the quark mass term produces explicit symmetry breaking, e.g.:

$$\partial_\mu(\bar{u}\gamma^\mu s) = i(m_u - m_s)\bar{u}s,$$
$$\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = i(m_u + m_d)\bar{u}\gamma_5 d. \tag{1.9}$$

The (mass)² of the pion, e.g., vanishes only if m_u and m_d are set equal to zero; in the real world it is proportional to $m_u + m_d$:

$$M_\pi^2 = (|m_u| + |m_d|)\cdot B + O(m^2). \tag{1.10}$$

The sensitivity of the pseudoscalar meson masses to the masses of the light quarks explains why the pion and kaon masses are so different: the mass of the kaon is given by

$$M_{K^+}^2 = (|m_u| + |m_s|)\cdot B + O(m^2) \tag{1.11}$$

where B is the same constant as in (1.10). The ratio $M_K^2 : M_\pi^2$ is large, because $|m_s|$ happens to be much larger than $|m_u|$, $|m_d|$.

The hidden, spontaneously broken, symmetry is not only responsible for the occurrence of Goldstone bosons — it fixes their low energy properties in terms of a single constant F_π which measures the matrix element

$$\langle 0|A_\mu^i|\pi^k(p)\rangle = i\delta^{ik}p_\mu F_\pi. \tag{1.12}$$

The pion must e.g. couple to the nucleon with strength

$$g_{\pi NN} = \frac{g_A M_N}{F_\pi} + O(m) \tag{1.13}$$

(Goldberger-Treiman relation). Another example is Weinberg's formula for the $\pi\pi$ scattering amplitude [6]

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2} + O(p^4). \tag{1.14}$$

This formula in particular allows one to predict the S- and P-wave scattering lengths. The P-wave scattering length e.g. is given by [6]

$$a_1^1 = \frac{1}{24\pi F_\pi^2}. \tag{1.15}$$

If the quark masses m_u and m_d are put equal to zero (in which case the pion is a genuine, massless Goldstone boson) the low energy theorems (1.13) and (1.15) must hold exactly. What happens in the real world? Since the symmetry underlying these low energy theorems is only broken by the masses m_u and m_d which are tiny, one should expect the theorems to hold rather accurately, even in the real world. Comparison of the experimental result

$$a_1^1|_{exp} = (0.038 \pm 0.002)\,(M_{\pi^+})^{-2} \tag{1.16}$$

with the theoretical prediction (1.15)

$$a_1^1|_{th} = 0.030(M_{\pi^+})^{-2} \tag{1.17}$$

however, shows a 25% discrepancy! This is puzzling for the following reason. Equation (1.9) shows that the breaking of SU(2) × SU(2) is measured by $m_u + m_d$ whereas the breaking of isospin symmetry is measured by $m_u - m_d$. Since m_d is almost twice as large as m_u the sum $m_u + m_d$ is not much larger than the difference $m_u - m_d$. If we find large violations of SU(2) × SU(2) why is isospin such a good symmetry? Alternatively, one may compare SU(2) × SU(2) with SU(3). Since $m_u + m_d$ is about 12 times smaller than $m_s - m_d$, we shoud expect the typical corrections to the SU(2) × SU(2) low energy theorems to be about 12 times smaller than the typical violations of SU(3) which are of order 20%. We should thus expect deviations of order 20%: 12 ≃ 2% rather than 25% as observed.

To see whether the observed deviations from the low energy theorems of current algebra indeed pose a problem we have to go beyond these crude estimates and calculate the perturbations generated by the quark mass term. Truong and his collaborators [7] have pointed out that the so-called unitarity corrections to the low energy theorems of current algebra turn out to be large in some cases. The unitarity corrections constitute an important part of these perturbations — they are however not the only contributions. In the following, I briefly sketch a method which allows one to determine the corrections to the low energy theorems in a systematic manner [8, 9]. I restrict myself to SU(2) × SU(2), i.e., expand the quantities of interest in powers of m_u and m_d at fixed m_s, m_c, … .

2. *Generating functional, Ward identities*

Let us first look at the unperturbed system ($m_u = m_d = 0$). Since the vector and axial vector currents are exactly conserved, the corresponding Ward identities lead to a closed system of constraints relating the Green's function

$$G_{m,n} = \langle 0|\mathrm{T}V_{\mu_1} \dots V_{\mu_m} A_{\nu_1} \dots A_{\nu_n}|0\rangle \tag{2.1}$$

to $G_{m,n-1}$ and to $G_{m-1,n}$. To solve these constraints it is convenient to look at the generating functional $Z(v, a)$ which collects all of these Green's functions. One adds external vector and axial vector fields to the Lagrangian

$$\mathscr{L}^0_{\mathrm{QCD}} \to \mathscr{L}^0_{\mathrm{QCD}} + \bar{q}v_\mu(x)\gamma^\mu q + \bar{q}a_\mu(x)\gamma^\mu\gamma_5 q \tag{2.2}$$

and considers the vacuum-to-vacuum amplitude in the presence of these external fields

$$e^{iZ(v,a)} = \langle 0 \text{ out } | 0 \text{ in}\rangle_{v,a}. \tag{2.3}$$

In the present context, $v_\mu(x)$ and $a_\mu(x)$ are 2 × 2 matrices acting on the flavour of u and d quarks:

$$v_\mu(x) = v^i_\mu(x)\frac{\tau^i}{2}; \quad a_\mu(x) = a^i_\mu(x)\frac{\tau^i}{2}. \tag{2.4}$$

The Green's functions are recovered from the generating functional by expanding $Z(v, a)$ in powers of v_μ, a_μ. The contribution quadratic in a_μ e.g. is the two-point-function of the axial current

$$Z(v, a) = \frac{i}{2}\int dx dy a_\mu^i(x) a_\nu^k(y) \langle 0|\mathrm{T} A^{\mu i}(x) A^{\nu k}(y)|0\rangle + \ldots \tag{2.5}$$

The Ward identities are equivalent to the statement that the generating functional is invariant with respect to the following SU(2) × SU(2) gauge transformation of the external fields

$$\begin{aligned} v'_\mu + a'_\mu &= V_\mathrm{R}(x)\,(v_\mu + a_\mu) V_\mathrm{R}^\dagger(x) + i V_\mathrm{R}(x)\partial_\mu V_\mathrm{R}^\dagger(x), \\ v'_\mu - a'_\mu &= V_\mathrm{L}(x)\,(v_\mu - a_\mu) V_\mathrm{L}^\dagger(x) + i V_\mathrm{L}(x)\partial_\mu V_\mathrm{L}^\dagger(x), \\ Z(v', a') &= Z(v, a). \end{aligned} \tag{2.6}$$

