DEPARTURE FROM CHIRAL SYMMETRY: ROLE OF UNITARITY AND ANALYTICITY*

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It is shown that the discrepancies between experimental data and current algebra calculations of Ke_4, $\eta \to 3\pi$ and S-wave $I = 0$ pion-pion scattering length done by Weinberg and others, are due to the neglect of the unitarity and analyticity in the S-wave $I = 0$ pion-pion channel (final state interaction).

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1

If we lived in a world where the pions (and more generally the pseudoscalar mesons η, K) have zero masses, i.e. the chiral world, there would be no difficulty in confronting the chiral low energy theorems derived from current algebra or the effective lagrangian and experimental results. This is so because the low energy theorems are strictly valid in the limit of the 4-momentum q_μ of the pion tend to zero which cannot be reached in the physical world, where the pion has a finite mass. In the real world, the pion mass is small, one could hope that the limit of $q_\mu \to 0$, could be approximated at zero pion kinetic energy. This is essentially the basis assumption of all the current algebra results obtained in the late 60's. Spectacular agreements with experimental results were obtained [1]. Recent experiments show, however, that there are substantial disagreements between the more precise experimental data and the theory for the following processes:

(*i*) Ke_4 decay: Current Algebra calculations for the emission of two pions in S and P waves are respectively too low compared with experimental data by 45% and 20% [2, 3];

(*ii*) $\eta \to 3\pi$ decay: Using Dashen's theorem to extract the η–π^0 tadpole mixing, the calculated $\eta \to 3\pi$ rate is too low by a factor of 3 compared with experimental data;

(*iii*) $\pi\pi \to \pi\pi$: The S-wave $I = 0$ $\pi\pi$ scattering length derived by Weinberg [4] is too low by 40% compared with experimental data obtained from Ke_4 decay [3]. The order

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of magnitude of the disagreement between theory and experiment is well above the precision of $5 \sim 10\%$ expected from the $SU(2) \times SU(2)$ chiral symmetry.

In this lecture, we should like to show the above mentioned discrepancies are due to the usual *incorrect* assumption that the unitarity in the 2π or 3π channel can be neglected. In fact, they all can universally be explained by including unitarity i.e. the pion-pion final state interaction in an essentially model independent way [5–7]. The solution presented here leaves unmodified the well known correct results of current algebra such as the Adler–Weissberger relation, the πN scattering lengths and the $K \to 2\pi$ and $K \to 3\pi$ relation etc. [1]. This is so because the low energy S-wave $I = 0$ pion pion interaction is stronger than the corresponding one for πN scattering.

To understand what is meant by the unitarity correction, let us consider the standard one soft pion emission [1] formula in terms of the matrix element $\langle B|O(0)|A\rangle$:

$$iR_\alpha \frac{f_\pi m_\pi^2}{m_\pi^2 - q^2} = \int d^4x e^{iqx} \delta(x_0) \langle B|[A_0^\alpha(x), O(0)]|A\rangle$$

$$+ iq_\mu \int d^4x e^{iqx} \langle B|T(A_\mu^\alpha(x) O(0))|A\rangle, \tag{1.1}$$

where q is the pion 4-momentum and A_μ is the axial current. The standard current algebra assumption is that the continuum contribution in the 2^{nd} term on the RHS of Eq. (1) can be neglected in the physical region. We shall show later that this assumption is not necessarily correct in general; in fact, it leads to incorrect results for problems listed above. To take into account of this term (and in general terms involved higher pion momenta) we can use its analytic property and unitarity condition which can be demonstrated to be valid. The reduction formula involving two pions or more are more complicated than Eq. (1.1) and can be straightforwardly written down. The resulting formulas are more involved but the basic ideas remain unchanged. We rely on the general principle such as unitarity and analyticity of the S-matrix and time reversal invariance to carry out our analysis. For problems involving more than one pion in the final state, we must take into account the two-body final state pion-pion interactions (for $\pi\pi \to \pi\pi$ and K_4 decay) and also 3 body final state interaction (for $\eta \to 3\pi$ decay).

The use of the analyticity and unitarity in combination with chiral theorems is neither really new, nor completely model independent. For a review of the literature up to 1974, see the review article of H. Pagels [8]. We wish to point out that the method presented here differs from the so-called "Chiral Perturbation Theory" [8, 9] discussed in the literature due to a more correct treatment of the strong interaction effect, in particular its threshold behaviour. Our approach is similar to the linear σ model, but is more general. As it will become clear later, our approximation can be checked using experimental data of the low energy pion-pion phase shifts.

