THE ANOMALY FREE CP[1] MODEL AND ITS *S*-MATRIX

By M. C. B. Abdalla* and A. Lima Santos

Instituto de Fisica, Universidade de São Paulo, Brasil

(Received December 28, 1983)

After showing that the CP[1] model factorizes we obtain the *S*-matrix for fundamental fields and bound states.

PACS numbers: 11.10.Lm

1. The anomaly free model

The classically established equivalence of the CP[1] and O(3) non linear sigma model [1], has never been shown quantum mechanically. This notice aims at calling attention to a fact that, in our opinion, solves the problem.

The CP^{n-1} model has a quantum anomaly preventing conservation of the non local (classically conserved) charge [2]. However, when coupled minimally to fermions, the anomaly cancels [3], the gauge field zero mass pole disappears, and the model turns out to be factorizable [4]. The *S*-matrix can be calculated, and compared with the $1/n$ expansion of the model [4]. Accordance is obtained in lowest order. The *S*-matrix has no pole in the physical sheet, and the *z*-field interacts via a repulsive force, preventing bound states. This situation is very different from the pure CP^{n-1} model, where long range forces confine partons in mesons. These long range forces, responsible for confinement, imply also the existence of the anomaly, as shown in an explicit calculation [2].

Briefly the CP^{n-1} model is the theory of an *n*-component complex *z*-field whose Lagrangian density is given by [1]

$$\mathscr{L} = \overline{D_\mu z} D_\mu z, \tag{1}$$

where

$$D_\mu z = \partial_\mu z - A_\mu z, \tag{2a}$$

$$A_\mu = -\frac{f}{n} \bar{z} \overleftrightarrow{\partial}_\mu z \tag{2b}$$

and the constraint

$$\bar{z}z = \sum_i \bar{z}_i z_i = n/2f. \tag{2c}$$

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* Present address: Niels Bohr Institute, 17, Blegdamsvej, 2100 Copenhagen Ø, Denmark.

At the classical level this model is known to possess an infinite number of conservation laws [5] and the simplest classically conserved non-local charge is given by:

$$Q^{ij} = \int dy_1 dy_2 \varepsilon(y_1 - y_2) J_0^{ik}(t, y_1) J_0^{kj}(t, y_2) - \frac{n}{2f} \int J_1^{ij}(t, y) dy, \tag{3}$$

where

$$J_\mu^{ij}(x) = z^i(x) \overleftrightarrow{\partial}_\mu \bar{z}^j(x) + 2A_\mu z^i(x) \bar{z}^j(x) \tag{4}$$

is the classical traceless Nöther current associated with the SU(n) rotations.

The classical integrability condition

$$\partial_\mu J_\nu^{ij} - \partial_\nu J_\mu^{ij} + 2\frac{2f}{n}[J_\mu(x), J_\nu(x)]^{ij} = 0 \tag{5}$$

is equivalent to $\dfrac{dQ^{ij}}{dt} = 0$.

At the quantum level we have problems because the charge (3) involves a product of two currents at the same point and so it is not well defined. To give a proper definition of the quantum non-local charge we must look at the short distance behavior of the product which appears in the commutator of Eq. (5). This has been done [2] and we are left with:

$$[J_\mu(x+\varepsilon), J_\nu(x)]^{ij} = C_{\mu\nu}^{\varrho}(\varepsilon) J_\varrho^{ij}(x) + D_{\mu\nu}^{\varrho\sigma}(\varepsilon) \partial_\sigma J_\varrho^{ij}(x) + E_{\mu\nu}^{\varrho\sigma}(\varepsilon) z_i \bar{z}_j F_{\varrho\sigma}(x), \tag{6}$$

where $C_{\mu\nu}^{\varrho}$, $D_{\mu\nu}^{\varrho\sigma}$ and $E_{\mu\nu}^{\varrho\sigma}$ are non zero and $F_{\varrho\sigma} = \partial_\varrho A_\sigma - \partial_\sigma A_\varrho$.

