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THE TOTAL ACTION OF THE ELECTROMAGNETIC FIELD CONTAINING THE INFRARED PART

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We obtain the leading asymptotic components of the Maxwell field at null infinity. Using these asymptotic expressions we derive the formula for the total action of the electromagnetic field in terms of the residue of the Fourier amplitude of the field at zero frequency.

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1. Introduction

Several authors have emphasized the difficulties connected with the variational formalism in the whole space-time for fields having the infrared part [1–3]. The problem is that the surface integrals which occur in the variational procedure cannot be neglected when the fields fall off too slowly at infinity. When the equations of motion hold, these surface terms are just the total action integral, if we take for variations of fields at infinity the fields themselves. For this reason it might be interesting to find the total action for fields with the infrared part.

The formula for the total action of the free electromagnetic field was obtained by Staruszkiewicz [4], whose proof is rather formal. The aim of this paper is to give the rigorous proof of Staruszkiewicz's result.

We want to avoid mathematically awkward, particularly in the infrared sector, procedure of taking limits "at infinity". For this reason we shall use Penrose's conformal transformations technique [5], which is a powerful and very elegant tool for investigating asymptotic phenomena.

2. *Asymptotic behaviour of free massless fields at null infinity*

For simplicity we shall perform our analysis for a real scalar field satisfying the equation

$$\Box \phi = 0. \tag{1}$$

For our purpose it is convenient to introduce the null spherical coordinates $(u, r, \vartheta, \varphi)$,

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where $u = t-r$ is the retarded time coordinate. We are interested in the asymptotic behaviour of the field ϕ along null geodesics labelled by (u, ϑ, φ) = const.

Any solution of equation (1) may be written in the form[1]

$$\phi(x) = \frac{1}{2\pi}\int \frac{d^3k}{\omega}\, a(\omega\vec{n})e^{-ikx} + \text{c.c.}, \tag{2}$$

where $\vec{n} = \dfrac{\vec{k}}{\omega}$, $\omega = |\vec{k}|$ and $kx = \omega x^0 - \vec{k}\cdot\vec{x}$.

We shall confine ourselves to the study of fields satisfying the condition

$$a^{\mathrm{R}}(\vec{n}) = \bar{a}^{\mathrm{R}}(\vec{n}), \tag{3}$$

where we have defined after Zwanziger [6]

$$a^{\mathrm{R}}(\vec{n}) \equiv \lim_{\omega\to 0} \omega a(\omega\vec{n}).$$

It is easy to see that if the condition (3) does not hold then, in general, the angular momentum of the field is infinite.

Let the z axis in momentum space coincide with the direction of $\vec{\sigma} = \dfrac{\vec{x}}{r}$. Then

$$\phi(u, r, \vec{\sigma}) = \frac{1}{2\pi}\int_0^\infty d\omega\omega \int_0^{2\pi} d\varphi \int_{-1}^{1} dz a(\omega\vec{n})e^{-i\omega u - i\omega r(1-z)} + \text{c.c.},$$

where $z = \vec{\sigma}\cdot\vec{n}$. After integrating by parts with respect to z we get

$$\phi(u, r, \vec{\sigma}) = I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{1}{ir}\int_0^\infty d\omega e^{-i\omega u} a(\omega\vec{\sigma}) + \text{c.c.},$$

$$I_2 = -\frac{1}{ir}\int_0^\infty d\omega e^{-i\omega(u+2r)} a(-\omega\vec{\sigma}) + \text{c.c.},$$

$$I_3 = -\frac{1}{ir}\int_0^\infty d\omega \int_0^{2\pi} d\varphi \int_{-1}^{1} dz e^{-i\omega[u+r(1-z)]}\frac{\partial}{\partial z} a(\omega\vec{n}) + \text{c.c.}.$$

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[1] As we investigate the infrared sector, the behaviour of Fourier amplitudes $a(k)$ for large ω is irrelevant for us. We shall assume throughout the paper that the amplitudes $a(k)$ fall off sufficiently fast for large ω to justify our calculations.

