TREATMENT OF CLOSED FERMION LOOPS IN QUANTUM ELECTRODYNAMICS*

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A careful treatment of closed fermion loops in quantum electrodynamics is developed starting from Schwinger's gauge invariant formulation of the current, involving a line integral, in an external field problem. The analysis is carried out for multiloop contributions and to all orders of perturbation theory. The derived *vanishing* property of closed fermion loops when any subset of the external photon momenta are set equal to zero has a very important role in the renormalization program as this does not necessitate the introduction of the unwanted light-light scattering contact term in the Lagrangian. This also permits taking the limit of the photon zero-mass at least for Euclidean nonexceptional external momenta. The functional approach, used in this work, is found to be particularly suited for treating the closed fermion loop problem.

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1. Introduction

The purpose of this work is to carry out a detailed study of closed fermion loops, with an arbitrary number of external photon lines, in quantum electrodynamics by using, in the process, Schwinger's (1953, 1951; Johnson 1965) gauge invariant formulation of the current in an external field problem. Closed fermion loops have always caused problems (cf. Jauch and Rohrlich 1976) due to their lack of transversality when obtained by a naive application of perturbation theory. Light-light scattering graphs involving four external photon lines (with or without radiative corrections), in particular, have a naive degree of divergence equal to zero, and one would then, according to the renormalization (Manoukian 1976, 1982) program, carry out subtractions over such subdiagrams. Consistency with the counter term formalism, however, would then require (see Manoukian 1979) the introduction of light-light scattering contact terms in the Lagrangian. A careful treatment of closed fermion loops shows, however, that the Schwinger formulation guarantees the *vanishing* of closed fermion loops when *any* subset of their external momenta are set equal to zero, and subtractions, over closed fermion loops, are then naturally intro-

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duced by this formalism. Subtractions over closed fermion loops in the renormalization scheme then may be formally "kept", since in any case they do not contribute to the final result, and most importantly *no* unwanted light-light scattering contact term is then to be introduced in the Lagrangian due to the redundancy of the corresponding subtractions in the subtraction formalism. As a matter of fact the introduction of such a contact term would necessary introduce *proper* subdiagrams involving *only* photon lines. According to an earlier analysis (Manoukian 1980) such proper subdiagrams may develop *singularities* in the limit of a zero photon-mass even for Euclidean nonexceptional momenta. We derive a general expression (see in particular Eq. (16)) for handling closed fermion-loops, their transversality, and their *vanishing* property with the vanishing of *any* subset of external photon momenta, are then established. The important role of the analysis in the subtraction formalism of renormalization is discussed without the need of introducing, in the intermediate steps, ultraviolet cut-offs or dimensional regularization; and the existence of the photon zero-mass limit in the renormalized theory with Euclidean nonexceptional momenta, are then respectively pointed out. Although the study of gauge invariance treatments has a long history, the present paper should fill a gap for its completeness in treating *all* processes (as opposed to just some processes or some low order contributions) in a unified manner and in handling the problem in the light of the renormalization program (cf. Manoukian 1983). The treatment also avoids dimensional regularization methods, as mentioned above. The paper should be also of pedagogical value, and in studying the problem we have taken full advantage of the path integral formulation of field theory. The treatment of nonabelian gauge theories will be carried out in a subsequent report.

2. *Treatment of closed fermion loops*

The vacuum to vacuum transition amplitude $\langle 0_+|0_-\rangle$ in the presence of external sources may be formally written in the form of a path integral (cf. Fradkin et al. 1970)

$$\langle 0_+|0_-\rangle = N \int [DA] \exp i \int (dx)(dx')\bar{\eta}(x)G(x,x';\ eA)\eta(x')$$

$$\times \exp - \int (dx) \int_0^e de' \operatorname{Tr}[\gamma_\sigma A^\sigma(x) G(x,x;e'A)] \exp i\mathscr{L}(A,J), \tag{1}$$

where N is a normalization factor independent of the external sources $\bar{\eta}$, η and J_μ. All the Green's functions of the theory may be obtained from the expression in (1) by functional differentiation with respect to the external sources. $\mathscr{L}(A,J)$ denotes $\mathscr{L}(A) + \int A^\mu J_\mu$, where $\mathscr{L}(A)$ is the free photon Lagrangian. $G(x,x';eA)$ satisfies the equation

$$\left(\frac{\gamma\partial}{i} + m - e\gamma_\mu A^\mu(x)\right) G(x,x';eA) = \delta(x-x'), \tag{2}$$

where $A^\mu(x)$ is a classical field. Under a gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x)$

$$G(x,x';e(A+\partial\lambda)) = \exp ie[\lambda(x') - \lambda(x)] G(x,x';eA). \tag{3}$$