Now we are able to define the quantum non local charge $Q^{ij} = \lim_{\delta\to 0} Q_\delta^{ij}$.

$$Q_\delta^{ij} = \frac{1}{n}\left\{ \int_{|y_1 - y_2| \geq \delta} dy_1 dy_2 \varepsilon(y_1 - y_2) J_0^{ik}(t, y_1) J_0^{kj}(t, y_2) - Z_\delta \int dy J_1^{ij}(t, y) \right\}, \tag{7}$$

where the dependence of Z in the cutoff δ is such as to cancel the linear divergences which appear in the commutator (6). So that, in order to obtain a well-defined charge Q we must have

$$Z_\delta = \frac{n}{2\pi} \ln\left(\frac{e^{\gamma - 1} m\delta}{2}\right). \tag{8}$$

The mass m is dynamically generated and given by $m^2 = \mu^2 e^{-n/2f}$ where μ is the renormalization point and γ is the Euler-Mascheroni constant.

In accordance with the confining properties of the theory one can verify using (6) and (7) that the quantum non-local charge is no longer conserved.

$$\frac{dQ^{ij}}{dt} = -\frac{2}{\pi} \int_{-\infty}^{\infty} z_i \bar{z}_j F_{10} dy. \tag{9}$$

This means that the model has an anomaly in its quantum non-local charge and because of this the model is not factorizable, and consequently has no factorizable S-matrix.

However, for $n = 2$ the picture changes. In this case the anomaly can be easily shown to be a total divergence, and we are able to construct a new quantum non-local conserved charge. We can then show that the CP^1 model has a quantum conserved non-local charge just redefining the old one (7).

The Nöther current (4) can be written as follows [6]:

$$J_\mu = -D_\mu X X^+ + X D_\mu X^+ = Y D_\mu Y^+ - D_\mu Y Y^+, \tag{10}$$

where the fields X and Y (for the CP^1 case) are the two-components fields:

$$X = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad Y = \begin{pmatrix} \bar{z}_2 \\ -\bar{z}_1 \end{pmatrix}, \tag{11}$$

satisfying

$$Y^+ X = 0, \tag{12a}$$

$$X^+ X = Y^+ Y = 1 \tag{12b}$$

and connected by

$$Y_i = \varepsilon_{ij} X_j^+, \quad (\varepsilon_{21} = -1). \tag{12c}$$

In general the following identity holds

$$X F^x_{\mu\nu} X^+ + Y F^y_{\mu\nu} Y^+ = \tfrac{1}{2} (\partial_\mu J_\nu - \partial_\nu J_\mu). \tag{13}$$

Now, the anomaly is just

$$(X_i X_j^+ - \tfrac{1}{2}\,\delta_{ij}) F^x_{\mu\nu} = (Y_j^+ Y_i - \tfrac{1}{2}\,\delta_{ij}) F^y_{\mu\nu}. \tag{14}$$

The above identity is a direct consequence of (11). So that we can take the mean value of both sides obtaining (13), which is the total divergence of the current (10).

At this point a standard definition for a conserved quantum non-local charge can be made and we conclude that the model is anomaly free.

This fact follows from a very simple criterion obtained in the context of group theory [7], for the so called non linear sigma models defined on symmetric Riemannian spaces. There a general criterion is shown for the presence or absence of anomalies in these quantum models. The conclusion is the following. Let a model be defined on a symmetric space M = G/H. One can have two possibilities:

1) The model is anomaly free if H is simple.
2) Anomalies are allowed if H contains notrivial ideals.

2. *Fundamental z-fields S-matrix*

In this section we construct, from the asymptotic field, a quantum non-local charge which has no contribution of the gauge field A_μ. To justify this procedure let us recall that the asymptotic part of a conserved current can be taken as the one which has the same commutation relations and vacuum expectation value as the interacting current [8]. The procedure as well as all the normalizations are completely analogous to Ref. [2].