Using the Riemann-Lebesgue lemma we obtain for large r

$$I_2 \underset{\substack{r\to\infty \\ (u,\vartheta,\varphi)=\mathrm{const}}}{=} \frac{\pi}{r}\, a^{\mathrm{R}}(-\vec{\sigma})+O\left(\frac{1}{r^2}\right),$$

$$I_3 \underset{\substack{r\to\infty \\ (u,\vartheta,\varphi)=\mathrm{const}}}{=} \frac{\pi}{r}\,[a^{\mathrm{R}}(\vec{\sigma})-a^{\mathrm{R}}(-\vec{\sigma})]+O\left(\frac{1}{r^2}\right).$$

Therefore

$$\phi(u, r, \vec{\sigma}) \underset{\substack{r\to\infty \\ (u,\vartheta,\varphi)=\mathrm{const}}}{=} \frac{\pi}{r}\, a^{\mathrm{R}}(\vec{\sigma})+\frac{1}{ir}\int_0^\infty d\omega(e^{-i\omega u}a(\omega\vec{\sigma})-\mathrm{c.c.})+O\left(\frac{1}{r^2}\right). \tag{4}$$

It follows from (4) that when the field has the infrared part then the leading $\left(\frac{1}{r}\right)$ component does not vanish at spacelike infinity ($u \to -\infty$). As we shall see in Section 4 this fact is responsible for non-vanishing of the total action integral for fields having $a^{\mathrm{R}}(\vec{n}) \neq 0$.

The asymptotic behaviour of ϕ at past null infinity, that is along $(v = t+r, \vartheta, \varphi) = \mathrm{const}$ geodesics, may be obtained in the similar way.

In order to write the analogue of the formula (4) for the Maxwell field we have to choose a basis since $O(r)$ statements for tensors make sense only with respect to the specified frame. We shall use the conventional null tetrad $(l, n, m, \overline{m})$ adapted to the null spherical coordinate system [7]. Namely

$$l = \frac{\partial}{\partial r}, \quad n = \frac{\partial}{\partial u}-\frac{1}{2}\frac{\partial}{\partial r}, \quad m = \frac{1}{\sqrt{2}\,r}\left(\frac{\partial}{\partial\vartheta}+\frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right),$$

where the only non-vanishing scalar products are $ln = 1$ and $m\overline{m} = -1$. In terms of this tetrad the Maxwell field is described by three independent complex components [7]

$$\phi_0 = F_{ab}l^a m^b, \quad \phi_1 = \tfrac{1}{2}F_{ab}(l^a n^b+\overline{m}^a m^b),$$

$$\phi_2 = F_{ab}\overline{m}^a n^b.$$

We also denote

$$A_{00} = A_a l^a, \quad A_{11} = A_a n^a, \quad A_{01} = A_a \overline{m}^a.$$

In the Lorentz gauge we may write the electromagnetic potential in the form analogous to (2)

$$A_\mu(x) = \frac{1}{2\pi}\int\frac{d^3k}{\omega}\, a_\mu(\omega\vec{n})e^{-ikx}+\mathrm{c.c.}.$$

Now, following the same procedure as for the scalar field we obtain[2]

$$A_{11}(u,r,\vec{\sigma}) \underset{\substack{r\to\infty\\(u,\vartheta,\varphi)=\text{const}}}{=} \frac{\pi}{r}\, a^{\mathrm{R}}_{\mu}(\vec{\sigma})n^{\mu} + \frac{1}{ir}\int_0^\infty d\omega(e^{-i\omega u}a_{\mu}(\omega\vec{\sigma})n^{\mu} - \text{c.c.}) + O\left(\frac{1}{r^2}\right), \tag{5a}$$