The function (Schwinger 1953, 1951; Johnson 1965)

$$\tilde{G}(x, x'; eA) = \exp\left[-ie \int_x^{x'} d\xi^\mu A_\mu(\xi)\right] G(x, x'; eA), \tag{4}$$

however, is gauge invariant as the line integral in its exponential part compensates the factor in the exponential in (3) under a gauge tıansformation. The Schwinger formalism is to define $G(x, x; eA)$, occuring in the trace in (1), as suitable limiting procedure $x \to x'$ of the gauge invariant object $\tilde{G}(x, x'; eA)$. The Green's function $G(x, x; eA)$ generates all the single closed fermion loops, and upon expansion of the exponential

$$\exp - \int (dx) \int_0^e de' \,\mathrm{Tr}\left[\gamma_\mu A^\mu(x) G(x, x; e'A)\right], \tag{5}$$

one may then generate all closed fermion loops contributions. The expression $G(x, x; eA)$ may be defined by (Schwinger 1953, 1951; Johnson 1965)

$$G(x, x; eA) = \lim_{\varepsilon \to 0} \underset{\substack{\varepsilon > 0 \\ \varepsilon < 0}}{\mathrm{Av}} \tilde{G}(x_+, x_-; eA), \tag{6}$$

where $x_\pm = x \pm \varepsilon/2$, $d\xi^\mu = (\varepsilon^\mu/2) d\lambda$, $-1 < \lambda < 1$, and Av denotes averaging over $\pm\varepsilon$. We define the Fourier transforms

$$S(x) = \int \frac{(dp)}{(2\pi)^4} \frac{e^{ipx}}{(\gamma p + m)}, \qquad A_\mu(x) = \int \frac{dQ}{(2\pi)^2} e^{iQx} A_\mu(Q). \tag{7}$$

Accordingly from (2) and (7) we obtain

$$\mathrm{Tr}\left[e\gamma_\mu A^\mu(x) G(x_+, x_-; eA)\right]$$
$$= \sum_{n=1}^{\infty} (e)^n \int \frac{dQ_1}{(2\pi)^2} \cdots \frac{dQ_n}{(2\pi)^2} A^{\mu_1}(Q_1) \ldots A^{\mu_n}(Q_n) e^{i(Q_1 + \ldots Q_n)}$$
$$\times e^{-i(Q_2 + \ldots + Q_n)\varepsilon/2} \int \frac{dp}{(2\pi)^4} e^{ip\varepsilon} \,\mathrm{Tr}\left[\gamma_{\mu_1} S(p) \gamma_{\mu_2} S(p - Q_2) \ldots \gamma_{\mu_n} S(p - Q_2 - \ldots - Q_n)\right]. \tag{8}$$

Also by using the well known expression

$$\int_{x_-}^{x_+} d\xi^\mu A_\mu(\xi) = \int \frac{dQ}{(2\pi)^2} e^{iQx} \frac{A_\mu(Q)\varepsilon^\mu}{2} \int_{-1}^{1} d\lambda \exp\left[i \frac{Q\varepsilon\lambda}{2}\right], \tag{9}$$

we obtain

$$\exp - ie \int_{x_-}^{x_+} d\xi^\mu A_\mu(\xi) = \sum_{m=0}^{\infty} \frac{(-ie/2)^m}{m!} \int \frac{dQ_1}{(2\pi)^2} \cdots \frac{dQ_m}{(2\pi)^2} \varepsilon^{\mu_1} \ldots \varepsilon^{\mu_m}$$
$$\times A_{\mu_1}(Q_1) \ldots A_{\mu_m}(Q_m) e^{i(Q_1 + \cdots + Q_m)x} \int_{-1}^{1} d\lambda_1 \ldots \int_{-1}^{1} d\lambda_m \exp \frac{i}{2}\left[Q_1\varepsilon\lambda_1 + \ldots + Q_m\varepsilon\lambda_m\right]. \tag{10}$$

Accordingly we have

$$\mathrm{Tr}\,[e\gamma_\mu A^\mu(x)\tilde{G}(x_+, x_-; eA)]$$

$$= \sum_{N=1}^{\infty} (e)^N \sum_{m=0}^{N-1} \frac{(-i/2)^m}{m!} \int \frac{dQ_1}{(2\pi)^2} e^{i(Q_1+\ldots+Q_N)x} A^{\mu_1}(Q_1) \ldots A^{\mu_N}(Q_N)$$

$$\times \int_{-1}^{1} d\lambda_1 \ldots \int_{-1}^{1} d\lambda_m \exp \frac{i}{2} [Q, \varepsilon\lambda, + \ldots + Q_m \varepsilon \lambda_m] \exp - \frac{i}{2} [Q_{m+2} + \ldots + Q_N]\varepsilon$$

$$\times \int \frac{dp}{(2\pi)^4} e^{ip\varepsilon} \varepsilon_{\mu_1} \ldots \varepsilon_{\mu_m} \mathrm{Tr}\,[\gamma_{\mu_{m+1}} S(p) \gamma_{\mu_{m+2}} S(p-Q_{m+2}) \ldots \gamma_{\mu_N} S(p-Q_{m+2} - \ldots - Q_N)]. \quad (11)$$

