MULTIDIMENSIONAL UNIVERSES, KALUZA-KLEIN, EINSTEIN SPACES AND SYMMETRY BREAKING*

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The aim of this paper is to give a review of facts and results on "multidimensional universes" which form the basis for Kaluza-Klein theories. The paper contains a survey of the theory of fiber bundles, a discussion of invariant metrics on groups and homogeneous spaces and a survey of results on topology and metric of multidimensional universes relevant for Kaluza-Klein theories.

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PLAN OF THE LECTURES

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1. *Introduction*

The following notes have been written after a set of lectures given at the Szczyrk Summer School; they are as close as possible to their oral counterpart; for this reason, the style is quite direct, also no attempt is made to give precise references at each step but they are collected at the end of the sections. The aim of these lectures was to present a review of the "multidimensional universes" where the old Kaluza-Klein idea holds true: we live in a world of dimension bigger than four but we do not realise it is so. This simple idea has enough power to restrict our attention to very particular topological spaces and very special metrics; however a detailed study of the subject can hardly be done without some basic knowledge of the relevant mathematical background, for this reason I decided to give first a survey of the theory of fiber bundles (this survey is rather based on intuitive grounds, even if precise definitions are sometimes given, however it is complete enough for our purpose), this is done in Section 2, then there is (Section 3) a discussion on invariant metrics on groups and homogeneous spaces, this section could be entitled "What can be the shape of our internal world?". Then comes a very short section on basic Riemannian geometry — almost without any formula. The important results about the structure (topology and metric) of these multidimensional universes is given in Section 5 where the physical ideas are also discussed; the impatient reader can go directly there; however, the mathematics of the subject being studied before (i.e. from Section 1 to 4) Section 5 is by itself short but "dense" and it is maybe unwise to step directly there. In Section 6 we show how to obtain many homogeneous Einstein metrics on groups and homogeneous spaces and study how they can lead to "spontaneous symmetry breaking". Actually I want to draw the attention of the reader on the fact that this Section 6 is special, in the following sense: the subsections devoted to Einstein metrics is certainly correct but the part (Section 6.3) devoted to the possible application to symmetry breaking is an unfinished and unpolished work (forgive the pun!); there exist difficulties (mentioned in the text) which, I hope, could be overcome.

Now it is maybe necessary to warn the "specialist" that there exist two entirely different kinds of theories under the keyword "Kaluza-Klein": in the first kind of theory (that we will discuss), dimensional reduction is automatic and is a consequence of the invariance of the metric under a group; in the second kind of theories, the metrics employed are usually not invariant and one has to perform some harmonic analysis and "integration over the internal space" to recover some 4-dimensional theory: this kind of theory will not be discussed at all.

2. *Generalities on fiber bundles*

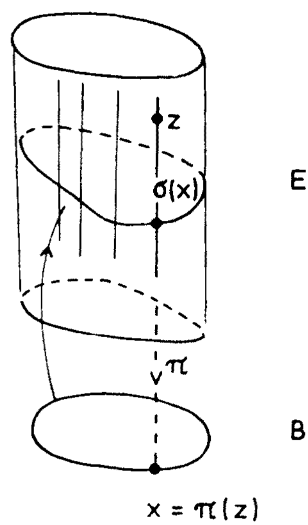
2.1. Intuitive aspects

Everybody knows what a topological space is, but one has to be cautious about the vocabulary, in particular,
a) a topological manifold is a special kind of topological space (each point possesses a neighborhood looking like an open set of R^n. Notice that the following object "∞" is not a manifold (it contains a cross ×).
b) a differential manifold is a special kind of a): it has to be "smooth".
c) a fiber bundle is a special case of b) and can be thought of as a collection of "fibers" glued together and parametrised by a "base space"; there is also an additional structure derived from the action of a group on the fibers.

Warnings:
* The same differentiable manifold can be sometimes given several fiber bundle structures.
* For the moment, we do not specify any particular metric on the spaces under consideration (no shape is specified yet).