(A formal "proof" of this property is easily obtained: one subjects the right- and left--handed components of the quark fields to the transformation

$$q_\mathrm{R}(x) \to V_\mathrm{R}(x) q_\mathrm{R}(x); \quad q_\mathrm{L}(x) \to V_\mathrm{L}(x) q_\mathrm{L}(x) \tag{2.7}$$

which formally leaves the theory invariant. Chiral transformations of this sort are in general however afflicted with anomalies [10]. In the case of SU(3) × SU(3) e.g. the generating functional is not invariant under the set of gauge transformations analogous to (2.7), but picks up a contribution from the anomaly. In the case of SU(2) × SU(2) there is no anomaly in the isovector currents — the generating functional is invariant under the gauge transformation (2.6).) I recommend it as an exercise to check that the gauge invariance of Z indeed implies the Ward identities relating say the 3-point-function $\langle 0|\mathrm{T} V_\lambda A_\mu A_\nu|0\rangle$ to $\langle 0|\mathrm{T} V_\mu V_\nu|0\rangle$, $\langle 0|\mathrm{T} A_\mu A_\nu|0\rangle$.

3. *Low energy expansion of Green's functions in chiral limit*

If the theory does not contain massless physical states then the Green's function admit a Taylor series expansion around $p = 0$, e.g.

$$i \int dx e^{ip(x-y)} \langle 0|\mathrm{T} A_\mu^i(x) A_\nu^k(y)|0\rangle = c_1^{ik} g_{\mu\nu} + c_2^{ik} p_\mu p_\nu + c_3^{ik} g_{\mu\nu} p^2 + \ldots \tag{3.1}$$

This expansion is equivalent to an expansion of the generating functional $Z(v, a)$ in powers of derivatives of the external fields:

$$Z = \int dx \{\tfrac{1}{2} c_1^{ik} a_\mu^i(x) a^{\mu k}(x) + \tfrac{1}{2} c_2^{ik} \partial_\mu a^{\mu i}(x) \partial_\nu a^{\nu k}(x) + \ldots\}. \tag{3.2}$$

Gauge invariance imposes restrictions on the coefficients $c_1, c_2, \ldots$: the coefficient c_1 e.g. must vanish, c_2 is equal to $-c_3$ etc. The leading term in the low energy expansion of Z is given by

$$\begin{aligned} Z &= c \int dx \operatorname{tr} \{F_{\mu\nu}^\mathrm{R} F^{\mu\nu\mathrm{R}} + F_{\mu\nu}^\mathrm{L} F^{\mu\nu\mathrm{L}}\} + \ldots, \\ F_{\mu\nu}^\mathrm{I} &= \partial_\mu F_\nu^\mathrm{I} - \partial_\nu F_\mu^\mathrm{I} - i[F_\mu^\mathrm{I}, F_\nu^\mathrm{I}]; \quad \mathrm{I} = \mathrm{R}, \mathrm{L}, \\ F_\mu^\mathrm{R} &= v_\mu + a_\mu; \quad F_\mu^\mathrm{L} = v_\mu - a_\mu. \end{aligned} \tag{3.3}$$

A single constant c thus determines the leading low energy behaviour of the 2-, 3- and 4-point-functions associated with the vector and axial currents, e.g.

$$i \int dx e^{ip(x-y)} \langle 0|\mathrm{T} V_\mu^i(x) V_\nu^k(y)|0\rangle = c\delta^{ik}(p_\mu p_\nu - g_{\mu\nu}p^2) + O(p^4),$$
$$i \int dx e^{ip(x-y)} \langle 0|\mathrm{T} A_\mu^i(x) A_\nu^k(y)|0\rangle = c\delta^{ik}(p_\mu p_\nu - g_{\mu\nu}p^2) + O(p^4). \tag{3.4}$$

In the chiral limit ($m_u = m_d = 0$) which we have been considering in the last section (and which must be taken if Z is to be gauge invariant under SU(2) × SU(2)) the spectrum of the theory however contains massless states — the Goldstone bosons. These states produce poles at $p^2 = 0$ and cuts starting there; the Green's functions do not admit a simple Taylor series expansion in powers of the momenta. The two-point function of the axial current, e.g. behaves like

$$i \int dx e^{ip(x-y)} \langle 0|\mathrm{T} A_\mu^i(x) A_\nu^k(y)|0\rangle = \delta^{ik} F_\pi^2 \frac{p_\mu p_\nu}{-p^2} + \ldots \tag{3.5}$$

near $p^2 = 0$: it is of order one rather than of order p^2 as in (3.4). Furthermore, since the momenta also occur in the denominator, the leading low energy behaviour of the generating functional is not given by a local expression such as (3.2), but starts with a nonlocal piece of the form

$$Z = \tfrac{1}{2} F_\pi^2 \int dx dy \partial^\mu a_\mu^i(x) \Delta_c(x-y) \partial^\nu a_\nu^i(y) + \ldots \tag{3.6}$$

This term by itself is not gauge invariant — it requires the presence of similar contributions e.g. in the 3-point functions. The structure of these contributions is determined by chiral symmetry (gauge invariance) alone. To determine the full generating functional at leading order in the low energy expansion one may consider any model Lagrangian which is invariant under SU(2) × SU(2) and for which the ground state spontaneously breaks this symmetry down to SU(2). Chiral symmetry guarantees that the leading low energy behaviour of the Green's functions associated with the vector and axial currents is the same as in QCD, provided only the value of F_π in the model is the same as it is in QCD [11, 12]. In this context the most convenient model is the nonlinear σ-model, because in this model the leading low energy behaviour is given by the tree graphs (graphs with n loops generate contributions which are suppressed by n powers of p^2, [12]). Since the set of all tree graphs is equivalent to the corresponding classical field theory this model offers a particularly simple construction of the leading term in the low energy expansion of the generating functional. The construction involves the following steps:

(*i*) Introduce a classical four-component field $U^A(x)$ subject to the constraint

$$U^{\mathrm{T}} U = (U^0)^2 + (U^1)^2 + (U^2)^2 + (U^3)^2 = 1. \tag{3.7}$$

Under SU(2) × SU(2) the field U^A transforms according to the vector representation $D^{(1/2,\,1/2)}$.

(*ii*) Define the covariant derivative $\nabla_\mu U^A$ by

$$\nabla_\mu U^0 = \partial_\mu U^0 + a_\mu^i U^i,$$
$$\nabla_\mu U^i = \partial_\mu U^i + \varepsilon^{ikl} v_\mu^k U^l - a_\mu^i U^0. \tag{3.8}$$

This definition insures that $\nabla_\mu U^A$ transforms like U^A.