The non-local charge can be written as:

$$Q^{ij} = -\tfrac{1}{2}\int_{-\infty}^{\infty} d\mu(p_1)d\mu(p_2)\bar{\varepsilon}(p_1-p_2):(a^i_{\text{in}}(p_1)a^{+k}_{\text{in}}(p_1)$$
$$-b^{+i}_{\text{in}}(p_1)b^k_{\text{in}}(p_1))\,(a^k_{\text{in}}(p_2)a^{+j}_{\text{in}}(p_2)-a^{+k}_{\text{in}}(p_2)a^j_{\text{in}}(p_2))$$
$$-\frac{1}{i\pi}\int d\mu(p)\ln\frac{(p^0+p)}{m}:\Big\{a^i_{\text{in}}(p)a^{+j}_{\text{in}}(p)-b^{+i}_{\text{in}}(p)b^j_{\text{in}}(p)$$
$$-\frac{\delta^{ij}}{2}(a^{+k}_{\text{in}}(p)a^k_{\text{in}}(p)-b^k_{\text{in}}(p)b^{+k}_{\text{in}}(p)\Big\}, \tag{15}$$

where a and b are creation and destruction operators obeying the usual commutator rules. The out-form of the same charge Q^{ij} differs from (15) just by the sign of the first term.

The action of the non-local charge Q^{ij} on the asymptotic states of two particles characterized by the rapidities θ_1, θ_2 and by the isospin indices c_l, d_l, c_l', d_l' is given by:

$$Q^{ij}|\theta_1c_1,\theta_2c_2\rangle_{\text{in}} = (M^{ij}_{\text{in}})d_1d_2c_1c_2|\theta_1d_1,\theta_2d_2\rangle_{\text{in}}, \tag{16a}$$

$$_{\text{out}}\langle\theta_1'c_1',\theta_2'c_2'|Q^{ij} = {}_{\text{out}}\langle\theta_1'd_1',\theta_2'd_2'|\,(M^{ij}_{\text{out}})_{c_1'c_2'd_1'd_2'}, \tag{16b}$$

where

$$(M^{ij}_{\text{in}})_{d_1d_2c_1c_2} = -\tfrac{1}{2}(\delta^{ic_1}\delta^{jd_2}\delta^{c_2d_1}-\delta^{ic_2}\delta^{jd_1}\delta^{c_1d_2})$$
$$-\frac{\theta_1}{i\pi}\left(\delta^{ic_1}\delta^{jd_1}\delta^{c_2d_2}+\frac{\delta^{ij}}{2}\delta^{c_1d_1}\delta^{c_2d_2}\right)-\frac{\theta_2}{i\pi}\left(\delta^{ic_2}\delta^{jd_2}\delta^{c_1d_1}+\frac{\delta^{ij}}{2}\delta^{c_1d_1}\delta^{c_2d_2}\right), \tag{17}$$

$$(M^{ij}_{\text{out}})_{c_1'c_2'd_1'd_2'} = \tfrac{1}{2}(\delta^{id_1'}\delta^{jc_2'}\delta^{d_2'c_1'}-\delta^{id_2'}\delta^{jc_1'}\delta^{d_1'c_2'})$$
$$-\frac{\theta_1}{i\pi}\left(\delta^{id_1'}\delta^{jc_1'}\delta^{c_2'd_2'}+\frac{\delta^{ij}}{2}\delta^{c_1'd_1'}\delta^{c_2'd_2'}\right)$$
$$-\frac{\theta_2}{i\pi}\left(\delta^{id_2'}\delta^{jc_2'}\delta^{c_1'd_1'}+\frac{\delta^{ij}}{2}\delta^{c_1'd_1'}\delta^{c_2'd_2'}\right). \tag{18}$$