Our method consists in constructing an S-matrix which satisfies the analyticity and unitarity and at the same time the chiral theorems. It is not a perturbative approach. More explicitly, if the chiral theorem gives a scale to the calculated matrix element in the limit of the pion momenta $q_1 \to 0$ and $q_2 \to 0$, then because of the unitarity in the two pion channel, this matrix element would not have the same value in the physical region (i.e. for

$s \equiv (q_1+q_2)^2 \geq 4m_\pi^2$). The correction factor at the dispersion energy squared s is given by

$$\frac{1}{D_I(s)} = \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta_I(s')d_{s'}}{s'(s'-s-i\varepsilon)}\right), \tag{1.2}$$

where δ_I is the isospin I pion-pion phase shift and we normalize the correction factor to unity in the two soft pion limit. Eq. (1.2) is not the only correction factor; other corrections being far away singularities are expected to be much less important. Using experimental data on δ_0, we have D_0^{-1} $(s = 4m_\pi^2) = 1.40$ which yields a substantial correction for the two soft pion S wave emission formula of current algebra. This is the basic idea of the enhancement factor needed to account for the experimental data of the $I = 0$ S-wave $\pi\pi$ scattering length, the Ke_4 S-wave $I = 0$ dipion matrix element. There is more enhancement in the $\eta \to 3\pi$ matrix element due to the 3-body nature of the problem.

2. *$K \to \pi\pi e\nu$ and related processes*

Let us first consider the Ke_4 decay [5]. The axial current matrix element for this process can be written as

$$\mathcal{M}_\mu = \langle \pi^+(q_1)\pi^-(q_2)\,|A_\mu^-(0)|K^+(p)\rangle$$

$$= \frac{1}{m_K}[F(s,t,u)\,(q_1+q_2)_\mu + G(s,t,u)\,(q_1-q_2)_\mu$$

$$+H(s,t,u)\,(p-q_1-q_2)_\mu],$$

where $s = (q_1+q_2)^2$, $u = (p-q)^2$, $t = (p-q_2)^2$, $k^2 = (p-q_1-q_2)^2$. Using the standard current algebra, neglecting terms of higher order than q_1^μ and q_2^μ, it is straightforward, although somewhat involved, to show:

$$F^0(s,t,u) = \frac{1}{4f_\pi}[3f_+(u)) + 3f_+(t) + f_-(u) + f_-(t)] - \tfrac{1}{2}\frac{f_K}{f_\pi^2},$$

$$F^1(s,t,u) = \frac{1}{4f_\pi}[3f_+(u) - 3f_+(t) + f_-(u) - f_-(t)],$$

$$G^1(s,t,u) = \frac{1}{4f_\pi}[f_+(u) + f_+(t) - f_-(u) - f_-(t)] + \tfrac{1}{2}\frac{f_K}{f_\pi^2},$$

$$G^0(s,t,u) = \frac{1}{4f_\pi}[f_+(u) - f_-(u) - f_+(t) + f_-(t)], \tag{2.1}$$

where $F = F^0+F^1$ and $G = G^1+G^0$, f_+, f_- are the Kl_3 form factors. It can be shown that Eq. (2.1), and a similar expression for H, satisfies all the boundary conditions imposed by the low energy theorems in the limit $q_1 \to 0$, $q_2 \to 0$ or both $q_1, q_2 \to 0$. F's and G's, as given by Eq. (2.1), are all real because we neglect the unitarity conditions in the two pion channel which are contained in the terms of higher momenta $q_1^\mu q_2^\nu M_{\mu\nu}$. In the physical region of Ke_4 decay, because f_+ and f_- have high mass singularities (K^* and κ resonances), $F^1 \approx 0$, $G^0 \approx 0$, $F^0(s, t, u) \approx f(s)$, $G^1(s, t, u) \approx g(s)$ are real and almost independent of s. The values of f and g are given in Table I. It is clear that there are large discrepancies between Eq. (2.1) and experiments:

(*i*) The calculated value of f is too low by 45%,

(*ii*) The calculated value of g is too low by only 20%,

(*iii*) The s dependence of f is correct in magnitude but has the wrong sign.

