POSSIBLE SUPERKINEMATICS

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The kinematical groups, classified by Bacry and Levy-Leblond (*J. Math. Phys.* **9**, 1605 (1967)), are extended to the supersymmetric ones with one Majorana bispinor generator. The problem of the positive definiteness of the norm in the representation space of the introduced supergroups is discussed.

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In the paper by Bacry and Levy-Leblond [1] the possible ten parameter kinematical groups are classified. They include the space-time translations, spatial rotations and inertial transformations connecting different inertial frames of reference. The kinematical groups are ordered by the contraction procedure which enables us to extract their physical meaning [1]. The classification has been obtained under the following assumptions
(*i*) space is isotropic (rotation invariance);
(*ii*) parity and time reversal are automorphism of the kinematical groups;
(*iii*) inertial transformations in any given direction form a noncompact one-parameter subgroup (weak causality requirement).
The generators H, $\vec{P}$ (space-time translations), $\vec{J}$ (rotations) and $\vec{K}$ (boosts) satisfies the basic commutation rules

$$[J_k, J_l] = i\varepsilon_{klm}J_m, \quad [J_k, P_l] = i\varepsilon_{klm}P_m,$$
$$[J_k, H] = 0, \quad [J_k, K_l] = i\varepsilon_{klm}K_m,$$
$$[H, P_k] = i\alpha K_k, \quad [H, K_k] = i\lambda P_k, \quad [P_k, K_l] = i\varphi\delta_{kl}H,$$
$$[P_k, P_l] = i\beta\varepsilon_{klm}J_m, \quad [K_k, K_l] = i\mu\varepsilon_{klm}J_m.$$

Here the values of the real parameters α, β, λ, μ, φ define eleven kinematical Lie algebras; they are quoted in Table I. The exhaustive discussion and physical interpretation of these kinematical groups is given in Ref. [1]. Let us point out some results of that discussion.

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TABLE I

Kinematical group		α	β	λ	μ	φ
de Sitter	SO(3, 2)	−1	−1	1	−1	1
de Sitter	SO(4, 1)	1	1	1	−1	1
Poincaré	P	0	0	1	−1	1
Para-Poincaré	P′	−1	−1	0	0	1
ISO(4)		1	1	0	0	1
Carroll	C	0	0	0	0	1
Expanding Universe	N+	1	0	1	0	0
Oscillating Universe	N−	−1	0	1	0	0
Galilei	G	0	0	1	0	0
Para-Galilei	G′	1	0	0	0	0
Static	S	0	0	0	0	0

(a) The kinematical algebras split in two classes: relativistic (relative-time) algebras with $\varphi = 1$ and nonrelativistic (absolute-time) ones with $\varphi = 0$. The latter admit the nontrivial central extension by one-dimensional algebra:

$$[P_i, K_j] = 0 \rightarrow [P_i, K_j] = -i\delta_{ij}I.$$

Note that for quantum-mechanical applications the projective representations of the symmetry groups are of interest.

(b) All the eleven kinematics can be regarded as belonging to two families

1. consisting of the de Sitter group SO(4, 1) and its rotation-invariant contractions (ISO(4), P, C, N+, G, G′, S);

2. consisting of the de Sitter group SO(3, 2) and its rotation-invariant contractions (P′, P, C, N−, G, G′, S).

In the present paper we discuss the extensions of the above kinematical groups to supersymmetric ones with one Majorana spinor generator Q_α. According to the isotropy assumption (*i*)

$$[\vec{J}, Q] = \tfrac{1}{2}\gamma^0\gamma^5\vec{\gamma}Q.$$

To solve the problem we follow the straightforward method which consists in

(a) writting the unknown commutators and anticommutators as linear forms of H, $\vec{P}$, $\vec{J}$, $\vec{K}$, Q;

(b) taking into account the assumption of parity (***P***) and time (*T*) invariance;

