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TOPONIUM PROPERTIES IN THE COULOMB-LINEAR POTENTIAL MODEL

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The fit to charmonium and bottomium mass spectra by the same Coulomb-linear potential of quark-antiquark interactions is presented. The predictions of mass spectrum and leptonic half-widths of the hypothetical toponium system are given for a wide range of the top quark mass.

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1. Introduction

Many potential models of quarkonia have been constructed (see e.g. [1–5]), and they turned out to be very successful in the fitting of the mass spectium, decay half-widths, transition rates etc. of charmonium (ψ) and bottomium (Υ) families. It was achieved with the help of solutions of Schroedinger equation for various types of potentials. The use of the non-relativistic Schroedinger equation not only for heavy, and hence only slightly relativistic quarkonia, but also for the lighter ones is justified by the fact that relativistic corrections are taken automatically into account by regarding the effective masses of quarks as free, phenomenological parameters (wich is consistent with the confinement, i.e.non--observability of free quarks, [6]). It was also shown that relativistic corrections cancel each other appreciably if the potential contains a negative constant part [7].

The most popular forms of the potential of interquark gluon interaction are:

(*i*) the Coulomb-linear potential [1, 2]

$$V(r) = -\frac{A}{r} + Br + C, \tag{1}$$

the first part describing short-distance, one-gluon exchange and the second part describing quark confinement at large distances,

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(*ii*) the logarithmic potential [3]

$$V(r) = C \ln \frac{r}{r_0}, \tag{2}$$

less singular at $r = 0$, predicting the independence of the mass spectrum of the quark mass;
(*iii*) the power-law potential [4]

$$V(r) = Ar^{\nu} + B, \tag{3}$$

where $\nu \cong 0.1$, unmotivated theoretically but offering a very good agreement with the observed spectra [5];
(*iv*) more complicated forms of the potential, inspired by QCD, including the logarithmic correction to the short-distance Coulomb term [8].

The possible refinement of various models, as well as taking into account the spin-orbit, hyperfine splittings etc., may lead to a small improvement in the case of charmonium and bottomium modelling. However, it is not necessary in the case of toponium (ζ) — hypothetical, very heavy quarkonium with hidden "top" quantum number, because we are interested in the approximate positions of energy levels in the spectrum and estimates of the decay widths which in turn are sensitive to the shape of the leading short-distance term of the potential. The potentials (*i*)-(*iv*) give for the mass of the top quark $m_t \gtrsim 20$ GeV very different predictions. This offers a possibility of deciding between the models after the anticipated discovery of the toponium.

In the present paper we consider the Coulomb-linear potential (*i*). This kind of potential is supported by the newest results in SU(2) lattice gauge theory [9]. The experimental data rule out the possibility that $m_{(\bar{t}t)} < 38.54$ GeV [10], and therefore we restrict ourselves to the interval $m_t \in (21 \text{ GeV}, 30 \text{ GeV})$ and construct a grid of models (see Chapter 3) with the values of parameters obtained from the fit to ψ and Υ quarkonia (see Chapter 2).

2. *Fitting of ψ and Υ families*

In our approach we attempt a simultaneous fit to the data, i.e. we assume the flavour independence of the gluon interactions and subsequently check the consistency of this assumption. The method of solution of the radial Schroedinger equation is analogous to that presented in [1]. The eigenvalues of binding energy computed by both methods coincide to within 0.1%, which is a fairly good test of the numerical procedure. For a given set of parameters: A, B, C, m_q, we obtain the binding energy E for both radially and non-radially excited states of quarkonium as well as the shape of the wave function and wave function squared at $r = 0$ for radial excitations (S-states). The binding energy gives the mass of the particular n-th state

$$M_n = 2m_q + E_n \tag{4}$$

and the wave function squared gives the leptonic width of S-states through the Van Royen-Weisskopf formula

$$\Gamma_{ee}(nS) = \frac{16\pi e_q^2 \alpha^2}{M_n^2} |\psi_{nS}(0)|^2, \tag{5}$$

where $e_q = 1/3$ or $2/3$ is the quark electric charge. It is known that the above non-relativistic formula is not generally applicable [2] and in the case of potential (1) gives the answer wrong by a constant factor of roughly 3. This is the reason for taking Γ_{ee} for charmonium ground state as an input quantity in our model.