Using the property $\varepsilon^\mu\, e^{ip\varepsilon} = -\dfrac{i\partial}{\partial p^\mu} e^{ip\varepsilon}$, integrating by parts over p, and taking the average over $+\varepsilon$ and $-\varepsilon$, we obtain for

$$\int (dx) \int_0^e \frac{de'}{e'} \mathrm{Av}\, \mathrm{Tr}\,[e'\gamma_\mu A^\mu(x) \tilde{G}(x_+, x_-; e'A)] \quad (12)$$

the expression

$$\sum_{N=1}^{\infty} \frac{(e)^N}{N} \int \frac{dQ}{(2\pi)^2} \cdots \frac{dQ_N}{(2\pi)^2} (2\pi)^4 \delta(Q_1 + \ldots + Q_N)$$

$$\times A^{\mu_1}(Q_1) \ldots A^{\mu_N}(Q_N) \sum_{m=0}^{\infty} \frac{(1/2)^m}{m!} \int \frac{dp}{(2\pi)^4} \left(\frac{e^{ip\varepsilon} + e^{-ip\varepsilon}}{2} \right) \int_{-1}^{1} d\lambda_1 \ldots \int_{-1}^{1} d\lambda_m$$

$$\times \exp[-\tfrac{1}{2}(\lambda_1 Q_1 + \ldots + \lambda_m Q_m)\partial/\partial p] \exp[\tfrac{1}{2}(Q_{m+2} + \ldots + Q_N)\partial/\partial p]$$

$$\times \frac{\partial}{\partial p^{\mu_1}} \cdots \frac{\partial}{\partial p^{\mu_m}} \mathrm{Tr}\,[\gamma_{\mu_{m+1}} S(p) \gamma_{\mu_{m+2}} S(p-Q_{m+2}) \ldots \gamma_{\mu_N} S(p-Q_{m+2} - \ldots - Q_N)]. \quad (13)$$

We note that an exponential factor $\exp[Q(\partial/\partial p)]$ merely acts as a translation operator of p to $p+Q$. We will eventually symmetrize over the pairs $(\mu_1, Q_1), \ldots, (\mu_N, Q_N)$. We note that in (13) for every term depending on p there exists a corresponding term with p replaced by $-p$. Also

$$\frac{\partial}{\partial p_{\mu_1}} \cdots \frac{\partial}{\partial p_{\mu_m}} \mathrm{Tr}\,[\gamma_{\mu_{m+1}} S(p) \ldots \gamma_{\mu_N} S(p-Q_{m+2} - \ldots - Q_N)], \quad (14)$$

involves N S-propagators and N gamma matrices: $\gamma_{\mu_1}, \ldots, \gamma_{\mu_N}$. Accordingly if $N =$ odd, then by using the fact that the trace of an odd number of gamma matrices is zero, we see that we have a complete cancellation between the p and $-p$ terms in (13) for N = odd in the limit $\varepsilon \to 0$ (of course this is the content of the classic Furry theorem). We may then restrict only the terms with N = even ($N = 2, 4, 6, \ldots$) in (13) for $\varepsilon \to 0$, and we obtain for

$$\int (dx) \int_0^e \frac{de'}{e'} \operatorname{Tr}\,[e'\gamma_\mu A^\mu(x)\tilde{G}(x, x; e'A)], \tag{15}$$

the expression

$$\sum_{N=2,4,\ldots} \frac{(e)^N}{N} \int \frac{dQ_1}{(2\pi)^2} \cdots \frac{dQ_N}{(2\pi)^2} (2\pi)^4 \delta(Q_1 + \ldots + Q_N) A^{\mu_1}(Q_1) \ldots A^{\mu_N}(Q_N)$$

$$\times \sum_{m=0}^{N-1} \frac{(1/2)^m}{m!} \int \frac{dp}{(2\pi)^4} \int_{-1}^{1} d\lambda_1 \ldots \int_{-1}^{1} d\lambda_m \exp\,[-\tfrac{1}{2}(\lambda_1 Q_1 + \ldots + \lambda_m Q_m)\partial/\partial p]$$

$$\times \exp\,[\tfrac{1}{2}(Q_{m+2} + \ldots + Q_N)\partial/\partial_p] \frac{\partial}{\partial p^{\mu_1}} \cdots \frac{\partial}{\partial p^{\mu_m}}$$

$$\times \operatorname{Tr}\,[\gamma_{\mu_{m+1}} S(p) \gamma_{\mu_{m+2}} S(p - Q_{m+2}) \ldots \gamma_{\mu_N} S(p - Q_{m+2} - \ldots - Q_N)]. \tag{16}$$