(*iii*) The Lagrangian

$$Z_1 = \tfrac{1}{2} F_\pi^2 \int dx \nabla_\mu U^{\mathrm{T}} \nabla^\mu U \tag{3.9}$$

is gauge invariant. The leading term in the low energy expansion of the generating functional is given by the extremum of Z_1 with respect to variations of U^A:

$$\nabla^\mu \nabla_\mu U^A - U^A (U^{\mathrm{T}} \nabla^\mu \nabla_\mu U) = 0. \tag{3.10}$$

This equation of motion determines the field $U^A(x)$ in terms of the external fields $v_\mu(x)$, $a_\mu(x)$ provided we specify appropriate boundary conditions. In Euclidean space the boundary condition on $U^A(x)$ for external fields of compact support is $U^0 \to 1$ as $|x| \to \infty$. In Minkowski space this amounts to the requirement that $U^A(x)$ only contains positive (negative) frequencies as $x^0 \to +\infty(-\infty)$ and that U^0 tends to 1. (A priori the direction of the unit vector U^A at infinity is arbitrary; the boundary condition $U^0 \to 1$ corresponds to the limit $M_\pi \to 0$ of the massive σ-model.)

I recommend it as an exercise to check that this prescription indeed reproduces the term of order $a_\mu a_\nu$ given in (3.6). One can also easily obtain the four-point function $\langle 0|\mathrm{T} A_\mu A_\nu A_\varrho A_\sigma|0\rangle$ by solving the classical equation of motion (3.10) to the necessary accuracy in the expansion of U^A in powers of the field a_μ. The residue of the pion poles contained in this Green's function determines the leading low energy behaviour of the $\pi\pi$ scattering amplitude in the chiral limit:

$$A(s, t, u) = \frac{s}{F_\pi^2} + O(p^4). \tag{3.11}$$

4. *Ward identities for m_u, $m_d \neq 0$*

In the preceding two sections the quark masses were turned off; $\mathrm{SU}(2) \times \mathrm{SU}(2)$ was an exact symmetry of the Lagrangian, broken only spontaneously by the asymmetry of the ground state. If the quark masses do not vanish, the Ward identities satisfied by the vector and axial vector Green's functions do not constitute a closed system; instead they relate these Green's functions to those containing the scalar and pseudoscalar operators $\bar{q}\lambda q$ and $\bar{q}\lambda\gamma_5 q$. The divergence of the axial current $\bar{u}\gamma_\mu\gamma_5 d$ e.g. is related to the density $\bar{u}\gamma_5 d$. To analyze the consequences of the Ward identities it is convenient to extend the generating functional by adding external scalar and pseudoscalar fields to the Lagrangian

$$\mathscr{L} = \mathscr{L}^0_{\mathrm{QCD}} + \bar{q} v_\mu(x) \gamma^\mu q + \bar{q} a_\mu(x) \gamma^\mu \gamma_5 q - \bar{q} s(x) q + \bar{q} p(x) i\gamma_5 q \tag{4.1}$$

where the new fields

$$s(x) = s^0(x)\mathbf{1} + s^i(x)\tau^i; \qquad p(x) = p^0(x)\mathbf{1} + p^i(x)\tau^i \tag{4.2}$$

are Hermitean colour neutral matrices in flavour space. The full QCD Lagrangian, including the quark mass term, is obtained by setting $v_\mu = a_\mu = p = 0$ and identifying the external field $s(x)$ with the mass matrix of the u and d quarks:

$$s(x) = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{4.3}$$

The expansion of the generating functional $Z(v, a, s, p)$ in powers of the external fields around the point $v = a = p = 0$, $s = \mathcal{M}$ generates all Green's functions associated with the vector, axial vector, scalar and pseudoscalar currents. If we instead expand the generating functional around the point $v = a = s = p = 0$, we obtain the Green's functions in the chiral limit; in particular, the quantity $Z(v, a, 0, 0)$ coincides with the object $Z(v, a)$ studied in the preceding two sections.

If the vacuum expectation values of the scalar operators $\bar{u}u$ and $\bar{d}d$ do not vanish in the chiral limit, then the Taylor expansion of $Z(v, a, s, p)$ around $s = 0$ contains a term linear in $s(x)$:

$$Z = -\int dx\{s_{11}(x)\langle 0|\bar{u}u|0\rangle_0 + s_{22}(x)\langle 0|\bar{d}d|0\rangle_0\} + \dots, \tag{4.4}$$

where the index 0 indicates that we are considering the chiral limit $m_u = m_d = 0$. Since in the chiral limit the vacuum is assumed to be invariant under SU(2) we must have

$$\langle 0|\bar{u}u|0\rangle_0 = \langle 0|\bar{d}d|0\rangle_0 \tag{4.5}$$

and the term linear in $s(x)$ may therefore be written in the form

$$Z = -\langle 0|\bar{u}u|0\rangle_0 \int dx \, \mathrm{tr}\, s(x) + \dots \tag{4.6}$$

A chiral transformation of the quark fields only leaves the Lagrangian (4.1) invariant if one transforms the external scalar and pseudoscalar fields accordingly. The Ward identities among the Green's functions associated with the vector, axial vector, scalar and pseudoscalar currents are equivalent to the statement that the generating functional $Z(v, a, s, p)$ is invariant under local $SU(2) \times SU(2)$ transformations of the external fields:

$$Z(v', a', s', p') = Z(v, a, s, p). \tag{4.7}$$

The transformation law for $v_\mu(x)$ and $a_\mu(x)$ is given in (2.6), the transformation law for $s(x)$ and $p(x)$ reads

$$s'(x) + ip'(x) = V_R(x)\{s(x) + ip(x)\}V_L^\dagger(x). \tag{4.8}$$

The contribution (4.6) clearly fails to satisfy this condition. The generating functional must therefore contain further contributions which, together with (4.6) are gauge invariant. In more familiar terms this amounts to the observation that a nonzero vacuum expectation value of $\bar{u}u$ is only consistent with the Ward identities if the two-point function $\langle 0|TA_\mu^3 \bar{u}i\gamma_5 u|0\rangle$ contains a pole at $p^2 = 0$. The presence of this pole in turn requires specific low energy singularities in the three-point functions etc.