2.2. Examples

2.2.1. The infinite cylinder



a) It is a collection of lines glued together and parametrised by a circle $B = S^1$.
b) These lines are "copies" of the set of real numbers R but the origin is not specified on each line. We say that the typical fiber is R.
c) The fibers are parametrised by the points of a circle B, therefore we have a map π (called the projection):

$$E \longrightarrow B$$

$$z \rightsquigarrow \pi(z) = x$$

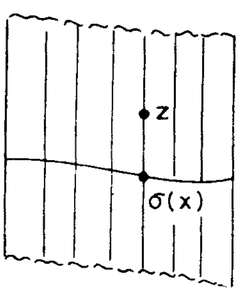
The set $\pi^{-1}(x) = E_x$ is called the fiber above x.
d) The additive group of real numbers acts on the fibers (by translation).
e) We can also cover the base (i.e. the circle B) by open sets $U_i =]\text{-}[$. Then $\pi^{-1}(U_i)$ is a strip diffeomorphic to $U_i \times R$ and we can think of E as a pasting of these strips.
f) The infinite cylinder is called a "trivial bundle" because it is diffeomorphic to (*base space*) × (*typical fiber*), here, $E = S' \times R$.
g) In order to specify the coordinates of a point $z \in E$, we need an origin on the fibers, therefore we need to "cut" the cylinder, therefore we need a map σ from the circle (base) into the cylinder. This map — called a section of E — has to satisfy the relation $\pi \circ \sigma$ = *Identity*. The role of this section will be to mark the origin on each fiber: $\sigma(x)$ *is* now the origin of the fiber above x and we can now say that the coordinates of $z \in E$ are (with

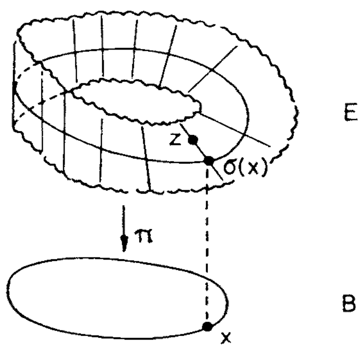
respect to the choice of σ), $z^\sigma = (x, \alpha)$ where $x = \pi(z) \in B$ and $\alpha \in R$. The choice of σ is of course arbitrary (call it gauge freedom!): if we had chosen another section σ' we would have found $z^{\sigma'} = (x, \beta)$ with $\beta \neq \alpha$.

Warnings:
* A fiber bundle is not necessarily trivial!
* It is not always possible to find a global section.

2.2.2. The infinite Moebius cylinder (also called open Moebius strip)

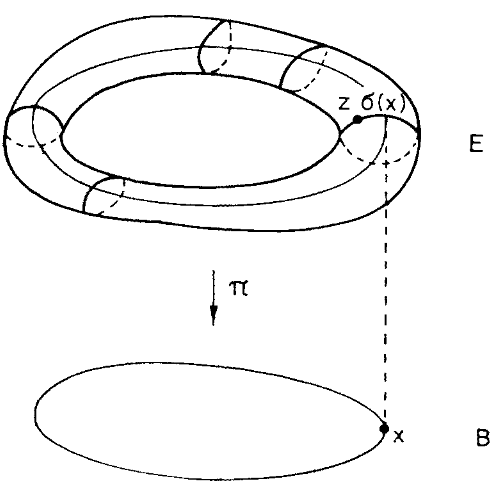
The previous example (infinite cylinder) could be constructed as follows: start with a usual strip and extend the segments "at infinity". Now, if we follow the same procedure but starting with a Moebius strip, we obtain the Moebius cylinder.





The figure crosses itself but this is due to the fact that it is impossible to embed this object in three dimensions. The properties a, b, c, d, e discussed in Section 2.1.1 are exactly the same but property f has to be modified: E is not a trivial bundle ($E \neq S' \times R$); however it is still a pasting of trivial bundles (see e), this property is called "local triviality".

2.2.3. The torus T^2



a) It is a collection of circles glued together and parametrised by another circle.
b) The fibers are copies of S^1 — or of the group U(1) — but the origin is not specified on each circle.
c) Fibers are parametrised by the points of another circle B and we have a map (projection) π:

$$E \longrightarrow B$$

$$z \rightsquigarrow \pi(z) = x$$

d) The group U(1) acts on the fibers.
e) We can also cover the base (the circle B) by open sets $U_i =]\!-\![$. Then $\pi^{-1}(U_i)$ is a por-

tion of cylinder diffeomorphic to $U_i \times S^1$ and we can think of E as a pasting of these pieces of cylinders.

f) The torus T^2 is trivial because $T^2 = (\textit{base}) \times (\textit{typical fiber}) = S' \times S'$.

g) The discussion runs as in 2.1.1, property g): to parametrise the points of E, we need to cut the torus. According to the choice of a section σ, we can write $z = (x, e^{i\theta})$ with $x = \pi(z)$.