Now the elastic scattering amplitude of two particles with rapidities θ_1 and θ_2 can be written as:

$$_{\text{out}}\langle\theta_1'c_1',\theta_2'c_2'|\theta_1c_1,\theta_2c_2\rangle_{\text{in}} = (4\pi)^2\delta(\theta_1'-\theta_1)\delta(\theta_2'-\theta_2)\,\{\delta^{c_1c_1'}\delta^{c_2c_2'}u_1(\theta)$$
$$+\delta^{c_1c_2'}\delta^{c_2c_1'}u_2(\theta)\}-(4\pi)^2\delta(\theta_1'-\theta_2)\delta(\theta_2'-\theta_1)\,\{\delta^{c_1c_1'}\delta^{c_2c_2'}u_1(\theta)+\delta^{c_1c_2'}\delta^{c_2c_1'}u_2(\theta)\}, \tag{19}$$

where $\theta = \theta_1-\theta_2$ and $u_1(\theta)$, $u_2(\theta)$ are restricted by the nonlocal conservation law obtained imposing that:

$$_{\text{out}}\langle\theta_1'c_1',\theta_2'c_2'|Q^{ij}|\theta_1c_1,\theta_2c_2\rangle_{\text{in}} = \begin{cases}(M^{ij}_{\text{out}})_{c_1'c_2'd_1'd_2'}\,{}_{\text{out}}\langle\theta_1'd_1',\theta_2'd_2'|\theta_1c_1,\theta_2c_2\rangle_{\text{in}}, & \text{(20a)}\\ {}_{\text{out}}\langle\theta_1'c_1',\theta_2'c_2'|\theta_1d_1,\theta_2d_2\rangle_{\text{in}}(M^{ij}_{\text{in}})_{d_1d_2c_1c_2}. & \text{(20b)}\end{cases}$$

This set of linear equations for $u_1(\theta)$ and $u_2(\theta)$ can be solved giving us one of the so called "factorization equations".

$$u_2(\theta) = -\frac{i\pi}{\theta} u_1(\theta). \tag{21}$$

Another factorization equation which relates $t_1(\theta)$ with $t_2(\theta)$ can be obtained by the usual crossing symmetry or in the same way as $u_1(\theta)$ and $u_2(\theta)$ just writing the elastic scattering amplitude of one particle and its anti-particle as follows:

$$\begin{aligned} {}_{\text{out}}\langle\theta_1' c_1', \theta_2' \bar{c}_2' | \theta_1 c_1, \theta_2 \bar{c}_2\rangle_{\text{in}} &= (4\pi)^2 \delta(\theta_1' - \theta_1)\delta(\theta_2' - \theta_2) \{\delta^{c_1 c_1'} \delta^{\bar{c}_2 \bar{c}_2'} t_1(\theta) \\ &+ \delta^{c_1 \bar{c}_2 c_1' \bar{c}_2'} t_2(\theta)\} - (4\pi)^2 \delta(\theta_1', \theta_2)\delta(\theta_2' - \theta_1) \{\delta^{c_1 c_1' \bar{c}_2 \bar{c}_2'} r_1(\theta) + \delta^{c_1 \bar{c}_2} \delta^{c_1' \bar{c}_2'} r_2(\theta)\}. \end{aligned} \tag{22}$$

From the above equations we obtain both relations among $t_1(\theta)$, $t_2(\theta)$ and $r_1(\theta)$, $r_2(\theta)$ which read

$$t_2(\theta) = -\frac{i\pi}{i\pi - \theta} t_1(\theta) \tag{23a}$$

and

$$r_1(\theta) = r_2(\theta) = 0. \tag{23b}$$

Finally we see that the relations (21) and (23) correspond to those of class II of Ref. [9].

3. *Bound states S-matrix for CP^1 model*

Now it turns out that CP^1 and the model coupled to fermions have the same factorization equations. We claim that the difference between the two models lies in the bound state spectrum. For the model with fermions there is no bound state pole [4], consequently the *S*-matrix for the partons is a complete one. On the other hand the pure model should have one bound state pole, in order that quantum mechanically, the equivalence between CP^1 and O(3) non linear sigma models holds [10].