By power counting and by an application of Gauss's theorem we learn that for $N = 6, 8, \ldots$, we have only to retain the $m = 0$ term in (16), as for $m = 0$,

$$\frac{\partial}{\partial p^{\mu_m}} \left\{ \frac{\partial}{\partial p^{\mu_1}} \cdots \frac{\partial}{\partial p^{\mu_{m-1}}} \operatorname{Tr}\,[\gamma_{\mu_{m+1}} S(p) \ldots \gamma_{\mu_N} S(p - Q_{m+2} - \ldots - Q_N)] \right\}, \tag{17}$$

integiates out to zero, over p, by Gauss's theorem. Also the exponential factor

$$\exp\,[\tfrac{1}{2}(Q_2 + \ldots + Q_N)\partial/\partial p] = 1 + O(\tfrac{1}{2}(Q_2 + \ldots + Q_N)\partial/\partial p), \tag{18}$$

may be replaced by one (for $N = 6, 8, \ldots$) as it also gives no contribution by an immediate application of Gauss's theorem. Upon symmetrization over the pairs $(\mu_1, Q_1), \ldots, (\mu_N, Q_N)$ we have for $N = 6, 8, \ldots$, the following terms contributing to (16):

$$\frac{(e)^N}{N} \int \frac{dQ_1}{(2\pi)^2} \cdots \frac{dQ_N}{(2\pi)^2} (2\pi)^4 \delta(Q_1 + \ldots + Q_N) A^{\mu_1}(Q_1) \ldots A^{\mu_N}(Q_N) \pi_{\mu_1 \ldots \mu_N}(Q_1, \ldots, Q_N), \tag{19}$$

where

$$\pi_{\mu_1 \ldots \mu_N}(Q_1, \ldots, Q_N) = \frac{1}{N!} \int \frac{dp}{(2\pi)^4} \sum_{[i_1 \ldots i_N]} \operatorname{Tr}\,[\gamma_{\mu_{i_1}} S(p) \gamma_{\mu_{i_2}}$$

$$\times S(p - Q_{i_2}) \ldots \gamma_{\mu_{i_N}} S(p - Q_{i_2} - \ldots - Q_{i_N})], \tag{20}$$

and $\sum_{[i_1 \ldots i_N]}$ denotes summation over all permutations of the indices in $(1, \ldots, N)$. We will come back to the general expression for $N = 6, 8, \ldots$, given in (19) later.

We now consider the term $N = 4$ in (16). That is we study the expression

$$\frac{(e)^4}{4}\int \frac{dQ_1}{(2\pi)^2}\cdots\frac{dQ_4}{(2\pi)^2}(2\pi)^4\delta(Q_1+\dots+Q_4)A^{\mu_1}(Q_1)\dots A^{\mu_4}(Q_4)$$

$$\times\sum_{m=0}^{3}\frac{(1/2)^m}{m!}\int\frac{dp}{(2\pi)^4}\int_{-1}^{1}d\lambda_1\dots d\lambda_m \exp\left[-\tfrac{1}{2}(\lambda_1 Q_1+\dots+\lambda_m Q_m)\partial/\partial p\right]$$

$$\times \exp\left[\tfrac{1}{2}(Q_{m+1}+\dots+Q_4)\partial/\partial p\right]$$

$$\times\frac{\partial}{\partial p^{\mu_1}}\cdots\frac{\partial}{\partial p^{\mu_m}}\mathrm{Tr}\left[\gamma_{\mu_{m+1}}S(p)\dots\gamma_{\mu_4}S(p-Q_{m+2}-\dots-Q_4)\right. \tag{21}$$

Since

$$\frac{\partial}{\partial p^{\mu_1}}\cdots\frac{\partial}{\partial p^{\mu_m}}\mathrm{Tr}\left[\gamma_{\mu_{m+1}}S(p)\dots\gamma_{\mu_4}S(p-Q_{m+2}-\dots-Q_4)\right]=O\left(\frac{1}{p^4}\right),$$

we may apply Gauss's theorem to replace the exponential factor

$$\exp\left[-\tfrac{1}{2}(\lambda_1 Q_1+\dots+\lambda_m Q_m)\partial/\partial p+\tfrac{1}{2}(Q_{m+2}+\dots+Q_4)\partial/\partial p\right]$$

simply by one as derivatives $(\partial/\partial p)^n$ of arbitrary order n will give zero contribution upon integration over p. The $\lambda_1, \dots, \lambda_m$ integrations may be then simply carried out to give the numerical factor $(2)^m$. Upon symmetrization over the pairs $(\mu_1, Q_1), \dots, (\mu_m, Q_m)$ we then obtain for (21)

$$\frac{(e)^4}{4}\int\frac{dQ_1}{(2\pi)^2}\cdots\frac{dQ}{(2\pi)^2}(2\pi)^4\delta(Q_1+\dots+Q_4)A^{\mu_1}(Q_1)\dots A^{\mu_4}(Q_4)\pi_{\mu_1\dots\mu_4}(Q_1,\dots Q_4), \tag{22}$$

with

$$\pi_{\mu_1\dots\mu_4}(Q_1,\dots,Q_4)=\frac{1}{4!}\int\frac{dp}{(2\pi)^2}\sum_{i=1}^{4}\sum_{[i_1\dots i_3]'}^{(i)}\{\cdot\}, \tag{23}$$