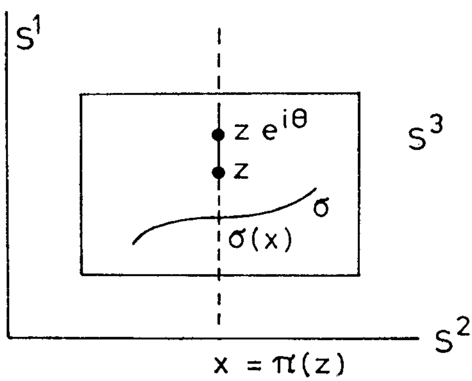
2.2.4. The manifold $S^2 \times S^1$

This manifold (which is not embeddable in 3 dimensions) can be thought of as a collection of circles S' parametrised by the points of a two-sphere S^2. It is a (trivial) bundle with base S^2 and typical fiber S^1; the group U(1) acts on the fibers.

2.2.5. The manifold $S^3 = \mathrm{SU}(2)$

It is first maybe useful to remind the reader that SU(2) has the topology of the three--sphere (proof: parametrise SU(2) by 2×2 matrices with complex coefficients α, β then, we obtain $\det(\;\;) = \mathrm{Re}^2(\alpha) + \mathrm{Im}^2(\alpha) + \mathrm{Re}^2(\beta) + \mathrm{Im}^2(\beta) = 1$, i.e., the equation of a 3-sphere).

Now, it is important to realise that SU(2) can be considered as a collection of circles S parametrised by the points of a two-sphere S^2, exactly as in the previous example; indeed, choose a subgroup $\mathrm{U}(1) \sim S$ in SU(2) and make a coset decomposition of SU(2) along U(1), i.e., define the equivalence relation $aRb \Leftrightarrow ab^{-1} \in \mathrm{U}(1) \Leftrightarrow a \in b\mathrm{U}(1)$, the set of equivalence classes is the homogeneous space $\mathrm{SU}(2)/\mathrm{U}(1) = S^2$ and we can write the decomposition $\mathrm{SU}(2) = \bigcup_{\bar{a} \in S^2} (a\mathrm{U}(1))$. Therefore, exactly as in the previous example we can consider $S^3 = \mathrm{SU}(2)$ as a fiber bundle with typical fiber S^1 and base S^2 but this time, it is not trivial because $S^3 \neq S^2 \times S^1$. Properties a, b, c, d, e, g could be discussed as previously. Since we cannot draw a picture of S^3, it is convenient to refer to the above fibering of S^3 by the following picture (easily generalisable!):



2.3. Definitions

2.3.1. Principal fiber bundles

The precise definition (which follows) becomes quite natural after the study of the previous examples. P has a structure of principal fiber bundle with total space P, base M, projection π and structure group G if

1) P and M are smooth manifolds, with $\left\{\begin{matrix} P \longrightarrow M \\ z \rightsquigarrow x \end{matrix}\right\}$ a smooth map onto.

2) Each $x \in M$ has a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic with $U \times G$ (local triviality).

3) G is a Lie group acting from the right on each fiber.

The third axiom means that fibers are just copies of the group G, however the origin is not specified on these fibers. The adjective "principal" means that the typical fiber is a Lie groups (all previous examples are principal bundles) but there are more general situations (next Section). One has to remember that the structure group G acts from the right, this is a convention but, fortunately, everybody agrees.

2.3.2. Associated fiber bundles

Starting with a group G, one can study actions of this group on other manifolds: on a vector space (we study linear realisations of G, i.e. representations) or on a manifold which is not a vector space (then we study non linear realisations of G, for example G action on homogeneous spaces G/H).

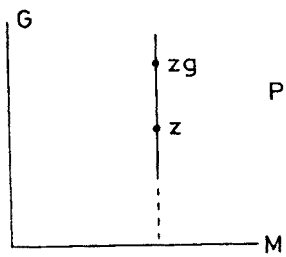
In an analogous way, starting with a principal bundle P, whose structural group is G and choosing a manifold F on which G acts (linearly or not), we can construct a new space by putting a copy of F above every point of the base of P, in other words we replace the typical fiber of P (the Lie group G) by a manifold on which G acts.