The approximate, guiding fit can be obtained with the help of WKB approximation which in our case reduces to the search for values of E satisfying the Bohr-Sommerfeld quantization rule

$$\oint \sqrt{m_q(E - V_{eff}(r))}\, dr = 2\pi\hbar(n+\tfrac{1}{2}), \qquad n = 0, 1, 2, \ldots \tag{6}$$

where the integral is over classically allowed region of motion ($E \geqslant V_{eff}$), and V_{eff} includes the centrifugal barrier [11]. Surprising is the high accuracy of the WKB approximation: it reproduces the energy levels in Coulomb-linear potential with an error less than 5 MeV for the ground state and less than 2 MeV for all the other states, to be compared with the characteristic energy spacing of the order of 1 GeV.

The values of parameters of potential (1) and effective quark masses which give the best fit to the observed ψ and Υ spectroscopy of triplet states ([12]) are as follows:

$$m = 1.65\ \text{GeV}, \qquad m = 5.04\ \text{GeV}$$

$$A = 0.470, \qquad B = 0.185\ \text{GeV}^2, \qquad C = -0.542\ \text{GeV}. \tag{7}$$

The value of $B = 0.937$ GeV/fm is very close to that obtained by other investigators (e.g. $B = 0.93$ in [2]), and to the predictions of lattice formulation of QCD in which $B \simeq 1$ GeV/fm is the string tension [13]. The value of A is in agreement with other published models. The coupling constant of strong interactions α_s, related to A through the relation $A = 4/3\,\alpha_s$, turns out $\alpha_s = 0.35$.

TABLE I

The comparison of the experimental data and the best theoretical fit to these data. Underlined are the input quantities. The asterisks denote the broad states above the charm (bottom) threshold. In the case of 2P triplet the C.O.G. energy is quoted

State	Design.	Experiment		Theory		$\frac{M_{th}-M}{M}$ (%)
		M (GeV)	Γ_{ee} (keV)	M_{th} (GeV)	Γ_{ee} (keV)	
1S	ψ	3.0969	4.80	3.0969	4.80	—
2S	ψ'	3.6860	1.96	3.6856	2.33	−0.011
3S*	ψ (4160)	4.1590	0.78	4.1195	1.63	−0.95
2P	χ_0, χ_1, χ_2	3.5219		3.5189		−0.086
3D	ψ''	3.7700		3.7990		+0.77
1S	Υ	9.4560	1.30	9.4560	1.30	—
2S	Υ'	10.016	0.51	10.012	0.52	−0.041
3S	Υ''	10.347	0.43	10.351	0.37	+0.042
4S*	Υ'''	10.569	0.28	10.627	0.31	+0.55

Table I presents the results of the simultaneous fit. It should be noted that for the masses of all the observed ψ and Υ states the agreement with the theory is better than 1% and for the states well below the threshold, i.e. the energy allowing the decay into mesons with open flavour, the agreement is better than 0.1%, which is a satisfying result.

As a conclusion we see that a unique potential (1) explains the features of both quarkonia, so the approximate flavour independence of the quark-antiquark potential, established in several ways [14], has been tested again.

3. *Properties of toponium*

The well known feature of the Coulomb-linear potential is that the excitation energy first decreases and then increases with the increasing m_q. The minimum for each (radial or non-radial) particular state is placed in the region from 3 to 10 GeV ([2]), so that in the region of our interest $m_t > 21$ GeV the energy levels rise up almost linearly with increasing m_t and quadratic corrections are negligible. The immediate reason for the linear behaviour of both masses and leptonic decay widths is that such behaviour is characteristic for the dominating Coulomb-part of potential, as can be shown by means of WKB approximation ([15])

$$M_n = 2m_t + \text{const}\,(n, l)m_t \sim m_t, \tag{8}$$

$$|\psi_n(0)|^2 = \text{const}\,(n, l)m_t^3 \tag{9}$$

and hence

$$\Gamma_{ee}(nS) = \text{const} \cdot \frac{|\psi_n(0)|^2}{M_n^2} \sim m_t. \tag{10}$$

Thus, it is not necessary to present the results of all the calculated models but only two of them, with top quark mass of 21 and 25 GeV (see Table II). The energy levels and half-widths for 21 GeV $< m_t <$ 30 GeV can easily be obtained by interpolation. We have added to these models additional two with the changed value of the coupling constant, namely with $A = 0.44$, to account for the possibility of the "running coupling constant", which according to QCD should decrease with the growing energy scale. It has been proved that for $0.44 < A < 0.47$ the linear interpolation of the data contained in Table II offers a sufficient accuracy in determination of toponium properties.