$$\{\cdot\} = \mathrm{Tr}\left[\gamma_{\mu_i}S(p)\gamma_{\mu_{i_1}}S(p-Q_{i_1})\dots\gamma_{\mu_{i_3}}S(p-Q_{i_1}-Q_{i_2}-Q_{i_3})\right]$$

$$+\frac{1}{m!}\sum_{m=1}^{3}\frac{\partial}{\partial p^{\mu_{i_1}}}\cdots\frac{\partial}{\partial p^{\mu_{i_m}}}\mathrm{Tr}\left[\gamma_{\mu_i}S(p)\gamma_{\mu_{i_{m+1}}}\right.$$

$$\left.\times S(p-Q_{i_{m+1}})\dots\gamma_{\mu_{i_3}}S(p-Q_{i_{m+1}}-\dots-Q_{i_3})\right], \tag{24}$$

where $\sum^{(i)}_{[i_1\dots i_3]'}$ is a sum over all permutations of the indices in $(1, \dots, \hat{i}, \dots N)$, with the hood sign $\wedge$ on i meaning that the latter is omitted. Because $N = 4$, Gauss's theorem cannot be applied to dismiss all of the terms involving the derivatives $(\partial/\partial p)$ in (24). For example for $i = 1$,

$$\sum_{[i_1 i_2 i_3]'}^{(i)} \{\cdot\} = \sum_{[i_2 i_3 i_4]} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_3}} S(p-Q_{i_2}) \ldots \gamma_{\mu_{i_4}} S(p-Q_{i_2}- \ldots -Q_{i_4})]$$

$$+ \sum_{[i_3 i_4]} \frac{\partial}{\partial p^{\mu_2}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_3}} S(p-Q_{i_3})\gamma_{\mu_{i_4}} S(p-Q_{i_3}-Q_{i_4})$$

$$+ \sum_{[i_2 i_4]} \frac{\partial}{\partial p^{\mu_3}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_2}} S(p-Q_{i_2})\gamma_{\mu_{i_4}} S(p-Q_{i_2}-Q_{i_4})$$

$$+\tfrac{1}{2} \sum_{[i_2 i_3 i_4]} \frac{\partial}{\partial p^{\mu_{i_2}}} \frac{\partial}{\partial p^{\mu_{i_3}}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_4}} S(p-Q_{i_4})]$$

$$+ \frac{\partial}{\partial p^{\mu_2}} \frac{\partial}{\partial p^{\mu_3}} \frac{\partial}{\partial p^{\mu_4}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)], \tag{25}$$

where $[i_2 i_3 i_4]$, $[i_3, i_4]$, $[i_2, i_4]$ denote permutations of the indices in (2, 3, 4), (3, 4), (2, 4), respectively. For $m = 1$ and $m = 2$, we consider the following terms in (24):

$$\frac{\partial}{\partial p^{\mu_2}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_3} S(p-Q_3)\gamma_{\mu_4} S(p-Q_3-Q_4)]$$

$$+ \frac{\partial}{\partial p^{\mu_2}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_4} S(p-Q_4)\gamma_{\mu_3} S(p-Q_3-Q_4)$$

$$+ \frac{\partial}{\partial p^{\mu_2}} \frac{\partial}{\partial p^{\mu_4}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_3} S(p-Q_3)], \tag{26}$$

occurring in (25). Each of the terms in (25) do not integrate out to zero by Gauss's theorem, the sum of these terms, however, do. To this end note by using the fact

$$S(p-Q) = (e^{-Q(\partial/\partial p)}-1)S(p)+S(p) = O\left(-Q\frac{\partial}{\partial p}\right) S(p)+S(p), \tag{27}$$

that from Gauss's theorem we may effectively set Q_3 and Q_4 in (26) equal to zero, to obtain equivalently to the sum of the terms in (26)

$$\frac{\partial}{\partial p^{\mu_2}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_3} S(p)\gamma_{\mu_4} S(p)] + \frac{\partial}{\partial p^{\mu_2}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_4} S(p)\gamma_{\mu_3} S(p)]$$

$$+ \frac{\partial}{\partial p^{\mu_2}} \frac{\partial}{\partial p^{\mu_4}} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_3} S(p)] = 0. \tag{28}$$