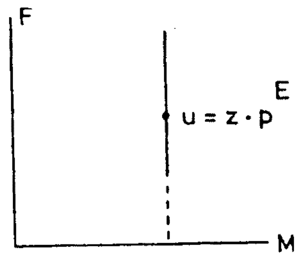
The precise definition is the following: consider a (left) group action on a manifold F—i.e. if $g \in G$, $p \in F$, we know what $gp \in F$ is —, then define the following equivalence relation in $P \times F$

$$(z, p) \sim (z', p') \Leftrightarrow \exists g \in G: \quad z' = zg \quad \text{and} \quad p' = g^{-1}p.$$

The quotient space $E = P \times F/\!\sim$ is called an associated bundle; it is called a vector bundle if F is a vector space. Let us call $u = z \cdot p = zg \cdot g^{-1}p$ an equivalence class (E is the set of u); the meaning of this notation is the following: the "geometrical object" u has "coordinates" p in the "frame" z (and it has coordinates $g^{-1}p$ in the frame zg). This somewhat abstract definition should be clarified by considering a well known example: let M be a manifold of dimension n and P the set of all possible frames of this manifold, it is clear that P is a bundle (intuitively it is a collection of fibers — the set of frames at one point of M — parametrised by the points of M); moreover, if we choose (arbitrarily) one frame z at one point x, any other frame z' can be obtained by an action of the group $\mathrm{GL}(n): z' = zg$; P is therefore a principal fiber bundle and the structure group is $\mathrm{GL}(n)$. Now, the group $\mathrm{GL}(n)$ acts also on the vector space R^n: if $g \in \mathrm{GL}(n)$ and $p \in R^n$, we know what $gp \in R^n$ is; therefore we can construct the associated (vector) bundle $P \times R/\!\sim\; = E$ via the above construction, it is clear that E is nothing else that the "tangent bundle of M" and an element $u = z \cdot p$ of E is a tangent vector (geometrical object) which has coordinates p in the frame z. The "abstract" definition given previously has the advantage of being general and useful, even if F is not a vector space.

A pictorial way of looking at bundles is given by Fig. 5.





Warning:
* The structure group G of P (principal fiber bundle) acts on P from the right, on F from the left but not at all on the associated bundle E.

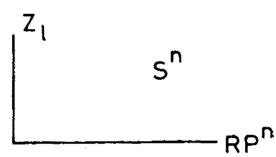
2.4. A few systematic examples

● A given Lie group G has usually many structures of principal fiber bundles (it can be sliced in many ways!): generalising the example 2.2.5, if one chooses a (closed) subgroup $H \subset G$, one can write G as an H bundle over G/H.
● Given a principal bundle P of base M and structure group G, we can always construct a lot of associated vector bundles of base M and fiber V (a vector space) by choosing V as a representation space for G; we can also construct a lot of associated (non vector) bundles E of base M and typical fiber G/H. The reader may convince himself that one can even recover P as an H-bundle over E (there are several ways of "slicing" P!).
● Hopf fibering of spheres: we present without proof a list of nice examples (useful later to build more examples).
From the embedding $S^n \subset R^{n+1}$ and from the usual construction of the real projective space RP^n as set of directions in R^{n+1}, i.e. from

the diagram

$$\begin{array}{ccc} R^{n+1} & \supset & S^n \\ \downarrow & & \downarrow \\ RP^n & = & S^n/Z_2 \end{array}$$

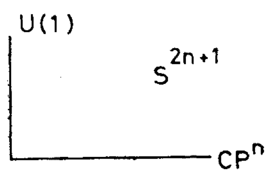
we obtain



in the same way,

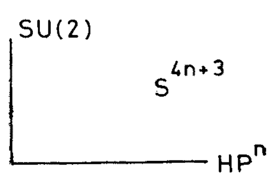
$$\text{from}\quad \begin{array}{l} C^{n+1} = R^{2n+2} \supset S^{2n+1} \\ \downarrow \\ CP^n = S^{2n+1}/U(1) \end{array}$$

we obtain



$$\text{from}\quad \begin{array}{l} H^{n+1} = R^{4n+4} \supset S^{4n+3} \\ \downarrow \\ HP^n = S^{4n+3}/SU(2) \end{array}$$

we obtain



The previous fibering of spheres are principal fiberings (the typical fiber is a group); let us quote also the (non principal) fibering of S^{15} as an S^7 bundle over S^8.