5. Low energy expansion for m_u, $m_d \neq 0$

In order to determine the structure of the low energy singularities required by a nonvanishing vacuum expectation value of $\bar{u}u$ and $\bar{d}d$ we merely have to extend the effective Lagrangian by coupling the σ-model to the external scalar and pseudoscalar fields. The crucial point in the effective Lagrangian method is that the nonlocal structure of the generating functional produced by the pion poles is converted into a local Lagrangian at the expense of introducing the additional field $U^A(x)$. (The nonlocal structures arise if one eliminates the field U^A by solving the relevant classical equations of motion.) To first order in the fields $s(x)$ and $p(x)$ the extended Lagrangian we are looking for is of the form

$$Z_1 = \frac{F_\pi^2}{2} \int dx \{\nabla_\mu U^T \nabla^\mu U + s^0(x) f^0 + s^i(x) f^i + p^0(x) g^0 + p^i(x) g^i\}, \tag{5.1}$$

where the functions f and g may depend on the fields v_μ, a_μ, U and their derivatives. The leading contribution in the low energy expansion arises from the terms with the least number of derivatives. (Note that $\nabla_\mu U^T \nabla^\mu U$ is the term of lowest dimension which can be constructed out of v_μ, a_μ, U and their derivatives in a gauge invariant manner; since $U^T U = 1$ there is no nontrivial invariant of dimension zero.) If the functions f and g do not contain derivatives of v_μ, a_μ, or U then gauge invariance implies that they are independent of v_μ and a_μ. Furthermore, the transformation law (4.8) shows that the vectors (s^0, p^i) and $(p^0, -s^i)$ transform in the same manner as the four-vector U^A. There are therefore only two gauge invariants that can be formed out of s, p and U: $s^0 U^0 + p^i U^i$ and $p^0 U^0 - s^i U^i$. Since the second invariant carries negative parity, the general effective Lagrangian of lowest dimension reads:

$$Z_1 = \frac{F_\pi^2}{2} \int dx \{\nabla_\mu U^T \nabla^\mu U + 4B(s^0 U^0 + p^i U^i)\}, \tag{5.2}$$

where B is a real constant. The leading low energy representation of the generating functional is obtained from this expression by eliminating the field $U^A(x)$ through the classical equations of motion which follow from $\delta Z_1 = 0$:

$$\nabla^\mu \nabla_\mu U^A - \chi^A = U^A \{U^T \nabla^\mu \nabla_\mu U - U^T \chi\}$$
$$\chi^A(x) = 2B\{s^0(x), p^i(x)\}. \tag{5.3}$$

To expand the generating functional in powers of the external fields we first need to determine the ground state, i.e. determine the field $U^A(x)$ in the absence of external perturbations ($v = a = p = 0$, $s = \mathcal{M}$). The ground state realizes the minimum of the Euclidean action:

$$B \cdot (m_u + m_d) \cdot U^0 = \text{maximum}. \tag{5.4}$$

If the product $B \cdot (m_u + m_d)$ is positive, the ground state is described by $U^0 = 1$. If $B \cdot (m_u + m_d)$ is negative, the minimum of the action occurs at $U^0 = -1$; in this case it is convenient to define a new pion field by $U^{A'} = -U^A$. This operation leaves the Lagrangian (5.2) invariant, except for a change of sign of the constant B. Without loss of generality we may therefore assume $B \cdot (m_u + m_d)$ to be positive and the ground state to be described by $U^0 = 1$:

$$B \cdot (m_u + m_d) > 0. \tag{5.5}$$

Computing the Taylor coefficient of the term linear in s at the point $v = a = s = p = 0$ one finds that the constant B determines the vacuum expectation values of $\bar{u}u$ and $\bar{d}d$:

$$\langle 0|\bar{u}u|0\rangle_0 = \langle 0|\bar{d}d|0\rangle_0 = -F_\pi^2 B. \tag{5.6}$$

Recall that we have chosen the quark field basis such that m_u, m_d and m_s have the same sign (see Section 1). If m_u and m_d are positive, the convention (5.5) requires the constant B to be positive and we thus get negative expectation values $\langle 0|\bar{u}u|0\rangle_0$, $\langle 0|\bar{d}d|0\rangle_0$. If the masses m_u and m_d are negative, the expectation values are positive.

The constant B also determines by how much m_u and m_d shift the pion mass away from $M_\pi = 0$. The position of the poles is determined by the equation of motion for the field $U^i(x)$. To lowest order in the external fields this equation becomes

$$\Box U^i + B\cdot(m_u+m_d)U^i = \partial^\mu a^i_\mu + 2Bp^i. \tag{5.7}$$

The pion mass is therefore given by

$$M_\pi^2 = B\cdot(m_u+m_d). \tag{5.8}$$

Comparison of the relations (5.6) and (5.8) leads to the familiar Gell-Mann-Oakes-Renner formula [13]

$$F_\pi^2 M_\pi^2 = -(m_u+m_d)\,\langle 0|\bar{u}u|0\rangle_0. \tag{5.9}$$

It is remarkable that the effective low energy Lagrangian does not take note of the mass difference between u and d: Z_1 is independent of the field $s^3(x)$ which contains the mass difference $m_u - m_d$. At leading order in the low energy expansion the Green's functions of QCD do therefore not show any sign of isospin breaking: the symmetry properties of the vacuum protect isospin symmetry. (This is due to the fact that the matrix element $\langle\pi|\bar{u}u-\bar{d}d|\pi\rangle$ vanishes in the chiral limit.)

In the low energy singularities of the Green's functions the quark masses m_u and m_d compete with the momenta: the pion pole factors $\sim (M_\pi^2-p^2)^{-1}$ vary rapidly in a region where the square of the momentum is of order $M_\pi^2 \sim (m_u+m_d)$. To describe these singularities coherently we should therefore not consider a low energy expansion in powers of the momenta at fixed m_u and m_d, but rather consider a simultaneous expansion in p and in m_u and m_d at fixed ratios m_u/p^2, m_d/p^2 [12]. The quark masses are contained in the external field $s(x)$; we must therefore treat this field as a small quantity of order p^2. In this manner of counting low energy dimensions the two contributions to the generating functional Z_1 given in (5.2) are both of order p^2. It is a simple matter to determine the leading low energy representation say of the axial vector two-point function or of the $\pi\pi$ scattering amplitude. One merely has to solve the classical equations of motion of the nonlinear σ-model coupled to external fields to the required accuracy. I recommend it as a very instructive exercise to show that the leading low energy contributions are given by

$$i\int dx e^{ip(x-y)}\langle 0|TA^i_\mu(x)A^k_\nu(y)|0\rangle = \delta^{ik}F_\pi^2 p_\mu p_\nu (M_\pi^2-p^2)^{-1} + O(p^2),$$