$$A_{01}(u,r,\vec{\sigma}) \underset{\substack{r\to\infty\\(u,\vartheta,\varphi)=\text{const}}}{=} \frac{\pi}{r}\, a^{\mathrm{R}}_{\mu}(\vec{\sigma})\overline{m}^{\mu} + \frac{1}{ir}\int_0^\infty d\omega(e^{-i\omega u}a_{\mu}(\omega\vec{\sigma})\overline{m}^{\mu}$$

$$- e^{i\omega u}\overline{a}_{\mu}(\omega\vec{\sigma})\overline{m}^{\mu}) + O\left(\frac{1}{r^2}\right), \tag{5b}$$

$$\phi_2(u,r,\vec{\sigma}) \underset{\substack{r\to\infty\\(u,\vartheta,\varphi)=\text{const}}}{=} \frac{1}{ir}\int_0^\infty d\omega\omega(e^{-i\omega u}a_{\mu}(\omega\vec{\sigma})\overline{m}^{\mu} - e^{i\omega u}\overline{a}_{\mu}(\omega\vec{\sigma})\overline{m}^{\mu}) + O\left(\frac{1}{r^2}\right). \tag{5c}$$

The detailed structure of other components is irrelevant for our purpose since they fall off like $\frac{1}{r^2}$ or faster. However, we note for completeness that their leading asymptotic behaviour may be determined from (5) via the asymptotic Maxwell equations.

3. The conformal compactification of Minkowski space-time

In this section, included mainly in order to fix the notation, we shall give a short summary of the conformal transformations techniques proposed by Penrose [5].

The key idea is to replace Minkowski space-time M with g_{ab} metric by a new, "unphysical" space-time $\hat{M}$ with a rescaled metric $\hat{g}_{ab} = \Omega^2 g_{ab}$ in such a way that $\hat{M}$ is a compact manifold with a boundary representing "infinity" in M. The boundary $\mathscr{S} \equiv \partial\hat{M}$ of $\hat{M}$, given by the equation $\Omega = 0$, consists of two disconnected parts $\mathscr{S} = \mathscr{S}^+ \cup \mathscr{S}^-$ each having the topology $R^1 \times S^2$. $\mathscr{S}^+$ and $\mathscr{S}^-$ are null surfaces and are called future and past null infinity respectively. For the standard choice of the conformal factor $\Omega = \frac{1}{r}$ the metric on $\{u = \text{const}\} \cap \mathscr{S}^+$ induced by $\hat{g}_{ab}$ is (minus) that of a unit sphere.

Now, consider the free Maxwell field F_{ab} in (M, g_{ab}). Because of the conformal invariance of the Maxwell equations, $\hat{F}_{ab} = F_{ab}$ satisfies the source-free equations on $(\hat{M}, \hat{g}_{ab})$. From the rescaling conventions for the tetrad vectors[3]

$$\hat{l}^{\mu} = \Omega^{-2}l^{\mu}, \quad \hat{n}^{\mu} = n^{\mu}, \quad \hat{m}^{\mu} = \Omega^{-1}m^{\mu}$$

it follows that

$$\hat{\phi}_0 = \Omega^{-3}\phi_0, \quad \hat{\phi}_1 = \Omega^{-2}\phi_1, \quad \hat{\phi}_2 = \Omega^{-1}\phi_2$$

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[2] The result equivalent to (5c) was obtained by Szczekowski [8].

[3] Hatted quantities refer to $(\hat{M}, \hat{g}_{ab})$ and are defined analogically to their counterparts in (M, g_{ab}).

and

$$\hat{A}_{00} = \Omega^{-2}A_{00}, \quad \hat{A}_{01} = \Omega^{-1}A_{01}, \quad \hat{A}_{11} = A_{11}.$$

The fields at $\mathscr{I}^+$ are defined to be

$$\hat{\phi}_k^0 \equiv \hat{\phi}_k|_{\mathscr{I}^+} \quad (k = 0, 1, 2), \tag{6a}$$

$$\hat{A}_k^0 \equiv \hat{A}_k|_{\mathscr{I}^+} \quad (k = 00, 01, 11). \tag{6b}$$

