ON CHAOTIC BEHAVIOUR OF CLASSICAL YANG-MILLS MECHANICS

By A. Górski

Institute of Physics, Jagellonian University, Cracow*

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It is shown that pure Yang-Mills mechanics is exponentially unstable even if an arbitrary, stable A_0^a component of the potential is included. However, stabilization of the theory is possible for the generalized Matinyan-Savvidi Ansatz on potentials when Higgs fields with sufficiently large vacuum expectation value are present.

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1. Introduction

The classical Yang-Mills mechanics (YMM) and its stochastic properties have been investigated recently [1, 2]. Using numerical methods it has been proven that, at least for a special Ansatz, YMM exhibits chaotic behaviour [2]. However, introducing extra Higgs scalars with sufficiently large vacuum expectation value deterministic behaviour ("ordered phase") can be reached [3].

This paper deals with the linearized (L-) stability of YMM. Exponential L-instability is an important indication of stochasticity. A useful constructive criterion for the onset of chaos in dynamical systems was proposed by Brumer and Duff and Toda [4]. They have used much more tractable L-stability instead of the global one. This method has been tested in several models. Moreover, it does need neither the explicit knowledge of solutions to the equations of motion nor any numerical calculations. In the following a similar method will be used to analyze YMM and to extend previous results.

Now, let us comment on the relation between chaos and exponential L-instability in dynamical systems. Usually a system is called chaotic (stochastic, random) when its metric entropy is positive: $h > 0$. The metric entropy is defined as $h \equiv \sum_n \lambda_+^n$, λ_+^n being a positive Lyapunov exponents[1]. Here, Lyapunov exponents are defined by

$$\lambda = \lim_{t\to\infty} \frac{\ln [\varrho(t)/\varrho(0)]}{t},$$

* Address: Instytut Fizyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland.

[1] For hamiltonian systems we have pairs of Lyapunov exponents $\lambda_\pm \equiv \pm\sqrt{\lambda^2}$.

where $\varrho^2 \equiv \delta A^2 + \delta \dot{A}^2$ and δA is a perturbation of the trajectory, dot means differentiation with respect to the time variable.

The model of YMM is extracted from the full Yang–Mills equations imposing the following Ansatz on the potentials[2]

$$\partial_k A^a_\mu = 0, \tag{1.1}$$

i.e. all gradient terms are assumed to vanish, fields are space-homogeneous. Let us recall that gradients of the potentials are not included in the most nonlinear (cubic) terms of the YM equations.

Whereas in the weak coupling (field) limit YM equations become almost linear, with small nonlinear perturbation only, we have, for homogeneous fields (1.1), highly nonlinear equations and strong self-coupling of potentials. This is just the case of YMM. Intuitively, this region can be defined as: $g^2\varrho\lambda^3 \gg 1$, where ϱ is the "gluon" density and λ is the wavelength. Hence, Ansatz (1.1) can also be associated with the infrared limit of the theory ($\partial_k \leftrightarrow p_k = 0$).

Ansatz (1.1) reduces the YM equations to a set of ordinary differential equations of the form

$$\ddot{A}^a{}_i + g\varepsilon^{abc}[\dot{A}^b{}_0 A^c{}_i + 2A^b{}_0 \dot{A}^c{}_i] + g^2\varepsilon^{abc}\varepsilon^{cde}[A^b{}_0 A^d{}_0 A^e{}_i - A^b{}_k A^d{}_k A^e{}_i] = gj^a{}_i, \tag{1.2}$$

$$\varepsilon^{abc} A^b{}_k \dot{A}^c{}_k - g\varepsilon^{abc}\varepsilon^{cde} A^b{}_k A^d{}_k A^e{}_0 = j^a{}_0, \tag{1.3}$$

where ε^{abc} is the totally antisymmetric symbol.

2. *Onset of chaos in YMM*

It has been shown that the equations of motion (1.2) exhibit stochastic behaviour when an extra restrictions on the potentials are imposed. They are of the form

$$A^a{}_0 \equiv 0, \tag{2.1}$$

and

$$A^a{}_k = O^a{}_k f^{(a)}(t), \tag{2.2}$$

where $O^a{}_k$ is an orthogonal constant matrix and there is no summation over the index a. Ansatz (2.2) may be viewed as a 3-dimensional section of the 9-dimensional space of solutions to equation (1.2). We shall show that for another 3-dimensional section of the form

$$A^a{}_k = \delta_{k3} A^a, \tag{2.3}$$

equations (1.3) are also unstable, even though $A^a{}_0$ can be nonzero, i.e. (2.1) is relaxed[3]. This means that the potentials $A^a{}_0$ are not able to stabilize YMM, even if they are treated

[2] Here Latin indices $a, b, c, \ldots$ are group indices, $i, j, k, \ldots$ are space indices and $\mu, \nu, \alpha, \ldots$ are Minkowski indices. The metric tensor signature is $(+---)$ and the gauge group is SU(2).