$$A(s,t,u) = \frac{1}{F_\pi^2}(s-M_\pi^2) + O(p^4). \tag{5.10}$$

6. *Terms of higher order in the low energy expansion*

In the preceding sections we have argued that the leading terms in the low energy expansion of the Green's functions associated with the vector, axial vector, scalar and pseudoscalar quark currents are determined by two constants F_π and B. To order p^2 the generating functional is given by the action of the classical non-linear σ-model coupled to external fields (tree graphs of the corresponding quantum theory). What are the higher order contributions? Consider, e.g., the $\pi\pi$ scattering amplitude $A(s, t, u)$. The leading contribution at low energies is given by $(s-M_\pi^2)/F_\pi^2$. Clearly this function cannot represent the full scattering amplitude. Unitarity implies that in the elastic region $4M_\pi^2 < s < 16M_\pi^2$ the scattering amplitude must obey a relation of the type

$$\operatorname{Im} A = |A|^2. \tag{6.1}$$

Since the leading contribution to A is a real function of order p^2 this relation requires a contribution of order p^4 with a non-vanishing absorptive part. In fact, Lehmann [14] has shown that in the chiral limit unitarity fixes the contribution of order p^4 up to two constants ($m_u = m_d = M_\pi = 0$):

$$A(s, t, u) = \frac{s}{F_\pi^2} + \frac{1}{96\pi^2 F_\pi^4}\left\{3s^2 \ln\left(\frac{a_1}{-s}\right) + t(t-u)\ln\left(\frac{a_2}{-t}\right) + u(u-t)\ln\left(\frac{a_2}{-u}\right)\right\}. \tag{6.2}$$

The two new constants which are needed, in addition to F_π, to specify the scattering amplitude at order p^4 are the scales a_1 and a_2 of the logarithms occurring in this expression.

More generally, the leading contributions to the Green's functions at low energies are given by the tree graphs of the non-linear σ-model. The tree graphs of a field theory are, however, not unitary. Unitarity requires that the theory must be quantized, that one includes graphs with loops. In the non-linear σ-model the couplings of the pion field contain derivatives. This property insures that graphs involving one loop are suppressed in comparison to the tree graphs by two powers of the external momenta, graphs containing two loops are suppressed by four powers of p, etc. [12]. The leading low energy behaviour is therefore given by the tree graphs [$O(p^2)$]; one loop graphs contribute at first non-leading order [$O(p^4)$], graphs with more than one loop only contribute if one extends the low energy expansion beyond first non-leading order. This feature is closely related to chiral symmetry which requires the Goldstone boson couplings to vanish at zero momentum (for processes that exclusively involve pions which furthermore all have small momenta of order p, the T-matrix is of order p^2). At low energies the interaction is therefore weak — it is this property which allows one to solve the constraints of unitarity, clustering and chiral symmetry in a perturbative manner by expanding the Green's functions in powers of the momenta.

The non-linear σ-model is however not renormalizable in four dimensions. Graphs involving loops require counter terms which are not present in the Lagrangian of this model. The model does therefore not specify the perturbative expansion in terms of the constants F_π and B which characterize the lowest order Lagrangian. Instead new, undetermined constants appear at every order of the perturbative expansion. In the present context this is not a disease of the non-linear σ-model which one ought to try to cure, it is a characteristic feature of the low energy expansion. The point is that the Ward identities of chiral symmetry only *relate* the low energy expansion coefficients of various Green's functions — they do not determine them completely. At leading order the general solution of the Ward identities contains two independent low energy constants (F_π, B). At first non-leading order the general solution of the Ward identities involves seven new low energy constants (see below). Even if the non-linear σ-model could be elevated to a mathematically consistent framework involving only a few parameters (one may, e.g., consider the renormalizable σ-model which apart from F_π and B contains a single new parameter, the mass of the σ-particle) there is no reason for this framework to produce the correct low energy expansion of the Green's functions to all orders in the momenta. (The only reason the σ-model is of interest in our context is that chiral symmetry fixes the leading low energy behaviour of the Green's functions in terms of two constants; the σ-model does contain two constants and does satisfy chiral symmetry in the tree graph approximation).

7. *Effective Lagrangian to order p^4*

To determine the general solution of the Ward identities at next-to-leading order, we need the effective Lagrangian to order p^4. The most general expression consistent with Lorentz invariance, parity and gauge invariance reads [8, 9]

$$Z_2 = \int dx\{l_1(\nabla_\mu U^T\nabla^\mu U)^2 + l_2(\nabla_\mu U^T\nabla_\nu U)(\nabla^\mu U^T\nabla^\nu U)$$
$$+ l_3(\chi^T U)^2 + l_4\nabla^\mu\chi^T\nabla_\mu U + l_5 U^T F^{\mu\nu}F_{\mu\nu}U + l_6\nabla^\mu U^T F_{\mu\nu}\nabla^\nu U$$
$$+ l_7(\tilde{\chi}^T U)^2 + h_1\chi^T\chi + h_2 \operatorname{tr} F_{\mu\nu}F^{\mu\nu} + h_3\tilde{\chi}^T\tilde{\chi}\}, \tag{7.1}$$

where χ^A and $\tilde{\chi}^A$ contain the external scalar and pseudoscalar fields:

$$\chi^A = 2B(s^0, p^i); \qquad \tilde{\chi}^A = 2B(p^0, -s^i). \tag{7.2}$$

The field strength tensor $F_{\mu\nu}$ which contains the external vector and axial vector fields and their derivatives is defined by

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)U^A = F^{AB}_{\mu\nu}U^B. \tag{7.3}$$

The constants h_1, h_2 and h_3 are irrelevant contact terms; h_1, e.g., contributes a momentum independent constant to the two-point-function of the scalar density $\bar{q}q$. The time-ordered product which occurs in this two-point-function is not unambiguous — the value of h_1 merely specifies the conventions used to define the time-ordered product. For this reason h_1, h_2 and h_3 do not occur in quantities of physical interest.

The Lagrangian (7.1) contains all counter terms necessary to give meaning to the one-loop graphs of the non-linear σ-model. The explicit expression for the generating functional, accurate to order p^4, reads

$$e^{iZ} = e^{iZ_2} \int_{\text{one loop}} d\mu[U]e^{iZ_1}, \tag{7.4}$$

where the functional integral over the field $U^A(x)$ is to be evaluated in the one loop approximation. To obtain the first two terms in the low energy expansion of the Green's functions one thus needs to evaluate (*i*) tree and one loop graphs of the nonlinear σ-model coupled to external fields (Z_1) and (*ii*)) tree graphs which contain one vertex of Z_2 together with any number of σ-model vertices. The sum of these contributions is finite, provided the loop integrals are cut off in a chirally symmetric fashion and provided the constants l_1, l_2 ..., h_3 are properly renormalized (since dimensional regularization preserves chiral symmetry, this method is suitable in the present context).