Accordingly all the terms involving the derivative $(\partial/\partial p)$ in (25) will cancel out upon integration over p by using Gauss's theorem with the exception of the term

$$\frac{\partial}{\partial p^{\mu_2}}\frac{\partial}{\partial p^{\mu_3}}\frac{\partial}{\partial p^{\mu_4}}\operatorname{Tr}[\gamma_{\mu_1}S(p)], \tag{29}$$

which does not integrate out to zero. The expression in (25) may be effectively replaced by

$$\sum_{[i_2 i_3 i_4]}\operatorname{Tr}[\gamma_{\mu_1}S(p)\gamma_{\mu_{i_2}}S(p-Q_{i_2})\ldots\gamma_{\mu_{i_4}}S(p-Q_{i_2}-\ldots-Q_{i_4})]$$
$$+\frac{\partial}{\partial p^{\mu_2}}\frac{\partial}{\partial p^{\mu_3}}\frac{\partial}{\partial p^{\mu_4}}\operatorname{Tr}[\gamma_{\mu_1}S(p)], \tag{30}$$

and the tensor $\pi_{\mu_1\ldots\mu_4}(Q_1,\ldots,Q_4)$ in (23) may be also rewritten as

$$\pi_{\mu_1\ldots\mu_4}(Q_1,\ldots,Q_4)=\frac{1}{4!}\int\frac{dp}{(2\pi)^4}\sum_{i=1}^{4}\sum_{[i,i_2,i_3]'}^{(i)}\{\cdot\}, \tag{31}$$

where $\{\cdot\}$ now is given by

$$\{\cdot\}=\operatorname{Tr}[\gamma_{\mu_1}S(p)\gamma_{\mu_{i_1}}S(p-Q_{i_1})\ldots\gamma_{\mu_{i_3}}S(p-Q_{i_1}-Q_{i_2}-Q_{i_3})$$
$$+\frac{1}{3!}\frac{\partial}{\partial p^{\mu_{i_1}}}\frac{\partial}{\partial p^{\mu_{i_2}}}\frac{\partial}{\partial p^{\mu_{i_3}}}\operatorname{Tr}[\gamma_{\mu_i}S(p)], \tag{32}$$

and $\sum_{[i_1 i_2 i_3]'}^{(i)}$, as before, is a sum over all permutations of the indices in $(1,\ldots,\hat{i},\ldots,4)$ with i omitted in the latter. It is surprizing that the presence of the extra term in (29) and (32) has *not* been emphasized in the literature.

We study the transversality property of $\pi_{\mu_1\ldots\mu_2}(Q_1,\ldots,Q_2)$ in (31). Consider the expression in (30) and contract it with, say, $Q_2^{\mu_2}$. Upon using the elementary property

$$S(p-Q)\gamma QS(p)=S(p-Q)-S(p)=(e^{-Q\partial/\partial p}-1)S(p)=O\left(-Q\frac{\partial}{\partial p}\right)S(p), \tag{33}$$

we obtain for the expression in (30) upon contracting it with $Q_2^{\mu_2}$, and upon integration over p:

$$\int\frac{dp}{(2\pi)^4}\sum_{[i_3 i_4]}\Big\{(e^{-Q_2(\partial/\partial p)}-1)\operatorname{Tr}[\gamma_{\mu_1}S(p)\gamma_{\mu_{i_3}}S(p-Q_{i_3})\gamma_{\mu_{i_4}}S(p-Q_{i_3}-Q_{i_4})]$$
$$+Q_2\frac{\partial}{\partial p}\operatorname{Tr}[\gamma_{\mu_1}S(p)\gamma_{\mu_{i_3}}S(p)\gamma_{\mu_{i_4}}S(p)]\Big\}. \tag{34}$$

Neither of the terms in the curly brackets in (34) integrates out to zero. Power counting, however, rigorously establishes that their *sum* integrates out to zero by cancellation and an application of Gauss's theorem. The presence of the extra factor in the curly brackets arising in (29) is then crucial. To contract the expression in (30) with $Q_1^{\mu_1}$, and integrate over p, we note that the terms in the first part in (30) may be obtained from

$$\sum_{[i_1 i_3 i_4]} \mathrm{Tr}\,[\gamma_{\mu_2} S(p)\gamma_{\mu_{i_1}} S(p-Q_{i_1})\gamma_{\mu_{i_3}} S(p-Q_{i_1}-Q_{i_3})\gamma_{\mu_{i_4}} S(p-Q_{i_1}-Q_{i_3}-Q_{i_4})] \tag{35}$$

by mere translations of the vector p in the above terms. Such translations are permitted by Gauss's theorem since $N = 4$. Accordingly the same analysis for the expression in (30) when contracted with $Q_2^{\mu_2}$ may be repeated with the expression (35) now contracted with $Q_1^{\mu_1}$. The other terms in (31) may be handled similarly. Accordingly $Q_j^{\mu_j}\pi_{\mu_1 \dots \mu_4}(Q_1, \dots, Q_4) = 0$, for $j = 1, 2, 3, 4$.