[3] Note that (2.1) cannot be obtained by a gauge transformation since the condition (2.3) is rather restrictive and no gauge freedom remains.

as being stable external fields, i.e. A^a_0 will not be subjected to any perturbations:

$$\delta A^a_0 = 0. \tag{2.4}$$

Substituting (2.3) to (1.2) and disturbing the potential: $A^a_k \to A^a_k + \delta A^a_k$ with $\delta A^a_k \sim \exp(\lambda t)$, the following algebraic equation for the eigenvalues λ (Lyapunov exponents) is obtained

$$\det |\lambda^2 - 2g\lambda S - g\dot{S} + g^2 S^2| = 0. \tag{2.5}$$

Here, the 3×3 matrix S is defined as

$$S^{ab} \equiv \varepsilon^{acb} A^c_0. \tag{2.6}$$

Equation (2.5) has the explicit form

$$f(\lambda) \equiv \lambda^6 + 2g^2\vec{A}_0{}^2\lambda^4 + 4g^2(\vec{A}_0 \cdot \dot{\vec{A}}_0)\lambda^3$$
$$+ g^2(\dot{\vec{A}}^2 + g^2\vec{A}_0{}^4)\lambda^2 - g^4[\vec{A}_0{}^2\dot{\vec{A}}_0{}^2 - (\vec{A}_0 \cdot \dot{\vec{A}}_0)^2] = 0. \tag{2.7}$$

Note that A^a_k potentials are not involved in (2.7), thanks to the Ansatz (2.3).

For non-Abelian $\vec{A}_0$ potentials, i.e. those satisfying the condition

$$\vec{A}_0 \times \dot{\vec{A}}_0 \neq 0, \tag{2.8}$$

the last term in (2.7) is negative. Then $f(\lambda = 0) < 0$ and for large enough $\lambda \in \mathrm{R}$ we have $f(\lambda) > 0$. Because of this there exists a positive real root of equation (2.7). Hence, the disturbance δA^a_k is growing exponentially with time and all non-trivial solutions of the form (2.3) are unstable.

In the special case, when the condition (2.8) is not satisfied, we have two zero solutions to (2.5) and the remaining four roots have to obey the equation

$$\lambda^4 + 2g^2\vec{A}_0{}^2\lambda^2 + 4g^2(\vec{A}_0 \cdot \dot{\vec{A}}_0)\lambda + g^2(\dot{\vec{A}}_0{}^2 + g^2\vec{A}_0{}^4) = 0. \tag{2.9}$$

This equation has no real solutions. Defining

$$\lambda \equiv a + ib, \qquad a, b \in \mathrm{R}, \tag{2.10}$$

we have, instead of (2.9), two equations for real and imaginary parts of λ:

$$g(a) \equiv a^4 + 2g^2\vec{A}_0{}^2a^2 - (\vec{A}_0\dot{\vec{A}}_0)a - \tfrac{1}{4}[g^2\dot{\vec{A}}_0{}^2 + g^4\vec{A}_0{}^4] = 0, \tag{2.11a}$$

$$b^2 = a^2 + 2g^2\vec{A}_0{}^2. \tag{2.11b}$$

Hence, $g(a = 0) < 0$ whereas, for large enough a, $g(a) > 0$, i.e. there exists a solution for a such that

$$a \equiv \operatorname{Re}\lambda > 0. \tag{2.12}$$

Thus we have proved that the solution (2.3) is always unstable, whatever the potential $\vec{A}_0$ is. In the following we investigate the cases in which YMM can be stabilized.