2.5. Group action on manifolds and the role of the normaliser

After what has been said previously it should be clear in the reader's mind that a space which can be written as $\big|\underline{{}^{G/H}\quad}_{M}$ can be obtained from a principal bundle $\big|\underline{{}^{G}\quad}_{M}$, however, it is maybe not clear (but it is important) that a space which can be written as $\big|\underline{{}^{H\backslash G}\quad}_{M}$ can be obtained from a principal bundle $\big|\underline{{}^{N/H}\quad}_{M}$ where N is the normaliser of H into G.

2.5.1. Normal subgroup. Normaliser. Centraliser

- Let N be a group and H a subgroup, in general $N/H = \{nH/n \in N\}$ is not a group because multiplication of classes $\bar{n} = nH$ is ambiguous, unless $nH = Hn$ for all n; in this last case H is called a normal subgroup of N and N/H is a group (left and right classes coincide).
- Let G be a group and H a subgroup ($H \subset G$). The normaliser N of H on G is defined as the biggest subgroup in which H is normal, i.e. $N = \{n/nH = Hn, n \in G\}$.

The normaliser N of H in G should not be confused with the centraliser Z of H in G defined as follows: $Z = \{z/z \in G, \forall\ h \in H, zh = hz\}$, Z contains in particular the center C of H.

For example, consider $H = \mathrm{SU}(2) \times \mathrm{U}(1)$ $G = \mathrm{SU}(5)$; then, up to discrete factors, $C = \mathrm{U}(1)$, $Z = \mathrm{SU}(3) \times \mathrm{U}(1)$, $N = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ and $N/H = \mathrm{SU}(3)$.

2.5.2. N/H as group of automorphisms of $H\backslash G$

Consider the set of *right* classes $H\backslash G = \{Hp/p \in G\}$, then it is natural to call "automorphism" of $H\backslash G$ a one to one map from $H\backslash G$ into itself commuting with the right G-action i.e. such that $\alpha(pg) = \alpha(p)g$. It is easy to see that there is a one to one correspondence between these automorphisms of $H\backslash G$ and multiplications (from the left) by elements of $n \in N$ (indeed $n[Ha] = [Hna]$) however, multiplication by n or by nh, $h \in H$ gives the same result therefore N/H can be considered as the group of automorphisms of $H\backslash G$ and N/H acts on $H\backslash G$ from the left.

2.5.3. A new class of associated bundles

Suppose that we start from a principal fiber bundle P of base M and structure group N/H, N being the normaliser of H in a group $G(H \subset N \subset G)$; then, following the general construction of associated bundles given in Sect. 2.3.2, we can, using the above action of N/H on HG, construct a new bundle E of base M and typical fiber $H\backslash G$.

Warning:

* The structure group N/H of P acts on P from the right (since P is a principal bundle), it acts on $H\backslash G$ from the left (Sect. 2.5.2) but does not act at all on the associated bundle E (Sect. 2.3.2). However the group G *still* acts on E (from the right).

In other words G plays the role of "active" transformations of E and N/H is the group of "passive" transformations (it does not move the points but changes their coordinates); when H is trivial then G and N/H are equal: active and passive transformations are described by the same group.

2.5.4. A theorem on group actions

Let us quote, without proof, the following theorem (which is quite natural after the previous discussion): let E be a manifold on which a group G acts regularly from the right (call H_u the little group or stabiliser at $u: H_u = \{g/gu = u\}$, then suppose that all the groups H are conjugated), then one can prove that E is a fiber bundle whose base M, the set of orbits, is a manifold and whose typical fiber is $H\backslash G$; moreover the structure group is N/H (i.e., E can be constructed as an associated bundle to a principal bundle of base M and fiber N/H).

Here we should pause to make a first physical interpretation: E is a "multidimensional universe", G is a group of global transformation, M is the space time, $H\backslash G$ is the internal space and N/H is the gauge group.

The basic reference for this Section is [1], more details about the "role of the normaliser" and about group action may be found in Sect. 2 of [2] and in [3].

3. *Special metrics on Lie groups and homogeneous spaces*

3.1. Intuitive aspects

When no metric is specified, one should think of a finite dimensional Lie group as a kind of surface with well defined topology (holes, handles, ...) but without given shape; however multiplication of points is defined, the notion of neighbourhood exists and this "protoplasmic" surface is smooth. Choosing a metric is choosing a shape; in most physical applications, the usual choice is the so called Killing metric but it should be realised that the corresponding "shape" — which is maximally symmetric — is by no means the only possible one. When we come to discuss the physical interpretation of multidimensional universes, we see that the Lie group under consideration (or, in some cases, the homogeneous space) coincides with what people call "the internal space"; at each point x of space time we have such an internal world, but its shape may change with the space-time point and we have therefore to know how to parametrise the metric of this internal space. For a given manifold, a metric (therefore a shape) can be maximally symmetric, partially symmetric or may have no symmetries at all; here we use the word symmetry with its intuitive meaning but it will be necessary later to become more precise.