4. *The total action of the Maxwell field*

The total action of the Maxwell field

$$S = -\frac{1}{16\pi}\int_M F_{\mu\nu}F^{\mu\nu}dV \tag{7}$$

takes the same form in terms of the rescaled quantities

$$S = -\frac{1}{16\pi}\int_{\hat{M}} \hat{F}_{\mu\nu}\hat{F}^{\mu\nu}d\hat{V} \tag{8}$$

because the invariant volume element on $\hat{M}$ transforms under conformal rescaling according to

$$d\hat{V} = \Omega^4 dV.$$

Since the Lagrangian density is a divergence when the Maxwell equations hold, we may write (8) as the surface integral

$$S = -\frac{1}{8\pi}\int_{\partial\hat{M}} \hat{F}_{\mu\nu}\hat{A}^\nu d\hat{\Sigma}^\mu. \tag{9}$$

We shall calculate (9) over $\mathscr{I}^+$. It is easy to show that the contribution from $\mathscr{I}^-$ is the same. We rewrite (9) in terms of (6)

$$S_{\mathscr{I}^+} = \frac{1}{8\pi}\int (\hat{\phi}_1^0\hat{A}_{11}^0 - \tfrac{1}{2}\,\mathrm{Re}\,\hat{\phi}_2^0\hat{A}_{01}^0)du\,d\sigma. \tag{10}$$

Here we have used the surface element on $\mathscr{I}^+$

$$d\hat{\Sigma}^\mu = \hat{n}^\mu du d\sigma.$$

When the Maxwell field fulfills the gauge condition $\hat{A}_{11}^0 = 0$, as (5a) does, then $\hat{\phi}_2^0 = -\frac{\partial}{\partial u}\hat{A}_{01}^0$ and (10) takes the simple form

$$S_{\mathscr{I}^+} = \frac{1}{8\pi}\int d\sigma|\hat{A}_{01}^0|^2\,\Big|_{u=-\infty}^{u=+\infty}. \tag{11}$$

It follows from (5b) that

$$\lim_{u\to\infty} \hat{A}^0_{01} = 0$$

and

$$\lim_{u\to-\infty} \hat{A}^0_{01} = 2\pi a^{\mathrm{R}}_\mu(\vec{\sigma})\overline{m}^\mu.$$

Therefore the total action $S = S_{\mathscr{S}^+} + S_{\mathscr{S}^-} = 2S_{\mathscr{S}^+}$ equals

$$S = \pi \int d\sigma a^{\mathrm{R}}_\mu(\vec{\sigma})a^{\mathrm{R}\mu}(\vec{\sigma}), \tag{12}$$

where we have used the identity

$$g_{\mu\nu} = 2(l_{(\mu}n_{\nu)} - m_{(\mu}\overline{m}_{\nu)}),$$

and the transversality condition

$$a_\mu(\vec{\sigma})l^\mu = 0.$$

The formula (12) is gauge invariant and Lorentz invariant, the first fact being obvious, the second following from Lorentz transformation properties of the measure $d\sigma$ and the residue of the amplitude $a^{\mathrm{R}}_\mu(\vec{\sigma})$. Namely, when the null vector k^μ undergoes the transformation

$$k'^\mu = \Lambda^\mu{}_\nu k^\nu$$

then

$$d\sigma' = \frac{d\sigma}{(\Lambda^0{}_\nu\sigma^\nu)^2}$$

and

$$a^{\mathrm{R}'}_\mu(\vec{\sigma}') = (\Lambda^0{}_\alpha\sigma^\alpha)\Lambda^\nu{}_\mu a^{\mathrm{R}}_\nu(\vec{\sigma}),$$

where we have introduced the four-index quantity

$$\sigma^\mu \equiv k^\mu/\omega = (1, \vec{\sigma}).$$

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