The original Ke_4 calculation by Weinberg can be obtained by setting $t = u = M_K^2$ which is outside of the physical region of the Ke_4 decay. Numerical results of Eq. (2.1) agree with previous calculations by Cronin [11] (effective Lagrangian approach without introducing scalar and meson vector mesons), Chounet et al. [11], and others (in all these calculations the pion-pion final state interactions are neglected).

TABLE I

Theory vs experiment Ke_4 matrix elements $\left(\lambda \text{ is defined by } f(s) = f(s = 4\mu^2)\left[1+\lambda\left(\frac{s-4m_\pi^2}{4m_\pi^2}\right)\right]\right)$

	Experiment	Calculation with unitarity neglected	Calculation with unitarity correction
$f(s = 4\mu^2)\sin\theta_c$	1.23 ± 0.03	0.83 ± 0.02	1.15
$g(s = 4\mu^2)\sin\theta_c$	1.04 ± 0.06	0.83 ± 0.02	0.98
λ	0.08 ± 0.02	-0.057	0.08

Our task is to remove simultaneously the discrepancies (*i*)–(*iii*) without introducing extra parameters. This is achieved by incorporating the unitarity condition and analyticity. As a consequence of the unitarity and time reversal invariance, f must have the phase of the S-wave $I = 0$ $\pi\pi$ interaction, g the P-wave $I = 1$ $\pi\pi$ interaction. Analyticity requires that the correction factor to be of the form given by Eq. (1.1), where we have used the fact that f and g are almost point-like. The corrected values of f and g and the energy dependence of f are given in Table I. It is seen that the agreement between experimental data and the current algebra calculation with the unitarity and analyticity correction is excellent.

As another application of the current algebra technique which is similar to Eq. (2.1), the following process is of great interest: $e^+e^- \to \pi_1^+\pi_2^-\pi^+\pi^-$. It was shown a long time ago that in the limit $q_1 \to 0$, *and* $q_2 \to 0$, this process is related to the pion form factor which is dominated by the ϱ resonance [12]. Experimental data on $e^+e^- \to \pi^+\pi^-\varrho$ near threshold is larger than the theoretical prediction (obtained by taking simultaneously $q_1 \to 0$ and $q_2 = 0$) by a factor of 15–20. It was pointed out by Pham Roiesnel and Truong [13] that the unitarity in the $\pi\varrho$ channel i.e. the $\pi\varrho$ final state interaction in the form of A_1 resonance, as observed in $\tau \to \pi\varrho\nu$ decay, must be taken into account. The technique used is the same as the one which leads to Eq. (2.1). Corresponding to f_+ and f_- in Eq. (2.1), we now have matrix element $\langle\pi\varrho|A_\mu|0\rangle$ which is dominated by A_1 resonance and provides an enhancement factor of $7\sim 10$. The remaining discrepancy of a factor of two is due presumably to the neglect of the unitarity condition in the 2π channel (unlike the Ke_4 calculation, here the u and t variables approach the mass of A_1 resonance which makes it very difficult to take into account of the pion-pion final state interaction).

3. $\eta\to 3\pi$ decay

A long standing problem for current algebra and approximate chiral symmetry has been the calculation of the three pion decay of η. If this decay proceeds through the 2nd order electromagnetic interaction, using current algebra and assuming that the $\eta\to 3\pi$ matrix element depends linearly on the "odd" pion energy, Sutherland [14] showed that this matrix element vanishes which is in contradiction with experimental facts.

A standard remedy for this situation is to allow for a "tadpole" isospin breaking mechanism which, in the language of the quark model, is associated with the up and down quark mass difference m_u-m_d [15]. A deeper understanding of the connection between the tadpole and the short distance behaviour was provided by Wilson [16]. From the U-spin argument of the SU(3) symmetry the $\eta-\pi^0$ mass mixing is related to the mass differences of K^+-K^0 and $\pi^+-\pi^0$. Dashen showed that in the SU(3) chiral limit, only tadpole contributes [17]. Using this theorem, Cabibbo and Maiani [18] calculated in 1969 the $\eta\to 3\pi$ decay rate and found $\Gamma(\eta\to 3\pi)\approx 62$ eV which was one order of magnitude smaller than the experimental $\eta\to 3\pi$ width available at that time. Subsequently the experimental value was changed to 204 ± 29 eV which is still a factor of 3 larger than the theoretical value. In 1975, Weinberg [19] recalculated this decay in the context of the U(1) problem and, of course, found the same result. This led Weinberg to question the experimental reliability of this quantity. Our viewpoint is that the experimental value of $\Gamma(\eta\to 3\pi)$ is correct; the discrepancy between the theory and experiment is due to the neglect of the unitarity condition in the 3π channel, i.e. final state interaction, just the same as in the Ke_4 decay. (If Weinberg's viewpoint was correct, one would have difficulties in explaining the ratio $\Gamma(\eta\to 3\pi)/\Gamma(\eta\to 2\gamma)$.)