We define bound state as [10, 11]:

$$\left|\pi^a_{\alpha_1\alpha_2} \frac{(\theta_1+\theta_2)}{2}\right\rangle = \tfrac{1}{2}\{|\alpha_1(\theta_1)\bar{\alpha}_2(\theta_2)\rangle - |\bar{\alpha}_2(\theta_1)\alpha_1(\theta_2)\rangle\}\lambda^a_{\alpha_1\alpha_2}, \tag{24}$$

where $\lambda^a_{\alpha_1\alpha_2}$ are the Pauli matrices.

We suppose that a bound-state is defined with a difference in the rapidity variables given by a constant α, which characterizes the bound-state pole [11]:

$$\theta_1 - \theta_2 = i\pi\alpha.$$

The bound state S-matrix is defined as

$$\begin{aligned} 4\langle\gamma_1' \bar{\gamma}_2' \delta_1' \bar{\delta}_2' | \alpha_1 \bar{\alpha}_2 \beta_1 \beta_2\rangle &= \sigma_1 \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}\delta_{\gamma_1'\delta_2'}\delta_{\gamma_2'\delta_1'} + \sigma_2 \delta_{\alpha_1\gamma_1'}\delta_{\alpha_2\beta_1}\delta_{\beta_2\delta_2'}\delta_{\gamma_2'\delta_1'} \\ &+ \sigma_3 \delta_{\alpha_1\beta_2}\delta_{\alpha_2\gamma_2'}\delta_{\beta_1\delta_1'}\delta_{\gamma_1'\delta_2'} + \sigma_4 \delta_{\alpha_1\delta_1'}\delta_{\alpha_2\beta_1}\delta_{\beta_2\gamma_2'}\delta_{\gamma_1'\delta_2'} + \sigma_5 \delta_{\alpha_1\beta_2}\delta_{\alpha_2\delta_2'}\delta_{\beta_1\gamma_1'}\delta_{\gamma_2'\delta_1'} \end{aligned}$$

$$+\sigma_6\delta_{\alpha_1\gamma_1'}\delta_{\alpha_2\gamma_2'}\delta_{\beta_1\delta_1'}\delta_{\beta_2\delta_2'}+\sigma_7\delta_{\alpha_1\delta_1'}\delta_{\alpha_2\delta_2'}\delta_{\beta_1\gamma_1'}\delta_{\beta_2\gamma_2'}+\sigma_8\delta_{\alpha_1\gamma_1'}\delta_{\alpha_2\delta_2'}\delta_{\beta_1\delta_1'}\delta_{\beta_2\gamma_2'}$$
$$+\sigma_9\delta_{\alpha_1\delta_1'}\delta_{\alpha_2\gamma_2'}\delta_{\beta_1\gamma_1'}\delta_{\beta_2\delta_2'}. \tag{25}$$

and the action of the non-local charge Q^{ij} on the bound states is well known.

$$Q^{ij}|\pi^a_{\alpha_1\alpha_2}\pi^b_{\beta_1\beta_2}\rangle_{\text{in}} = (M^{ij}_{\text{in}})_{\alpha_1'\alpha_2'\beta_1'\beta_2'\alpha_1\alpha_2\beta_1\beta_2}|\pi^a_{\alpha_1'\alpha_2'}\pi^b_{\beta_1'\beta_2'}\rangle_{\text{in}} \tag{26a}$$

$$_{\text{out}}\langle\pi^a_{\gamma_1'\gamma_2'}\pi^b_{\delta_1'\delta_2'}|Q^{ij} = {}_{\text{out}}\langle\pi^a_{\gamma_1\gamma_2}\pi^b_{\delta_1\delta_2}|\,(M^{ij}_{\text{out}})_{\gamma_1\gamma_2\delta_1\delta_2\gamma_1'\gamma_2'\delta_1'\delta_2'}, \tag{26b}$$