3. Stabilization of YMM

It has been shown that YMM can be put in ordered phase when extra Higgs fields with sufficiently large vacuum expectation value are introduced [3]. This has been done numerically with the assumptions (2.1), (2.2) and additional Matinyan–Savvidi (MS) Ansatz:

$$f^{(3)} \equiv 0. \tag{3.1}$$

The equations of motion in the presence of Higgs fields result from the Hamiltonian

$$H = H_{\text{YM}} + \tfrac{1}{2}(\dot{B}_a{}^2 + \dot{\sigma}^2) + \frac{g^4}{4}(A_i^a A_i^a)\left[\frac{B_a{}^2}{2} + \left(\frac{\sigma}{\sqrt{2}} + \eta\right)^2\right]$$
$$+\lambda^2\left[\tfrac{1}{2}B_a{}^2 + \left(\frac{\sigma}{\sqrt{2}} + \eta\right)^2 - \eta^2\right], \tag{3.2}$$

where the Higgs field ϕ is defined as

$$\phi \equiv \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} iB_1 + B_2 \\ \sqrt{2}\,\eta + \sigma - iB_3\end{pmatrix}, \tag{3.3}$$

and H_{YM} is the pure YM part of the Hamiltonian.

The constraint equations, analogous to (1.3), are

$$\varepsilon^{abc}A^b{}_i\dot{A}^c{}_i - \frac{\eta}{\sqrt{2}}\dot{B}_a + \tfrac{1}{2}[\sigma\dot{B}_a - B_a\dot{\sigma} - \varepsilon^{abc}B_b\dot{B}_c] = 0. \tag{3.4}$$

Here, it will be shown that YMM becomes L-stable for sufficiently large vacuum expectation value η of the Higgs field for generalized MS Ansatz, i.e. when the restriction (3.1) is relaxed.

Using the notation $f^{(1)} \equiv x, f^{(2)} \equiv y, f^{(3)} \equiv z$ and imposing on the Higgs field the condition

$$B_a = \sigma = 0, \tag{3.5}$$

we have the field equations in the form

$$\ddot{x} + g^2x(y^2+z^2) + \tfrac{1}{2}g^2\eta^2 x = 0,$$
$$\ddot{y} + g^2y(x^2+y^2) + \tfrac{1}{2}g^2\eta^2 y = 0,$$
$$\ddot{z} + g^2z(x^2+y^2) + \tfrac{1}{2}g^2\eta^2 z = 0, \tag{3.6}$$

and the Hamiltonian

$$H = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \tfrac{1}{2}g^2(x^2y^2+x^2z^2+y^2z^2) + \tfrac{1}{4}g^2\eta^2(x^2+y^2+z^2). \tag{3.7}$$

It can be seen easily that after appropriate rescalling we have in the theory only one dimensionless parameter

$$\pi \equiv \frac{g^2}{4}\left(\frac{\eta}{\mu}\right)^4, \tag{3.8}$$

where μ^4 is defined as the Hamiltonian.

The linearized equations for perturbations are

$$\delta\ddot{x}+g^2\left(y^2+z^2+\frac{\eta^2}{2}\right)\delta x+2g^2xy\delta y+2g^2xz\delta z = 0,$$

$$\delta\ddot{y}+2g^2xy\delta x+g^2\left(x^2+z^2+\frac{\eta^2}{2}\right)\delta y+2g^2yz\delta z = 0,$$

$$\delta\ddot{z}+2g^2xz\delta x+2g^2yz\delta y+g^2\left(x^2+y^2+\frac{\eta^2}{2}\right)\delta z = 0. \tag{3.9}$$

This implies the following algebraic equation for the eigenvalues λ:

$$\begin{aligned}&\lambda^6/g^6+2(x^2+y^2+z^2+\tfrac{3}{4}\eta^2)\lambda^4/g^4+[\tfrac{3}{4}\eta^4+2(x^2+y^2+z^2)\eta^2\\&+x^4+y^4+z^4-(x^2y^2+x^2z^2+y^2z^2)]\lambda^2/g^2+[\tfrac{1}{8}\eta^6+\tfrac{1}{2}(x^2+y^2+z^2)\eta^4\\&+\tfrac{1}{2}(x^4+y^4+z^4-x^2y^2-x^2z^2-y^2z^2)\eta^2+18x^2y^2z^2-3x^4(y^2+z^2)\\&-3y^4(x^2+z^2)-3z^4(x^2+y^2)] = 0.\end{aligned} \tag{3.10}$$

The condition for L-stability is that all solutions to (3.10) should satisfy [5]

$$\mathrm{Re}\,\lambda \leqslant 0. \tag{3.11$'$}$$

For equation (3.10) condition (3.11′) is satisfied if and only if

$$\mathrm{Re}\,\lambda = 0 \tag{3.11}$$

i.e.