Using the same $I=0$ S-wave pion-pion interaction as in Ke_4 decay [5], Roiesnel and Truong [6] showed that the $\eta\to 3\pi$ rate and the odd pion spectrum agree well with the experimental data. Their calculation of spectra and rate is rather involved because of the nature of the three body final state interaction. In the following, we present a simplified method which circumvents this problem. The price that we have to pay, is not to be able to predict the odd pion π^0 spectrum and instead have to use the experimental data of the spectrum and extrapolate it to the unphysical point of the soft pion π^0. This is similar to the method used long time ago to relate $K\to 2\pi$ and $K\to 3\pi$ rate. Here we want to relate the matrix element $\langle 2\pi|v(0)|\eta\rangle$ to the $\eta\pi^0$ mixing, where v is a pseudoscalar density to be explained later.

Let us first review how the standard calculation with the effect of the final state interaction neglected, can be carried out. We can analyze the $\eta\to 3\pi$ in the framework of Quantum

Chromodynamics language:

$$\mathscr{L} = \mathscr{L}_0 + m_s\bar{s}s + m_u\bar{u}u + m_d\bar{d}d, \tag{3.1}$$

where $\mathscr{L}$ is SU(3)×SU(3) chiral invariant and the mass terms break the chiral invariance. To the RHS of Eq. (5.1) we have to add a pure electromagnetic interaction which can be neglected in the $\eta \to 3\pi$ decay [14]. The matrix element for $\eta \to 3\pi$ can be written as

$$\mathscr{M} = c_3 \langle 3\pi | u | \eta \rangle,$$

where $c_3 = \frac{1}{2}(m_u - m_d)$, and $u_a \equiv \bar{q}\frac{\lambda^a}{2}q$, q is the quark column matrix; let us define also $v_a \equiv \bar{q}\gamma_5\frac{\lambda^a}{2}q$, and $v = \frac{1}{\sqrt{3}}(\sqrt{2}v_0 + v_8)$. Taking the π^0 soft, we need the equal time commutation relation:

$$i[Q_5^3, u_3] = (\sqrt{2}\,v_0 + v_8)\frac{1}{\sqrt{3}} \equiv v, \tag{3.2}$$

$$\mathscr{M}(\eta \to \pi^0(\text{soft})\, 2\pi) = \left(\frac{\sqrt{2}\,c_3}{f_\pi}\right)\langle \pi^+\pi^- | v | \eta \rangle. \tag{3.3}$$

On the other hand in the limit of soft π^+ or soft π^-, from the $[Q_5^\pm, u_3] = 0$, the matrix element vanishes. Assuming the matrix element of $\eta \to \pi^+\pi^-\pi^0$ is a linear function of π^0 energy E_0 (this assumption can be shown to be correct but the proof is somewhat tedious), we can write

$$\mathscr{M}(\eta \to \pi^+\pi^-\pi^0) = \left(1 - \frac{2E_0}{m_\eta}\right)\frac{\sqrt{2}\,c_3}{f_\pi}\langle \pi^+\pi^- | v | \eta \rangle. \tag{3.4}$$

Contracting another pion and using the commutation relation $[Q_5^i, v^j] = -d_{ijk}v_k$, we can write:

$$\langle \pi^+\pi^- | v | \eta \rangle = -\frac{\sqrt{2}}{f_\pi}\langle \pi^0 | u_3 | \eta \rangle \tag{3.5}$$

and hence

$$\mathscr{M}(\eta \to \pi^+\pi^-\pi^0) = +\frac{2}{f_\pi^2}\frac{\delta m^2}{\sqrt{3}}\left(1 - \frac{2E_0}{m_\eta}\right), \tag{3.6}$$

where we have used the Dashen's theorem to show $c_3 = \frac{\delta m^2}{\sqrt{3}}$ with $\delta m^2 = m^2(K^+) - m^2(K^0) - m^2(\pi^+) + m^2(\pi^0)$. Eq. (3.6) gives a correct value for the slope of the odd pion spectrum but a wrong value for the partial width of $\Gamma(\eta \to \pi^+\pi^-\pi^0) = 65$ eV, compared with the experimental value of 204 ± 29 eV.