where

$$(M^{ij}_{\text{in}})_{\alpha_1'\alpha_2'\beta_1'\beta_2'\alpha_1\alpha_2\beta_1\beta_2} = \tfrac{1}{2}(\delta^{i\alpha_2'}\delta^{j\beta_1'}\delta^{\alpha_2\beta_1}-\delta^{i\beta_1}\delta^{j\alpha_2}\delta^{\alpha_2'\beta_1'})\delta^{\alpha_1\alpha_1'}\delta^{\beta_2\beta_2'}$$
$$+\tfrac{1}{2}(\delta^{i\beta_2'}\delta^{j\alpha_2}\delta^{\beta_2\alpha_2'}-\delta^{i\alpha_2'}\delta^{j\beta_2}\delta^{\alpha_2\beta_2'})\delta^{\alpha_1\alpha_1'}\delta^{\beta_1\beta_1'}$$
$$+\tfrac{1}{2}(\delta^{i\beta_1}\delta^{j\alpha_1'}\delta^{\alpha_1\beta_1'}-\delta^{i\alpha_1}\delta^{j\beta_1'}\delta^{\beta_1\alpha_1'})\delta^{\alpha_2\alpha_2'}\delta^{\beta_2\beta_2'}$$
$$+\tfrac{1}{2}(\delta^{i\alpha_1}\delta^{j\beta_2}\delta^{\alpha_1'\beta_2'}-\delta^{i\beta_2'}\delta^{j\alpha_1'}\delta^{\alpha_1\beta_2})\delta^{\alpha_2\alpha_2'}\delta^{\beta_1\beta_1'}$$
$$+\alpha\delta^{ij}\delta^{\alpha_1\alpha_1'}\delta^{\alpha_2\alpha_2'}\delta^{\beta_1\beta_1'}\delta^{\beta_2\beta_2'}-\phi_1\delta^{i\alpha_1}\delta^{j\alpha_1'}\delta^{\alpha_2\alpha_2'}\delta^{\beta_1\beta_1'}\delta^{\beta_2\beta_2'}$$
$$+\phi_2\delta^{i\alpha_2'}\delta^{j\alpha_2}\delta^{\alpha_1\alpha_1'}\delta^{\beta_1\beta_1'}\delta^{\beta_2\beta_2'}-\phi_3\delta^{i\beta_1}\delta^{j\beta_1'}\delta^{\alpha_1\alpha_1'}\delta^{\alpha_2\alpha_2'}\delta^{\beta_2\beta_2'}$$
$$+\phi_4\delta^{i\beta_2'}\delta^{j\beta_2}\delta^{\alpha_1\alpha_1'}\delta^{\alpha_2\alpha_2'}\delta^{\beta_1\beta_1'} \tag{27a}$$

with $\phi_i = \dfrac{\theta_i}{i\pi}$

and

$$(M^{ij}_{\text{out}})_{\gamma_1\gamma_2\delta_1\delta_2\gamma_1'\gamma_2'\delta_1'\delta_2'} = -\tfrac{1}{2}(\delta^{i\delta_1'}\delta^{j\gamma_1}\delta^{\gamma_1'\delta_1}-\delta^{i\gamma_1'}\delta^{j\delta_1}\delta^{\delta_1'\gamma_1})\delta^{\gamma_2\gamma_2'}\delta^{\delta_2\delta_2'}$$
$$-\tfrac{1}{2}(\delta^{i\gamma_1'}\delta^{j\delta_2'}\delta^{\gamma_1\delta_2}-\delta^{i\delta_2}\delta^{j\gamma_1}\delta^{\gamma_1'\delta_2'})\delta^{\gamma_2\gamma_2'}\delta^{\delta_1\delta_1'}$$
$$-\tfrac{1}{2}(\delta^{i\gamma_2}\delta^{j\delta_1}\delta^{\delta_1'\gamma_2'}-\delta^{i\delta_1'}\delta^{j\gamma_2'}\delta^{\delta_1\gamma_2})\delta^{\gamma_1\gamma_1'}\delta^{\delta_2\delta_2'}$$
$$-\tfrac{1}{2}(\delta^{i\delta_2}\delta^{j\gamma_2'}\delta^{\gamma_2\delta_2'}-\delta^{i\gamma_2}\delta^{j\delta_2'}\delta^{\gamma_2'\delta_2})\delta^{\gamma_1\gamma_1'}\delta^{\delta_1\delta_1'}$$
$$+\alpha\delta^{ij}\delta^{\gamma_1\gamma_1'}\delta^{\gamma_2\gamma_2'}\delta^{\delta_1\delta_1'}\delta^{\delta_2\delta_2'}-\phi_1\delta^{i\gamma_1'}\delta^{j\gamma_1}\delta^{\gamma_2\gamma_2'}\delta^{\delta_1\delta_1'}\delta^{\delta_2\delta_2'}$$
$$+\phi_2\delta^{i\gamma_2}\delta^{j\gamma_2'}\delta^{\gamma_1\gamma_1'}\delta^{\delta_1\delta_1'}\delta^{\delta_2\delta_2'}-\phi_3\delta^{i\delta_1'}\delta^{j\delta_1}\delta^{\gamma_1\gamma_1'}\delta^{\gamma_2\gamma_2'}\delta^{\delta_2\delta_2'}$$
$$+\phi_4\delta^{i\delta_2}\delta^{j\delta_2'}\delta^{\gamma_1\gamma_1'}\delta^{\gamma_2\gamma_2'}\delta^{\delta_1\delta_1'}. \tag{27b}$$