$$\mathrm{Re}\,\lambda^2 \leqslant 0. \tag{3.12}$$

Equation (3.10) can be written as

$$a_0\Lambda^3+a_1\Lambda^2+a_2\Lambda+a_3 = 0, \tag{3.13}$$

where $\Lambda \equiv \lambda^2$ and a_i are the coefficients defined by (3.10). It is easy to check that the coefficients a_0, a_1, a_2 are positive. To investigate the real part of λ (i.e. to check the condition (3.12)) we use the Hurwitz criterion [6]. According to this criterion all solutions to (3.13) have negative real parts if and only if

$$a_0 > 0, \quad a_1 > 0, \tag{3.14a}$$

and

$$\Delta_2 \equiv \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} < 0, \quad a_3\Delta_2 > 0. \tag{3.14b}$$

Hence we have to examine the conditions (3.14b). To satisfy them there should be:

$$\Delta_2 \equiv \eta^6+4r^2\eta^4+(5r^2-3\delta^4)\eta^2+2r^6-3r^2\delta^4-27x^2y^2z^2 > 0, \tag{3.15a}$$

and

$$8a_3 \equiv \eta^6+4r^2\eta^4+4(r^4-3\delta^4)\eta^2+8[-3r^2\delta^4+27x^2y^2z^2] > 0, \tag{3.15b}$$

where r and δ are defined by

$$r^2 \equiv x^2+y^2+z^2, \tag{3.16a}$$

$$\delta^4 \equiv x^2y^2+x^2z^2+y^2z^2. \tag{3.16b}$$

Inequalities (3.15) are third order algebraic inequalities with respect to $\eta^2 \in \mathbf{R}$ ($\eta^2 \geqslant 0$). Due to the obvious relation

$$r^4-3\delta^4 \geqslant 0 \tag{3.17}$$

all coefficients in (3.15), except the free terms, are positive. Hence, the sufficient conditions to satisfy (3.14) are

$$4r^2\eta^4+2r^6-3r^2\delta^4-27x^2y^2z^2 > 0, \tag{3.18a}$$

and

$$4r^2\eta^4-3r^2\delta^4 > 0. \tag{3.18b}$$

Now, we have to use the inequalities implied by the form of the Hamiltonian μ^4. From (3.7) we have the condition on δ:

$$\delta^4 \leqslant \tfrac{1}{2}\left(\frac{2}{g}\right)^2 \mu^4, \tag{3.19}$$

Using (3.19) we can see that (3.18) is surely satisfied if

$$4r^2\eta^4-9r^2\,\tfrac{1}{2}\left(\frac{2}{g}\right)^2 \mu^4 \geqslant 0, \tag{3.20a}$$

and

$$4r^2\eta^4-3r^2\,\tfrac{1}{2}\left(\frac{2}{g}\right)^2 \mu^4 \geqslant 0. \tag{3.20b}$$

Taking into account (3.8) we have the following condition on the dimensionless parameter π, to obey (3.14),

$$\pi \geqslant 3 \tag{3.21'}$$

and, in effect,

$$\pi_{crit} < 3. \tag{3.21}$$

Hence for large enough values of the parameter π, the YMM can be stabilized by the Higgs fields. The same inequalities (3.20) can also be used in the special case when Ansatz (3.1) is imposed ($z = 0$). In this case transition to the ordered phase was established in the computer experiment for the critical value of the parameter π [3]:

$$\pi_{crit} \approx 0.2 \tag{3.22}$$

This value is consistent with our result (3.21). The inequality (3.21) is rather crude because of the rough estimations (3.18) and (3.19). However, our analysis is independent of the form of solutions to the equations of motion (3.6).

Let us mention that for dynamical (i.e. time-dependent) Higgs fields and one-component YM potential qualitatively similar numerical results were obtained[4].

4. *Conclusions*

In this paper L-stability of YMM has been analyzed. It has been shown that pure YMM is exponentially L-unstable. Even arbitrary temporal component of the potential, $A^a{}_0$, treated as a stable external field, is not able to stabilize the space components of the potential. However, YMM can become stable if extra Higgs fields are introduced, with sufficiently large vacuum expectation values. The transition to the stable phase has been shown to occure at $\pi = \pi_{crit} < 3$. This is consistent with, and extends, the results obtained recently by several authors [2, 3], where transition from stochastic to deterministic phase was observed in computer experiments.

These results suggest that Higgs fields are necessary to have ordered (stable) phase in the theory. On the other hand, beyond the Higgs phase gauge fields, and in particular their vacuum field configuration, behave stochastically. This seems to be encouraging conclusion, especially with regard to the recent results, where randomness of the field has been suggested to be relevant to the confinement phenomenon [7].

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