What goes wrong with this calculation [21]? Eq. (3.6) is obviously wrong because it is real, whereas it is expected to be complex due to the final state interaction. As it is mentioned above, because the treatment of the 3-body final state interaction is complicated we restrict ourselves to study the following "unphysical" amplitude:

$$g(s = m_\eta^2) = \lim_{q_{\pi^0}\to 0} \mathcal{M}(\eta \to \pi^+\pi^-\pi^0), \tag{3.7}$$

where $s = (q_{\pi^+}+q_{\pi^-})^2$. The experimental value of the RHS of Eq. (3.7) can be obtained from experimental data by a linear extrapolation of the observed spectrum to the unphysical point:

$$|g(s = m_\eta^2)| \left(\frac{2\delta m^2}{\sqrt{3}f_\pi^2}\right)^{-1} = 1.6 \pm 0.1, \tag{3.8}$$

while standard current algebra calculation, Eq. (3.6), yields unity.

Let us examine carefully Eq. (3.5). While it is exact in the chiral SU(3) limit, we have to make correction for this relation for physical values of π's and η's. The most important correction is the unitarity condition in the S-wave 2π channel, as it was shown in the Ke_4 decay [5]. As a consequence of the unitarity and time reversal invariance, the matrix element $\langle\pi^+\pi^-|v|\eta\rangle$ must have the phase of the $I = 0$ S-wave pion-pion interaction [20]. This is so because v is isoscalar and carries no momentum. Imposing in addition the analyticity requirement, Eq. (3.5) is now modified to

$$\langle\pi^+\pi^-|v|\eta\rangle = -\frac{\sqrt{2}}{f_\pi}\langle\pi^0|u_3|\eta\rangle \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\delta_0(s')ds'}{s'(s'-s-i\varepsilon)}\right), \tag{3.9}$$

where we normalize at $s = 0$ corresponding to the chiral SU(3) limit ($m_\eta \approx m_\pi \approx 0$), and have assumed the self-energy correction of the η propagation function can be neglected (as it can be justified by the success of the anomaly formula for $\eta \to 2\gamma$). Because v carries no momentum, the variable s denotes here also the η mass). Evaluating the RHS of Eq. (3.9) at $s = m_\eta^2$ corresponding to the η on its mass shell, and using Eqs. (3.3) and (3.6) we have:

$$g(s = m_\eta^2)\left(\frac{2\delta m^2}{\sqrt{3}f_\pi^2}\right)^{-1} = 1.50e^{i45^\circ} \tag{3.10}$$

which agrees well with Eq. (3.8). It should be noticed that the phase of g at $s = m_\eta^2$ is the phase $\delta_0 \approx 45°$. Its effect, therefore, cannot be neglected, as it is done in the naive current algebra calculation which, not surprisingly, leads to erroneous results. In a more realistic calculation, we have to calculate the pion spectrum, the $\eta \to 3\pi$ phase and also have to include the SU(3) breaking effect and the η and η′ mixing. Including also these effects, the final result yields $\Gamma(\eta \to \pi^+\pi^-\pi^0)$ which is about 20% too large, compared with the experimental value [6].

In the above discussion, we have ignored the U(1) problem in Quantum Chromodynamics which results in identifying the operator v with the divergence of a singlet current. The remedy for this situation is well known [19]. The above calculation is unaffected by the U(1) problem.

It is obvious that the success of the current algebra for relating $K \to 2\pi$ and $K \to 3\pi$ is unaffected by the above consideration, because $K \to 2\pi$ amplitude is directly measured by experiment and is not calculated from K–π mixing, as in the $\eta \to 3\pi$ problem. Similarly to the $\eta \to 3\pi$ calculation, it is possible to show that the "odd" pion spectrum in $K \to 3\pi$ is linear to a good accuracy, and the phase of $K \to 3\pi$ can be reliably calculated which should be quite useful in the study of the violation of *CP* in K_L and K_S system.