The last important prediction concerning toponium is the number of narrow, observable resonances below the top threshold, decaying only through the 2 or 3 gluon annihilation. The energy threshold for the production of mesons with nonvanishing top quantum number with respect to the ground-state energy is given by

$$E_{\text{thr.}} = 2M_{t\bar{u}} - M_{t\bar{t}}, \tag{11}$$

where the subscripts denote the quark content of each meson. The problem in calculating masses of light mesons (the reduced mass is close to the mass of u or d quark) is that these systems are highly relativistic. It can be shown that one can properly reconstruct both

TABLE II

The mass spectra and leptonic half-widths of toponium for 4 parameter sets. $B = 0.184\ \text{GeV}^2$, $C = -0.542$ GeV. The masses of mesons are in GeV, and the half-widths in keV

State	$m = 21$ GeV, $A = 0.47$		$m = 25$ GeV, $A = 0.47$		$m = 21$ GeV, $A = 0.44$		$m = 25$ GeV, $A = 0.44$	
	M	Γ_{ee}	M	Γ_{ee}	M	Γ_{ee}	M	Γ_{ee}
1S	40.352	13.23	48.126	15.48	40.500	10.91	48.298	12.85
2S	41.372	2.31	49.288	2.51	51.414	1.99	49.337	2.12
3S	41.701	1.26	49.637	1.26	41.732	1.13	49.667	1.13
4S	41.924	0.91	49.861	0.90	41.948	0.87	49.882	0.82
5S	42.108	0.75	50.037	0.74	42.126	0.72	50.055	0.68
6S	42.269	0.67	50.189	0.64	42.280	0.63	50.205	0.59
7S	42.407	0.62	50.326	0.58	42.421	0.57	50.341	0.54
2P	41.331		49.258		41.375		49.305	
3P	41.669		49.606		41.695		49.635	
4P	41.894		49.832		41.913		49.852	
5P	42.075		50.009		42.092		50.027	
6P	42.233		50.161		42.248		50.177	
7P	42.376		50.300		42.389		50.314	
3D	41.615		49.601		41.641		49.587	
4D	41.846		49.786		41.864		49.808	
5D	42.029		49.967		42.044		49.984	
6D	42.188		50.122		42.204		50.138	
7D	42.335		50.261		42.347		50.275	

the masses of light mesons with quark content uu, dd, ss, etc., and the experimental values of E in charmonium and bottomium, if in the case of relativistic systems the potential (1) contains an additional piece $V_0 = -0.55$ GeV. Within this approach, for values of m_c, m_b, B, C, listed in (7) we obtain

$$E_{thr} \cong 0.793 + (0.23A - 0.0506)m_t, \tag{12}$$

where masses and energies are in GeV.

In particular, in our standard model, $A = 0.47$ and from the Table II we can infer that in the whole range $m_t \in (21\ \text{GeV}, 30\ \text{GeV})$ below the top threshold there are 6 P-states, 5 D-states, and 6 (7) S-states, for m_t smaller (greater) than about 22.5 GeV.

4. *Conclusions*

We have presented the model of quarkonia based on the Coulomb-linear confining potential and non-relativistic Schroedinger equation. The single potential reproduces the spectroscopy of charmonium and bottomium systems with an accuracy of 1%. The values of model parameters obtained from the fit are consistent with those of other Coulomb-linear models (see e.g. [2]). They are used to predict the mass spectrum and the leptonic half-widths of the hypothetical toponium system for a wide range of the top quark mass. 5 D-states, 6 P-states, and 6 or 7 S-states of toponium are expected to exist below the top threshold.

The discovery of toponium is crucial for the determination of the short distance part of the potential of quark-antiquark interaction and for the question of the validity of the Van Royen-Weisskopf formula.

All tne computations were done on the PDP 11/45 computer at N. Copernicus Astronomical Center in Warsaw.

Note added in proof. The recently discovered (K. Han et al., *Phys. Rev. Lett.* **49**, 1612 (1982), C. Klopfenstein et al., ibid. **51**, 160 (1983)) non-radial excitations of bottomium are reproduced very satisfactorily by our model with parameters given by (7). For the 2P triplet the predicted C.O.G. mass is 9918 GeV (experimental value of 10256 GeV). This gives the relative errors +0.2% and +0.1% respectively, comparable to the fit quality for other charmonium and bottomium states.

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