(c) imposing the generalized Jacobi identities [2]

$$[X, [Y, Q]]+[Y, [Q, X]]+[Q, [X, Y]] = 0,$$

$$[X\{Q, Q'\}]+\{[Q, X], Q'\}+\{[Q', X], Q\} = 0,$$

$$[Q, \{Q', Q''\}]+[Q', \{Q'', Q\}]+[Q'', \{Q, Q'\}] = 0,$$

where X, Y are bosonic generators. To do this we adopt the following form of the ***P*** and ***T*** transformation rules for real Majorana spinors

$$P^{-1}QP = i\gamma_0 Q, \quad T^{-1}QT = \gamma_0\gamma_5 Q.$$

Here we use the Majorana representation for γ-matrices ($\vec{\gamma}$ – symmetric imaginary, γ^0 – antisymmetric imaginary and γ^5 – antisymmetric imaginary). Remember also that the physical T-inversion is antiunitary in the space of states, that is it realizes as an anti-automorphism of the Lie superalgebra. After steps (a) and (b) we have

$$[H, Q] = h\gamma^0 Q, \quad [\vec{P}, Q] = p\vec{\gamma}Q, \quad [\vec{K}, Q] = ik\gamma^0\vec{\gamma}Q,$$
$$\{Q_\alpha, Q_\beta\} = a\delta_{\alpha\beta}H + b(\gamma^0\vec{\gamma})_{\alpha\beta}\vec{P} + ic\vec{\gamma}_{\alpha\beta}\vec{K} + d(\gamma^5\vec{\gamma})_{\alpha\beta}\vec{J},$$

where a, b, c, d, h, p and k are real. Using the Jacobi identities we obtain the following constraints

$$\begin{aligned}
4p^2+\beta &= 0, & \lambda c+2bh &= 0,\\
4k^2+\mu &= 0, & \lambda a+2bk &= 0,\\
\varphi h+2pk &= 0, & \varphi c+2ap &= 0,\\
\lambda p+2hk &= 0, & \varphi b+2ak &= 0,\\
\alpha k-2ph &= 0, & d-2bp &= 0,\\
\alpha b-2ch &= 0, & d-2hc &= 0,\\
\alpha a-2cp &= 0, & d-2ah &= 0.
\end{aligned}$$

From $4p^2+\beta = 0$ it is obvious that there is no extension of the de Sitter SO(4, 1) group and the ISO(4) one accordingly to the common expectations [3]. The extensions of other groups are summarized in Table II. If the extensions with $\{Q, Q\} = 0$ are neglected then all other superalgebras are obtained by extension of the SO(3, 2)-family. Note that all superalgebras of this family can be obtained from the de Sitter SO(3, 2) extension by a composition of the Wigner–Inönü contractions listed below:

– supersymmetric speed-space contraction

$$\vec{P} \to \varepsilon\vec{P}, \quad \vec{K} \to \varepsilon\vec{K}, \quad Q \to \sqrt{\varepsilon}Q, \quad \varepsilon \to 0,$$

– supersymmetric speed-time contraction

$$H \to \varepsilon H, \quad \vec{K} \to \varepsilon\vec{K}, \quad Q \to \sqrt{\varepsilon}Q, \quad \varepsilon \to 0,$$

– supersymmetric space-time contraction

$$\vec{P} \to \varepsilon\vec{P}, \quad H \to \varepsilon H, \quad Q \to \sqrt{\varepsilon}Q, \quad \varepsilon \to 0,$$

– supersymmetric general contraction

$$\vec{P} \to \varepsilon\vec{P}, \quad H \to \varepsilon H, \quad \vec{K} \to \varepsilon\vec{K}, \quad Q \to \sqrt{\varepsilon}Q, \quad \varepsilon \to 0,$$

Note that the supersymmetric charge contraction

$Q \to \varepsilon Q, \quad \varepsilon \to 0$ leads to $\{Q, Q\} = 0$.