4. *The low energy pion pion scattering*

Under the assumptions that the low energy pion-pion amplitude can be expanded in a power series which is limited to linear terms of the variables s, t, u where $s = (p_\alpha + p_\beta)^2$, $t = (p_\alpha - p_\gamma)^2$, $u = (p_\alpha - p_\delta)^2$, where α, β, γ, δ are the isospin indices of the pions, and that the σ commutator is an isoscalar, Weinberg showed that the low energy pion-pion scattering can be written as [4]:

$$T_{\pi\pi} = \frac{1}{2f_\pi^2}\left[\delta_{\alpha\beta}\delta_{\gamma\delta}(s-m_\pi^2)+\delta_{\alpha\delta}\delta_{\beta\gamma}(u-m_\pi^2)+\delta_{\alpha\gamma}\delta_{\beta\delta}(t-m_\pi^2)\right] \tag{4.1}$$

and hence the scattering lengths a_I of isospin I are given by:

$$a_0 = \tfrac{7}{4} L = 0.155\, m_\pi^{-1}, \tag{4.2a}$$

$$a_2 = \frac{L}{2} = -0.045\, m_\pi^{-1}, \tag{4.2b}$$

$$a_1 = \frac{L}{3} = 0.03\, m_\pi^{-1}. \tag{4.2c}$$

a_0 is accurately measured in Ke_4 experiment [3] and was found to be $0.26 \pm 0.5\, m_\pi^{-1}$ which is much larger than Eq. (4.2a) [21]. Attempts were made to improve Weinberg calculation by incorporating the unitarity correction as a perturbative effect and a high energy S wave $I = 0$ σ resonance (or ε resonance) lead to a negligible correction to the Weinberg value [22].

We would like to point out here that the unitarity correction cannot be treated perturbatively and must be taken into account simultaneously with the effect of a high energy σ resonance. Because of the constraints of the current algebra and unitarity, this resonance has a wide width (proportional to m_σ^3) which makes the low energy S-wave $I = 0$ phase shift δ_0 large, and it leads to a substantial larger value of a_0 than that predicted by Weinberg. In the following we study this subtle effect and want to show that the S-wave $I = 0$ partial wave, instead of being given by Eq. (4.1), i.e. $f_0(s) = \frac{L}{4}(2s - m_\pi^2)$, is now given by

$$f_0^0(s) = \frac{L}{4}(2s-m_\pi^2)\exp\left(\frac{(s-s_0)}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\delta_0(s')ds'}{(s'-s_0)(s'-s)}\right), \tag{4.3}$$

where $s_0 = \frac{1}{2} m_\pi^2$. The denominator function D is normalised to unity at s_0 instead of $s = 0$. For $s \geqslant 4m_\pi^2$ they both differ by less than 5%, and hence the same correction factor explains Ke_4, $\eta \to 3\pi$ and a_0.

To see how it is possible to arrive at Eq. (4.3) let us consider the linear σ model; the low energy $\pi\pi$ scattering amplitude is now given by [22]:

$$T_{\pi\pi}(s, t, u) = \frac{1}{2f_\pi^2}\left[\delta_{\alpha\varrho}\delta_{\gamma\delta}(s-m_\pi^2)\frac{m_\sigma^2-m_\pi^2}{m_\sigma^2-s}+(s\leftrightarrow t)+(s\leftrightarrow u)\right]. \qquad (4.4)$$

From Eq. (4.4) at $s = 4m_\pi^2$, we obtain an enhancement factor of $(m_\sigma^2-m_\pi^2)/m_\sigma^2-4m_\pi^2$ which could be substantial if m_σ was sufficiently small. There is, however, no such a low energy resonance. This problem can be avoided by taking into account the self energy correction to the propagator of the σ which effectively pushes the σ mass to a much higher energy and hence gives a satisfactory agreement with experimental data. A self energy correction in the σ propagator does not lead to a completely unitarized $\pi\pi$ amplitude. It is simpler and more accurate to consider the partial wave amplitude $f_0^0(s)$. We assume that the Weinberg expansion:

$$f_0^0(s) = \frac{L}{4}(2s-m_\pi^2) \qquad (4.5)$$

is accurate in the neighbourhood of $s = \frac{1}{2} m_\pi^2$. This is essentially Khuri's result [24]. But Khuri's result is accurate only in this neighbourhood. Our task is to unitarize $f_0^0(s)$, taking into account the boundary condition (4.5) and the existence of a high energy resonance σ (or ε). This can be done using the well known N/D method, with a twice subtracted D function. Let us set:

$$N(s) \simeq \frac{L}{4}(2s-m_\pi^2), \qquad (4.6)$$

where we neglect the left-hand cut contribution to the N function. The D function, normalized to unity at $s = m_\pi^2/2$, in the elastic unitarity approximation, is given by:

$$D(s) = 1+b_0\left(s-\frac{m_\pi^2}{2}\right)+\frac{L}{2}\left(s-\frac{m_\pi^2}{2}\right)\left(h(s)-h\left(\frac{m_\pi^2}{2}\right)-i\varrho(s)\right), \qquad (4.7)$$

where $h(s) = \frac{2}{\pi}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\ln\left(\frac{\sqrt{s}+\sqrt{s-4m_\pi^2}}{2m_\pi}\right)$ and we have subtracted twice the D--function to ensure the convergence of the dispersion integral and to introduce a σ resonance in a simple manner.

With $b_0 = -0.04m_\pi^{-2}$, δ_0^0 computed from Eq. (4.7) agrees well with the experimental data up to $s = 40m_\pi^2$ and δ_0^0 remains below 90° for $s \leqslant 1$ GeV^2. The scattering length a_0^0 computed from Eq. (4.6) and (4.7) is $0.21m_\pi^{-1}$ instead of $0.155m_\pi^{-1}$ as given by Weinberg. The reason for this enhancement is simple: the D^{-1} function, instead of unity as given by Weinberg expansion, becomes 1.4 because of the effect of the σ resonance and also of the effect of the square root threshold singularity [26, 23].

In actual calculation, it is necessary to justify the neglect of the left-hand cut. To do this, a more sophisticated approach is needed by using Roy's equation, where crossing symmetry is exact [25]. It can be shown [7] that for $s \leqslant 25m_\pi^2$, the assumption of neglecting of the left-hand cut is reasonably accurate. This is so because these singularities are represented by subtraction constants which are accurate for a low energy theory. As a related subject, we would like to point out that the effect of the variation of $D^{-1}(s)$ function is not only limited to Ke_4, $\eta \to 3\pi$ and a_0^0, but is also important in the process $\pi^- p \to \pi^+\pi^- n$, where the original hope was to study the nature of the σ commutator [27]. It is clear from our calculation that this cross section cannot simply be computed from the effective lagrangian in the tree approximation. It is important to do the unitarity correction in the $I = 0$ S-wave 2π channel, which can approximately be achieved by multiplying the effective lagrangian matrix element by $D_0^{-1}(s)$ normalized to unity at $s = m_\pi^2/2$ or $s = 0$. This would enhance the matrix element by roughly a factor of 1.4 at the threshold of $\pi^- p \to \pi^+\pi^- n$ process. It is only after this correction taken into account that one can study the effect of the σ commutator.

5. $\eta' \to \eta\pi\pi$

Another topic related to PCAC is the process $\eta'(p') \to \eta(p)\, \pi^+(q_1)\pi^-(q_2)$, which is apparently in contradiction with PCAC and some ideas of the current algebra [28].

(*i*) From PCAC, it is expected this matrix element has a zero in the limit $q_1 \to 0$ or $q_2 \to 0$. The experimental dipion spectrum for this decay as a function of $s = (q_1+q_2)^2$ does not exhibit this behavior. The dipion spectrum in $\psi' \to \psi\pi\pi$ shows, however, clearly the Adler zero.

(*ii*) Taking the limit of q_1 and $q_2 \to 0$ simultaneously, the $\eta' \to \eta\pi\pi$ matrix element is proportional to the sum of the "up" and "down" quark masses which are small. Estimate of the partial width for this process indicates that it is a factor of 50 too small compared with its experimental value.