The occurrence of counter terms which are not linear in the external field χ^A (or contain derivatives thereof) is related to the problems one encounters if one calculates the Green's functions of the *pion field* in the standard manner [15]. The external field technology we are using here avoids these problems as it retains the full symmetry of the theory at every stage of the calculation. Note also that an effective Lagrangian which only allows one to deal with on-shell matrix elements [12] does not determine the manner in which the low energy parameters depend on the quark masses. In our framework all low energy constants refer to the massless theory; the quark masses enter as explicit symmetry breaking parameters contained in the external fields:

$$\begin{aligned} \chi^0 &= 2Bs^0 = B(m_u+m_d)+ \ldots, \\ \chi^3 &= -2Bs^3 = B(m_d-m_u)+ \ldots . \end{aligned} \tag{7.5}$$

8. *Results*

With the recipe given in the last section it is in principle straightforward to calculate the low energy expansion of the Green's functions to first nonleading order (see Ref. [9] for details). From the resulting expression for the two-point-function $\langle 0|TA_\mu A_\nu|0\rangle$ e.g. one finds that the corrections of order p^4 shift the mass of the charged pion from

$$\mathring{M}_\pi^2 = B \cdot (m_u+m_d) \tag{8.1}$$

to

$$M_{\pi+}^2 = \mathring{M}_\pi^2 \left\{1 - \frac{\mathring{M}_\pi^2}{16\pi^2 \mathring{F}_\pi^2} \ln \frac{\mu_3}{\mathring{M}_\pi} + O(\mathring{M}_\pi^4)\right\}, \tag{8.2}$$

where μ_3 is the renormalization group invariant scale of the constant l_3. This expression shows that the expansion of the pion mass in powers of the quark masses m_u, m_d involves nonanalytic contributions [16]. The logarithmic term is proportional to $(m_u+m_d)^2 \log(m_u+m_d)$; the second derivative of M_π^2 with respect to the quark masses is infinite in the chiral limit. If one attempts to calculate M_π^2 directly by treating the quark mass term in the Lagrangian as a perturbation, one runs into an infrared divergence at second order. This divergence is due to the presence of massless particles (pions) in the unperturbed system and is by no means specific to the quark mass expansion of the pion mass. Similar logarithms show up in the quark mass expansion of the pion decay constant or of the scattering lengths (see below).

The (mass)^2 of the neutral pion differs from $M^2_{\pi^+}$ by a contribution of order $(m_u - m_d)^2$, measured by the low energy constant l_7:

$$M^2_{\pi^0} = M^2_{\pi^+} - (m_u - m_d)^2 \cdot \frac{2B}{F^2_\pi} \cdot l_7 + O(m^3). \tag{8.3}$$

The quark mass expansion of the pion decay constant reads [16]

$$F_\pi = \mathring{F}_\pi \left\{ 1 + \frac{\mathring{M}^2_\pi}{8\pi^2 \mathring{F}^2_\pi} \ln \frac{\mu_4}{\mathring{M}_\pi} + O(\mathring{M}^4_\pi) \right\}. \tag{8.4}$$

In this expression μ_4 stands for the renormalization group invariant scale of the low energy constant l_4. In some observables the chiral logarithms occur even in bare form, without a protecting factor of M^2_π. The electromagnetic charge radius, e.g., is given by

$$\langle r^2 \rangle_{\mathrm{e.m.}} = \frac{1}{8\pi^2 F^2_\pi} \left\{ \ln \frac{\mu_6}{M_\pi} - \tfrac{1}{2} \right\} + O(M^2_\pi), \tag{8.5}$$

where μ_6 is the renormalization group invariant scale of l_6. In the limit m_u, $m_d \to 0$ the charge radius tends to infinity [17]! It is not difficult to understand why this is so: in the chiral limit the pion cloud which surrounds any particle — in particular the cloud surrounding the pion — becomes long range, because there is no Yukawa factor $\exp(-M_\pi r)$ to cut it off. The charge distribution only falls off like a power of the distance, the mean square radius of the distribution diverges. The effect is even more dramatic in the scalar form factor

$$\langle \pi' | \bar{u}u + \bar{d}d | \pi \rangle = F_S(t). \tag{8.6}$$

The scalar radius which measures the slope of $F_S(t)$:

$$F_S(t) = F_S(0) \{ 1 + \tfrac{1}{6} \langle r^2 \rangle^\pi_S t + \ldots \} \tag{8.7}$$

contains a chiral logarithm with a coefficient that happens to be six times larger than the coefficient in the electromagnetic radius:

$$\langle r^2 \rangle^\pi_S = \frac{6}{8\pi^2 F^2_\pi} \left\{ \ln \frac{\mu_4}{M_\pi} - \tfrac{13}{24} \right\} + O(M^2_\pi). \tag{8.8}$$

This implies that the scalar radius is rather sensitive to the value of the pion mass. One therefore expects the slope of the analogous form factor $\langle \pi | \bar{u}s | K \rangle$ to be quite different, roughly

$$\langle r^2 \rangle^\pi_S \simeq \langle r^2 \rangle^{\pi K}_S + \frac{6}{8\pi^2 F^2_\pi} \ln \frac{M_K}{M_\pi}. \tag{8.9}$$

The radius $\langle r^2 \rangle^{\pi K}_S$ is measured in the decay $K \to \pi \mu \nu$:

$$\langle r^2 \rangle^{\pi K}_S = 6\lambda_0 = 0.23 \pm 0.05\ \mathrm{fm}^2. \tag{8.10}$$

The estimate (8.9) then implies $\langle r^2\rangle_S^\pi = 0.6$–$0.7\ \mathrm{fm}^2$. A more systematic analysis of the problem within SU(3)×SU(3) leads to [18]

$$\langle r^2\rangle_S^\pi = 0.6\pm 0.15\ \mathrm{fm}^2. \tag{8.11}$$

We thus predict very strong deviations from SU(3) in the scalar radii. Note that the value of the scalar radius determines the deviation of F_π from its value in the chiral limit (compare (8.4) and (8.8)):

$$F_\pi = \mathring{F}_\pi \left\{1+\tfrac{1}{6} M_\pi^2\langle r^2\rangle_S^\pi + \frac{13}{192\pi^2}\frac{M_\pi^2}{F_\pi^2} + O(M_\pi^4)\right\}. \tag{8.12}$$

With the value (8.11) we conclude that F_π must be larger than $\mathring{F}_\pi$ by about 6%.