If we set $Q_2 = 0$ in (30) we obtain

$$\begin{aligned}
&\sum_{[i_2 i_3 i_4]} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_2}} S(p-Q_{i_2}) \dots \gamma_{\mu_{i_4}} S(p-Q_{i_2}-Q_{i_3}-Q_{i_4})]|_{Q_2=0} \\
&\quad + \frac{\partial}{\partial p^{\mu_2}} \sum_{[i_3 i_4]} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_3}} S(p-Q_{i_3})\gamma_{\mu_{i_4}} S(p-Q_{i_3}-Q_{i_4})] \\
&= -\frac{\partial}{\partial p^{\mu_2}} \sum_{[i_3 i_4]} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_3}} S(p-Q_{i_3})\gamma_{\mu_{i_4}} S(p-Q_{i_3}-Q_{i_4}) \\
&\quad + \frac{\partial}{\partial p^{\mu_2}} \sum_{[i_3 i_4]} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_3}} S(p)\gamma_{\mu_{i_4}} S(p)].
\end{aligned} \tag{36}$$

Again by Gauss's theorem the sum of the expressions in (36) integrate out to zero as we may effectively set Q_3 and Q_4 equal to zero in them. A similar analysis may be carried out for all the terms in (31). Accordingly $\pi_{\mu_1 \dots \mu_4}(Q_1, \dots, Q_4)|_{Q_j=0} = 0$ for $j = 1, 2, 3, 4$.

Now we consider the case with $N = 6, 8, \dots$ in (20). Consider the expression

$$\int \frac{dp}{(2\pi)^4} \mathrm{Tr} \sum_{[i_2 \dots i_N]} [\gamma_{\mu_1} S(p)\gamma_{\mu_{i_2}} S(p-Q_{i_2}) \dots \gamma_{\mu_{i_N}} S(p-Q_{i_2}- \dots -Q_{i_N})], \tag{37}$$

contributing to $\pi_{\mu_1 \dots \mu_N}(Q_1, \dots Q_N)$ in (20). Upon contracting the expression in (37) with $Q_2^{\mu_2}$, say, we obtain by using in the process (33),

$$\int \frac{dp}{(2\pi)^4} \sum_{[i_3 \dots i_N]} (e^{-Q_2(\partial/\partial p)}-1)\, \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_3}} S(p-Q_{i_3}) \dots \gamma_{\mu_{i_N}} S(p-Q_{i_3}- \dots -Q_{i_N})], \tag{38}$$

and the integrand integrates out to zero since $N = 6, \dots$, without the need of an additional factor as for the case $N = 4$. The other terms in (20) may be handled similarly and we obtain $Q_j^{\mu_j}\pi_{\mu_1 \dots \mu_N}(Q_1, \dots Q_N) = 0$, for $j = 1, \dots, N$. If we set $Q_2 = 0$ in (37) we obtain for the integrand in the latter

$$-\frac{\partial}{\partial p^{\mu_2}} \sum_{[i_3 \dots i_N]} \mathrm{Tr}\,[\gamma_{\mu_1} S(p)\gamma_{\mu_{i_3}} S(p-Q_{i_3}) \dots \gamma_{\mu_{i_N}} S(p-Q_{i_3}- \dots -Q_{i_N})], \tag{39}$$

which integrates out to zero by Gauss's theorem without the need of an extra term. The other terms in (30) are handled similarly. That is

$$\pi_{\mu_1\ldots\mu_N}(Q_1, \ldots, Q_N)|_{Q_j=0} \quad \text{for} \quad j = 1, \ldots, N.$$

The situation for $N = 2$ is well known (Johnson 1965) but will be discussed for completeness and for the convenience of the reader. From (16) we have for $N = 2$,

$$\frac{e^2}{2}\int \frac{dQ_1}{(2\pi)^2}\frac{dQ_2}{(2\pi)^2}(2\pi)^4\delta(Q_1+Q_2)A^{\mu_1}(Q_1)A^{\mu_2}(Q_2)\pi_{\mu_1\mu_2}(Q_1, Q_2), \tag{40}$$

$$\pi_{\mu_1\mu_2}(Q_1, Q_2) = \int \frac{dp}{(2\pi)^4}\left\{\exp\left(\frac{Q_2}{2}\frac{\partial}{\partial p}\right)\mathrm{Tr}\,[\gamma_{\mu_1}S(p)\gamma_{\mu_2}S(p-Q_2)]\right\}$$

$$+\tfrac{1}{2}\int_{-1}^{1} d\lambda \exp\left(-\lambda\frac{Q_1}{2}\frac{\partial}{\partial p}\right)\frac{\partial}{\partial p^{\mu_1}}\mathrm{Tr}\,[\gamma_{\mu_2}S(p)]. \tag{41}$$

By an application of Gauss's theorem, the integrand in (41), appearing in the curly brackets, may be replaced by

$$\mathrm{Tr}\left[\gamma_{\mu_1}S\left(p+\frac{Q_2}{2}\right)\gamma_{\mu_2}S\left(p-\frac{Q_2}{2}\right)\right]+\left[1+\tfrac{1}{24}\left(Q_2\frac{\partial}{\partial p}\right)^2\right]\frac{\partial}{\partial p^{\mu_1}}\mathrm{Tr}\,[\gamma_{\mu_2}S(p)]. \tag{42}$$