What is wrong with PCAC and current algebra? We want to show that it is an accident in nature, that the term proportional to q_μ in Eq. (1.1) dominates. To see this, let us define $u = (p'-q_1)^2$, $t = (p'-q_2)^2$. Let us assume the $\eta\pi$ states are dominated by the scalar meson $\delta(980)$ which is almost degenerate in mass with the $\eta'(960)$. In the limit $q_1 \to 0$, $u = (p'-q_1)^2 = (p+q_2)^2 = m_{\eta'}^2$, the matrix element vanishes by the virtue of the Adler theorem, but its coefficient is large due to the δ propagator. In the physical region, where u is fairly different from $m_{\eta'}^2$ and m_δ^2, the product of the Alder zero and pole of δ gives rise to a constant matrix element.

Using a nonet scheme of scalar mesons δ, ε etc., and the derivative couplings for the pseudoscalar mesons to satisfy the Adler's theorem, the predicted width for η' is in agreement with the subsequent experimental measurements. The dipion, $\eta\pi$ spectra are in very good agreement with the experimental data [28].

6. *Proton decay*

We end this lecture by discussing a much more fashionable subject, namely, the proton decay, within the framework of the Grand Unification Theory SU(5). It was first shown by Tomozawa [30] that it is more reliable to use current algebra to calculate the amplitude $p(p) \to \pi_0(q)+e^+(k)$, in the limit $q_\mu \to 0$. This limit is a poor approximation for this decay, because the energy π^0 is large. We must therefore make corrections, using the same technique discussed above.

Consider the matrix element $\langle\pi^0|\mathscr{L}_{e^+}(0)|p\rangle$ as an analytic function of k^2 with a cut starting at $k^2 = (m_p+m_\pi)^2$. Current algebra low energy theorem is given at $k^2 = m_p^2$, while the matrix element for the physical process is evaluated at $k^2 = m_e^2 \simeq 0$. This is a long extrapolation, so corrections must be made, using analyticity and unitarity, in the form of the resonance approximation for the πN system (S_{11} and P_{11}).

The final result for the rate $p \to \pi^0 e^+$ changes little, while there is a strong suppression for the mode $p \to \eta e^+$ due to some accidental cancellations. The proton lifetime is found to be 10^{29} years, which is more than two orders of magnitude faster than the experimental limit.

7. *Conclusion*

We have presented here the resolution of some old-fashioned but fundamental problems which were left unsolved or forgotten in the fast moving field of the particle physics. It is useful to ask whether these problems can be solved by the more fashionable Monte Carlo method of the lattice gauge theory of QCD. The answer is definitely no, at least in the foreseeable future, because of the euclidean nature of the method used for the lattice gauge theory.

Editorial note. This article was proofread by the editors only, not by the author.

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$\varrho(s) = \left(\frac{s-4m_\pi^2}{s}\right)^{1/2}$, the well known square root threshold singularity, becomes a step function in the limit $m_\pi^2 \to 0$. Because of the unitarity, this would give rise to a logarithm function for the real part of the amplitude which is used in the chiral perturbation theory. It was correctly noted by Lehman in his calculation of pion-pion scattering (*Phys. Lett.* **41B**, 529 (1972)), this approximation should not be serious for physical quantities which are sufficiently far from the 2π threshold. Unfortunately, we are dealing here with physical quantities which are on or very near to the threshold. For a discussion of the singularity $m_\pi^2 \log m_\pi^2$ in connection with the linear σ model, see for example G. S. Guralnick et al., *Phys. Rev.* **D7**, 2467 (1973).

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$$\mathcal{M}(s, t, u) = A\left(\frac{1}{m_\delta^2 - t} + \frac{1}{m_\delta^2 - u}\right) + \frac{B}{m_\varepsilon^2 - s} + C + Ds,$$

where C represents the contact terms and D the bremstrahlung terms. Because $m_\delta^2, m_\varepsilon^2 \gg s, t, u$ we expand the pole terms, use $s+t+u = m_\eta^2 + 3m_\pi^2$ and keep linear terms in s, t, u, we have

$$\mathcal{M}(s, t, u) = A' + B's.$$

Using current algebra constraints on $\mathcal{M}$ to calculate A', B' we arrive at Eq. (3.6) and hence there is no possible enhancement with δ pole. Some enhancement would be possible, if we took into account the quadratic terms. These terms could not be large because of the matrix restriction on the pion spectrum. Our result on the role of δ pole is in contradiction with a recent calculation by P. Minkowski, *Phys. Lett.* **116B**, 373 (1982). This is presumably due to the unsymmetrical treatment of the charged pions in this article.

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