Finally, I come to the predictions for the $\pi\pi$ scattering lengths which motivated our analysis. To leading order in the low energy expansion $[O(p^2)]$ only the S and P wave phase shifts are different from zero [6]. At order p^4 all partial waves receive a contribution. For those threshold parameters which are nonzero in leading order the low energy theorems of current algebra receive corrections of relative order M_π^2, for the remaining (infinite set of) threshold parameters we obtain new low energy theorems. Unfortunately, these predictions involve four new low energy constants: the renormalization group invariant scales μ_1, μ_2, μ_3, μ_4 of $l_1, \ldots, l_4$. The scales μ_1 and μ_2 are related to the constants a_1 and a_2 which appear in Lehmann's low energy representation of the chiral amplitude, cf. (6.2). These constants are measured in the D-wave scattering lengths:

$$a_2^0 = (720\pi^3F_\pi^4)^{-1}\left\{\ln\frac{\mu_1}{M_\pi} + 4\ln\frac{\mu_2}{M_\pi} - \tfrac{53}{16}\right\} + O(M_\pi^2),$$

$$a_2^2 = (720\pi^3F_\pi^4)^{-1}\left\{\ln\frac{\mu_1}{M_\pi} + \ln\frac{\mu_2}{M_\pi} - \tfrac{103}{80}\right\} + O(M_\pi^2). \tag{8.13}$$

As mentioned above, the constant μ_4 is related to the scalar radius of the pion. Since the fourth low energy constant μ_3 only appears in the S-wave scattering lengths, we may express the corrections to the current algebra theorems for a_1^1, b_0^0 and b_0^2 in terms of the D-wave scattering lengths and of the scalar radius. The improved low energy theorem for a_1^1 e.g. reads

$$a_1^1 = \frac{1}{24\pi F_\pi^2}\left\{1+\tfrac{1}{3} M_\pi^2\langle r^2\rangle_S^\pi + \tfrac{19}{576}\frac{M_\pi^2}{\pi^2F_\pi^2}\right\}$$
$$+\tfrac{10}{3} M_\pi^2(a_2^0 - \tfrac{5}{2} a_2^2) + O(M_\pi^4). \tag{8.14}$$

The corrections of order M_π^2 increase the value of a_1^1 from the current algebra prediction 0.030 to 0.037, to be compared with the experimental value 0.038 ± 0.002. The predictions for the parameters b_0^0, b_0^2 and for the combination $2a_0^0 - 5a_0^2$, which is also independent of μ_3, contain similar corrections — the predictions agree remarkably well with the data. In fact, the data allow one to turn it around and to use the observed threshold parameters to measure the scalar radius of the pion. We rewrite the improved low energy theorems for a_1^1, b_0^0, b_0^2 and $2a_0^0-5a_0^2$ in the form

$$1+\tfrac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi = 24\pi F_\pi^2\{a_1^1-\tfrac{10}{3}M_\pi^2(a_2^0-\tfrac{5}{2}a_2^2)\}-\tfrac{19}{576}\frac{M_\pi^2}{\pi^2F_\pi^2},$$

$$1+\tfrac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi = 4\pi F_\pi^2\{b_0^0-10M_\pi^2(a_2^0+5a_2^2)\}-\tfrac{39}{64}\frac{M_\pi^2}{\pi^2F_\pi^2},$$

$$1+\tfrac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi = -8\pi F_\pi^2\{b_0^2-10M_\pi^2(a_2^0+\tfrac{1}{2}a_2^2)\}+\tfrac{89}{320}\frac{M_\pi^2}{\pi^2F_\pi^2},$$

$$1+\tfrac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi = \frac{4\pi}{3}\frac{F_\pi^2}{M_\pi^2}\{2a_0^0-5a_0^2\}-\tfrac{41}{192}\frac{M_\pi^2}{\pi^2F_\pi^2}. \tag{8.15}$$

Inserting the values of the threshold parameters given by Petersen [9] one obtains the results quoted in the Table. For comparison I also give the corresponding lowest order predictions of current algebra — if there were no corrections of order M_π^2 the entries in the last column of the table should all be equal to one. Note that the value found for $1+\frac{1}{3}M^2\langle r^2\rangle_S^\pi$ is indeed somewhat larger than one in all four cases. The mean value of this quantity, $1.12\pm\pm0.04$, implies $\langle r^2\rangle_S^\pi = 0.7\pm0.2\ \text{fm}^2$ in good agreement with the SU(3)×SU(3) estimate $\langle r^2\rangle_S^\pi = 0.6\pm0.15\ \text{fm}^2$ given above.

TABLE

Scattering lengths interpreted as measurements of the scalar radius of the pion

Leading contribution	$1+\frac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi$	Soft pions
a_1^1	1.12 ± 0.11	1.28 ± 0.07
b_0^0	1.13 ± 0.19	1.40 ± 0.17
b_0^2	1.18 ± 0.10	0.92 ± 0.09
$2a_0^0-5a_0^2$	1.10 ± 0.05	1.15 ± 0.05
Mean value	1.12 ± 0.04	
Prediction	1.10 ± 0.02	1

The improved low energy theorem for the S-wave scattering length a_0^0:

$$a_0^0 = \frac{7}{32\pi}\frac{M_\pi^2}{F_\pi^2}\left\{1+\frac{5}{42\pi^2}\frac{M_\pi^2}{F_\pi^2}\left[\ln\frac{\mu_1}{M_\pi}+2\ln\frac{\mu_2}{M_\pi}-\tfrac{3}{8}\ln\frac{\mu_3}{M_\pi}\right.\right.$$
$$\left.\left.+\tfrac{21}{10}\ln\frac{\mu_4}{M_\pi}+\tfrac{21}{16}\right]+O(M_\pi^4)\right\} \tag{8.16}$$

contains the renormalization group invariant scales μ_1, μ_2, μ_3, μ_4 of all four low energy constants $l_1, \ldots, l_4$. To evaluate the corrections of order M_π^2 we therefore need additional information to pin down μ_3. A crude estimate may be obtained [9] on the basis of

SU(3) × SU(3); although this estimate does not lead to a very precise value for μ_3 the uncertainty affects the prediction for a_0^0 by less than 1%. The corrections of order M_π^2 increases the lowest order prediction ($a_0^0 = 0.16$) to

$$a_0^0 = 0.20 \tag{8.17}$$

to be compared with the experimental value $a_0^0 = 0.26 \pm 0.05$.