The non-local conservation law

$$(M^{ij}_{\text{out}})_{\gamma_1\gamma_2\delta_1\delta_2\gamma_1'\gamma_2'\delta_1'\delta_2'}\langle\gamma_1'\bar{\gamma}_2'\delta_1'\bar{\delta}_2'|\alpha_1\bar{\alpha}_2\beta_1\bar{\beta}_2\rangle$$
$$\equiv \langle\gamma_1\bar{\gamma}_2\delta_1\bar{\delta}_2|\alpha_1'\bar{\alpha}_2'\beta_1'\bar{\beta}_2'\rangle\,(M^{ij}_{\text{in}})_{\alpha_1'\alpha_2'\beta_1'\beta_2'\alpha_1\alpha_2\beta_1\beta_2} \tag{28}$$

implies the following equations

$$\sigma_1 = \frac{\sigma_2}{\phi+\alpha-1}, \tag{29a}$$

$$\sigma_2 = \frac{\sigma_6}{\phi-1-\alpha}, \tag{29b}$$

$$\sigma_3 = \frac{\sigma_6}{\phi+\alpha-1}, \tag{29c}$$

$$\sigma_4 = \sigma_5 = 0, \tag{29d}$$

$$\sigma_7 = \frac{\sigma_6}{\phi^2}, \tag{29e}$$

$$\sigma_8 = \sigma_9 = -\frac{\sigma_6}{\phi}, \tag{29f}$$

where $\phi = \dfrac{\phi_1+\phi_2}{2} - \dfrac{\phi_3+\phi_4}{2}$.

We have then as solution for the S-matrix

$$\begin{aligned} {}_{ij}S_{kl} = & \left[\frac{2\phi(\phi-1)+(1-\alpha^2)}{\phi(\phi-\alpha-1)(\phi+\alpha-1)}\right]\sigma_6\delta_{ij}\delta_{kl} \\ & + \left[\frac{\phi(\phi-1)^2+(\phi-1)(1-\alpha^2)}{\phi(\phi-\alpha-1)(\phi+\alpha-1)}\right]\sigma_6\delta_{ik}\delta_{jl} \\ & - \left[\frac{2\phi(\phi-1)^2+(\phi-1)(1-\alpha^2)}{\phi^2(\phi-\alpha-1)(\phi+\alpha-1)}\right]\sigma_6\delta_{il}\delta_{jk}. \end{aligned} \tag{30}$$

Now we see that we must fix α in order to satisfy crossing and for $\alpha = 1$ the final S-matrix becomes the one from the O(3) non-linear sigma model [10, 11].

The authors wish to thank E. Abdalla for stimulating discussions. The work of M.C.B. Abdalla and A. Lima was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP).

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