Now, let us consider the problem of the positive definiteness of the norm in the representation spaces of the kinematical supergroups. For quantum-mechanical applications

TABLE II

Kinematical superalgebra	a	b	c	d	h	p	k
SO(3, 2)	1	−1	1	1	1/2	−1/2	1/2
	0	0	0	0	1/2	−1/2	1/2
P	1	−1	0	0	0	0	1/2
	0	0	0	0	0	0	1/2
P′	1	0	1	0	0	−1/2	0
	0	0	0	0	0	−1/2	0
C	1	0	0	0	0	0	0
	0	0	0	0	0	0	0
N⁺	0	0	0	0	1	0	0
	0	0	0	0	0	0	0
N⁻	0	1	1	0	−1/2	0	0
	0	0	0	0	1	0	0
	0	0	0	0	0	0	0
G	0	1	0	0	0	0	0
	0	0	0	0	1	0	0
	0	0	0	0	0	0	0
G′	0	0	1	0	0	0	0
	0	0	0	0	1	0	0
	0	0	0	0	0	0	0
S	1	b	c	0	0	0	0
	−1	b	c	0	0	0	0
	0	b	c	0	0	0	0
	0	0	0	0	1	0	0

unitary representations are of interest. It is obvious that in the case $\{Q, Q\} = 0$ the norm in the underlying Hilbert space cannot be positive definite. In fact for vectors of the form $Q_\alpha|\psi\rangle$ it vanishes: $2\|Q_\alpha|\varphi\rangle\|^2 = \langle\psi|\{Q_\alpha, Q_\alpha\}|\psi\rangle = 0$. In less trivial cases the positivity condition reads

$$2\|Q_\alpha|\varphi\rangle\|^2 = (a\langle\psi|H|\psi\rangle + b(\gamma^0\vec{\gamma})_{\alpha\alpha}\langle\psi|\vec{P}|\psi\rangle + ic\vec{\gamma}_{\alpha\alpha}\langle\psi|\vec{K}|\psi\rangle + d(\gamma^5\vec{\gamma})_{\alpha\alpha}\langle\psi|\vec{J}|\psi\rangle) > 0$$

for all $|\psi\rangle$ belonging to the domain of the superalgebra. Explicitly, with use of the γ-matrices given in Appendix the above inequalities have the following form

$$a\langle\psi|H|\psi\rangle - b\langle\psi|P_3|\psi\rangle + c\langle\psi|K_2|\varphi\rangle - d\langle\varphi|J_1|\psi\rangle > 0,$$
$$a\langle\psi|H|\psi\rangle + b\langle\psi|P_3|\psi\rangle + c\langle\psi|K_2|\varphi\rangle + d\langle\psi|J_1|\psi\rangle > 0,$$
$$a\langle\psi|H|\psi\rangle + b\langle\psi|P_3|\psi\rangle - c\langle\psi|K_2|\varphi\rangle - d\langle\psi|J_1|\psi\rangle > 0,$$
$$a\langle\psi|H|\psi\rangle - b\langle\psi|P_3|\psi\rangle - c\langle\psi|K_2|\psi\rangle + d\langle\psi|J_1|\psi\rangle > 0.$$

Thus

$$a\langle\psi|H|\psi\rangle > 0$$

and

$$a\langle\psi|H|\psi\rangle \pm d\langle\psi|J_1|\psi\rangle > 0,$$

$$a\langle\psi|H|\psi\rangle \pm c\langle\psi|K_2|\psi\rangle > 0,$$

$$a\langle\psi|H|\psi\rangle \pm b\langle\psi|P_3|\psi\rangle > 0.$$

Consequently for superalgebras with $a \leqslant 0$ there does not exist representation space with positive definite norm. To this class belong all supersymmetric extensions of N^+, N^-, G, G′ and those from remaining ones with $a = 0$ and $a = -1$. In the SO(3,2) case with $a = 1$ the last inequalities can be satisfied simultaneously for unitary representations with H bounded below [4]. The same conclusion holds for P, P′, C and S. Therefore only for supersymmetric extensions of SO(3,2), P, P′ and S with $a = 1$ there exist a class of unitary representations in the Hilbert space with positive definite norm.

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APPENDIX

The Majorana representation of γ-matrices

$$\gamma_0 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3.$$

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