We note that the latter is symmetrical. Upon contracting the expression in (41) with Q_2, we obtain for the latter

$$\mathrm{Tr}\,[\gamma_{\mu_1}(e^{-(Q_2/2)(\partial/\partial p)}-e^{(Q_2/2(\partial/\partial p))})S(p)]+\mathrm{Tr}\left[Q_2\frac{\partial}{\partial p^{\mu_2}}\left[1+\tfrac{1}{24}\left(Q_2\frac{\partial}{\partial p}\right)^2\right]\gamma_{\mu_1}S(p)\right]$$

$$= \mathrm{Tr}\left[\left(-Q_2\frac{\partial}{\partial p}\right)-\tfrac{1}{24}\left(Q_2\frac{\partial}{\partial p}\right)^3-\frac{1}{2^4 5!}\left(Q_2\frac{\partial}{\partial p}\right)^5+\ldots\right]\gamma_{\mu_1}S(p)$$

$$+\mathrm{Tr}\left[Q_2\frac{\partial}{\partial p}+\tfrac{1}{24}\left(Q_2\frac{\partial}{\partial p}\right)^3\right]\gamma_{\mu_1}S(p). \tag{43}$$

By cancellation and from Gauss's theorem we then obtain $Q_2^{\mu_2}\pi_{\mu_1\mu_2}(Q_1, Q_2) = 0$. If we set $Q_2 = 0$, the expression in (41) becomes identically equal to zero. Thus we have treated all the cases.

We have seen that Schwinger's gauge invariant formulation of the current guarantees the facts that $Q_j^{\mu_j}\pi_{\mu_1\ldots\mu_N}(Q_1, \ldots, Q_N) = 0$ and $\pi_{\mu_1\ldots\mu_N}(Q_1, \ldots, Q_N)|_{Q_j=0} = 0$ for $j = 1, \ldots, N$. We note that upon integration over the photon field in (1), the $A^{\mu_1}(Q_1), \ldots, A^{\mu_N}(Q_N)$ in

$$\frac{(e)^N}{N}A^{\mu_1}(Q_1)\ldots A^{\mu_N}(Q_N)\pi_{\mu_1\ldots\mu_N}(Q_1, \ldots, Q_N) \tag{44}$$

provide virtual and external lines for Green's functions. Transversality then implies, in particular, that for any virtual photon line hitting a closed fermion loop, a Q_μ term in a photon propagator will not contribute and we may effectively work in the Feynman gauge.

We may summarize by saying that

$$\pi_{\mu_1 \ldots \mu_N}(Q_1, \ldots, Q_N) = \int \frac{dp}{(2\pi)^4} I_{\mu_1 \ldots \mu_N}(Q_1, \ldots Q_N; p), \tag{45}$$

where for $N = 2$, $I_{\mu_1\mu_2}$ is given by the expression in (42), for $N = 4$ $I_{\mu_1 \ldots \mu_4}$ is given by the expressions through (31) and (32), and for $N = 6, 8, \ldots$ $I_{\mu_1 \ldots \mu_N}$ is given by the integrand in (37).

Since $\pi_{\mu_1 \ldots \mu_N}(Q_1, \ldots, Q_N)$ vanishes when any subset of the $Q_1, \ldots, Q_N$ vanish, we may effectively rewrite (45) as

$$\pi_{\mu_1 \ldots \mu_N}(Q_1, \ldots, Q_N) = \int \frac{dp}{(2\pi)^4} Q_{(N)} I_{\mu_1 \ldots \mu_N}(Q_1, \ldots, Q_N; p), \tag{46}$$

where

$$\theta_{(N)} = 1 + \sum_{j=1}^{N} \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant N} (-1)^j \theta_{i_1} \ldots \theta_{i_j}, \tag{47}$$

and θ_i is the operation of setting $Q_i = 0$. We note, in particular, that $\theta_i \theta_{(N)} = 0$ by cancellation of the terms in (47). The expression (46) provides a rigorous definition for closed fermion loops in dealing with the renormalization (Manoukian 1976, 1982) program without, introducing in the process, ultraviolet cut-offs or dimensional regularization. Also we note that the multiloop contributions are obtained from the "multiplication" of single closed fermion loop expressions (46). Accordingly a multiloop contribution will again vanish if any subset of the remaining external momenta are set equal to zero. The vanishing of closed fermion loops with the vanishing of any of their external photon momenta implies that their degree of divergence is reduced from their naive one by a number equal to the number of their external photon lines. This, very welcome, vanishing property of closed fermion loops implies that we may, without loss of generality, still carry out subtractions of renormalization over them, since in any case the latter are redundant, *without* the necessity of introducing a light-light scattering contact term in the Lagrangian. Due to the absence of a contact light-light scattering term in the Lagrangian then in turn implies that any graph in the Feynman rules cannot contain a *proper sub*diagram composed solely of photon lines. According to an earlier analysis (Manoukian 1980) the photon zero-mass limit may be then taken for Euclidean nonexceptional momenta. A study in the above spirit will be carried out for nonabelian gauge theories in a subsequent report.

Editorial note. This article was proofread by the editors only, not be the author.

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