As mentioned above, the information contained in the representation of the scattering amplitude to order p^4 is not restricted to the S- and P-wave threshold parameters. The low energy representation fixes the scattering lengths of all partial waves to leading order in an expansion in powers of the quark masses. For the isospin one channel, e.g. one obtains [8] the following sequence of low energy theorems (l = 3, 5, ...).

$$a_l^l = \frac{(M_\pi)^{4-2l}}{512\pi^3 F_\pi^4} \frac{l!(l-3)!}{[(2l+1)!!]^2} (13l^2+5l-22)\{1+O(M_\pi^2)\}. \tag{8.18}$$

In contrast to the predictions for the S- and P-waves these low energy theorems only provide us with the leading terms in the quark mass expansion. As we have seen the corrections of relative order M_π^2 to the S- and P-wave threshold parameters are substantial and one should therefore expect sizeable corrections also to the above lowest order predictions for the higher partial waves.

Why are the corrections of order M_π^2 so large? A substantial part of these corrections is due to the fact that the low energy theorems of current algebra involve the value $\mathring{F}_\pi$ of the pion decay constant in the chiral limit rather than the physical value F_π. The difference is determined by the scalar radius $\langle r^2 \rangle_S^\pi$ (see (8.12)). The lowest order predictions are proportional to F_π^{-2} and hence systematically underestimate the result by $(F_\pi/\mathring{F}_\pi)^2 \simeq 1.13$. The reason why this effect amounts to a correction of order 13% (instead of one or two percent as suggested by the rule of thumb given in the first section) is that the scalar radius contains a chiral logarithm with a large coefficient — the correction is of order $M_\pi^2 \log M_\pi^2$ rather than of order M_π^2. In fact, the threshold parameters contain further chiral logarithms which do not come from $\langle r^2 \rangle_S^\pi$. To estimate the effect of these logarithmic contributions one simply replaces the four renormalization group invariant scales $\mu_1, \ldots, \mu_4$ by a common scale of order 1 GeV. One finds that these logarithmic contributions are indeed responsible for the bulk of the corrections: with the exception of the P-wave parameters a_1^1, b_1^1 the simple recipe $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$ GeV reproduces the improved low energy theorems rather well [8]. The bulk of the corrections therefore comes from the threshold region of the two pion cuts which is at the origin of these logarithmic contributions. In the case of the P-wave parameters a_1^1, b_1^1 the corrections are sizeable for a different reason: in this channel the vicinity of the ϱ-meson pole produces a sizeable enhancement (for details see [9]). The deviations from the lowest order predictions are much larger than indicated by the rule of thumb because the perturbation produced by the quark mass term is enhanced by small energy denominators: both the crossed channel thresholds and the ϱ-pole are close enough to produce substantial enhancements of this perturbation.

I conclude that the analysis reported here confirms the standard picture of spontaneously broken chiral symmetry. The low energy theorems associated with the hidden symmetry are borne out by experiment. In particular, the data on $\pi\pi$ scattering show clear evidence for the presence of the nonanalytic contributions required by chiral symmetry. Indeed the data allow one to measure the scalar radius of the pion, which (together with the D-wave scattering lengths) determines the deviations from the low energy theorems of current algebra.

Similar chiral symmetry breaking effects due to the long range nature of the meson clouds are present in all hadronic low energy parameters. To compare the relevant low energy theorems (η-decay, σ-term, ...) with data it is important to first check whether the lowest order predictions are stable with respect to higher order corrections. In particular, one needs to work out the nonanalytic contributions whose form is unambiguously determined by chiral symmetry.

In order for a lattice calculation to achieve the accuracy of the low energy expansion discussed in this paper, it must include the fermion determinant on a lattice whose size is large compared to M_π^{-1} (the nonanalytic contributions are due to $q\bar{q}$ pairs which walk away rather far from the source which creates them). In particular, one should not be worried if Monte Carlo calculations carried out in the quenched approximation tend to produce too high a value for the mass of the proton: the long range part of the meson cloud lowers the proton mass by about 140 MeV [2].

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REFERENCES

[1] S. Weinberg, *Phys. Rev. Lett.* **31**, 494 (1974).
[2] J. Gasser, H. Leutwyler, *Phys. Rep.* **87C**, 77 (1982).
[3] V. Baluni, *Phys. Rev.* **D19**, 2227 (1979); R. J. Crewther, P. Di Vecchia, G. Veneziano, E. Witten, *Phys. Lett.* **88B**, 123 (1979).
[4] H. Leutwyler, CERN preprint TH-3739.
[5] R. Dashen, *Phys. Rev.* **183**, 1245 (1969); R. Dashen, M. Weinstein, *Phys. Rev.* **183**, 1291 (1969); H. Pagels, *Phys. Rep.* **16C**, 219 (1979).
[6] S. Weinberg, *Phys. Rev. Lett.* **17**, 616 (1966).
[7] T. N. Truong, *Phys. Lett.* **99B**, 154 (1981); C. Roiesnel, T. N. Truong, *Nucl. Phys.* **B187**, 293 (1981). See also the contribution by T. N. Truong to this meeting.
[8] J. Gasser, H. Leutwyler, *Phys. Lett.* **125B**, 321, 325 (1983).
[9] J. Gasser, H. Leutwyler, CERN preprint TH-3689 (1983); *Ann. Phys.* (N.Y.), in print.
[10] S. L. Adler, W. A. Bardeen, *Phys. Rev.* **182**, 1517 (1969); W. A. Bardeen, *Phys. Rev.* **184**, 1848 (1969); J. Wess, B. Zumino, *Phys. Lett.* **37B**, 95 (1971).
[11] S. Coleman, J. Wess, B. Zumino, *Phys. Rev.* **177**, 2239 (1969); C. Callan, S. Coleman, J. Wess, B. Zumino, *Phys. Rev.* **177**, 2247 (1969).
[12] S. Weinberg, *Physica* **96A**, 327 (1979).
[13] M. Gell-Mann, R. J. Oakes, B. Renner, *Phys. Rev.* **175**, 2195 (1968); S. L. Glashow, S. Weinberg, *Phys. Rev. Lett.* **20**, 224 (1968).
14] H. Lehmann, *Acta Phys. Austriaca* Suppl. **11**, 139 (1973).

[15] T. Appelquist, C. Bernard, *Phys. Rev.* **D23**, 425 (1981); J. Honerkamp, *Nucl. Phys.* **B36**, 130 (1972).
[16] L.-F. Li, H. Pagels, *Phys. Rev. Lett.* **26**, 1204 (1971); P. Langacker, H. Pagels, *Phys. Rev.* **D8**, 4595 (1973).
[17] M. A. B. Beg, A. Zepeda, *Phys. Rev.* **D6**, 2912 (1972).
[18] J. Gasser, H. Leutwyler, to be published.
[19] M. M. Nagels et al., *Nucl. Phys.* **B147**, 189 (1979).