3.2. Left and right invariant vector fields on a Lie group G

Let us call $\mathscr{G}$, the tangent space at the origin $e \in G$, i.e., the Lie algebra of G; let us also choose arbitrarily a base $\{X_\alpha\}$ in this vector space. These vectors can be considered as first degree differential operators acting on functions defined on G and they satisfy the commutation relations $[X_\alpha, X_\beta] = C^\gamma_{\alpha\beta} X_\gamma$. Also let us call $\{Y^\alpha\}$ the dual basis of $\{X_\alpha\}$, they satisfy $Y^\alpha[X_\beta] = \delta^\alpha_\beta$. Now, using matrix notations, let us define $\{\varepsilon_\alpha\}$ the base of left invariant vector fields on G which satisfy

$$\varepsilon_\alpha(e) = X,$$
$$g_1\varepsilon_\alpha(g_2) = \varepsilon_\alpha(g_1 g_2),$$
$$[\varepsilon_\alpha, \varepsilon_\beta](g) = C^\gamma_{\alpha\beta}\varepsilon_\gamma(g),$$

$\{\omega^\alpha\}$ the dual basis of $\{\varepsilon_\alpha\}: \omega^\alpha[\varepsilon_\beta] = \delta^\alpha_\beta$.
$\{e_\alpha\}$ a basis of right invariant vector fields on G which satisfy

$$e_\alpha(e) = -X_\alpha$$
$$e_\alpha(g_2)g_1 = e_\alpha(g_2 g_1)$$
$$[e_\alpha, e_\beta](g) = -C^\gamma_{\alpha\beta}e_\gamma(g)$$

$\{\sigma^\alpha\}$ the dual basis of $\{e_\alpha\}$: $\sigma^\alpha[e_\beta] = \delta^\alpha_\beta$. Of course, $\varepsilon_\alpha(g)$, for example, denotes a vector of the tangent space to G at $g \in G$. One can prove that $[e_\alpha(g), \varepsilon_\beta(g)] = 0$. To fix the ideas let us give an example with SU(2) parametrised by three Euler angles θ, φ, ψ; at the point $g = (\theta, \varphi, \psi)$, consider the following:

$$\varepsilon_1 = \cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi\left(\cot\theta \frac{\partial}{\partial\varphi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\psi}\right); \qquad \omega^1 = \cos\varphi d\theta + \sin\varphi \sin\theta d\psi$$

$$\varepsilon_2 = \sin\varphi \frac{\partial}{\partial\theta} + \cos\varphi\left(\cot\theta \frac{\partial}{\partial\varphi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\psi}\right); \qquad \omega^2 = \sin\varphi d\theta - \cos\varphi \sin\theta d\psi$$

$$\varepsilon_3 = \frac{\partial}{\partial\psi}; \qquad \omega^3 = d\varphi + \cos\theta d\psi$$

$$e_1 = \cos\psi \frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta \frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right); \qquad \sigma^1 = \cos\psi d\theta + \sin\psi \sin\theta d\varphi$$

$$e_2 = \sin\psi \frac{\partial}{\partial\theta} + \cos\psi\left(\cot\theta \frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right); \qquad \sigma^2 = \sin\psi d\theta - \cos\psi \sin\theta d\varphi$$

$$e_3 = \frac{\partial}{\partial\varphi}; \qquad \sigma^3 = d\psi + \cos\theta d\varphi$$

One can easily check that $\omega^i[\varepsilon_j] = \sigma^i[e_j] = \delta^i{}_j$, that $[\varepsilon_1, \varepsilon_2] = \varepsilon_3$ etc., that $[e_1, e_2] = -e_3$ etc. and that $[\varepsilon_i, e_j] = 0$.

3.3. Metrics on Lie groups and their isometries

3.3.1. Example with SU(2)

A metric is a rule allowing us to compute the scalar product of two vectors at one point $g \in G$. If y^i are the coordinates of g in some chart, the tangent space at g is spanned by the operators $\left\{\frac{\partial}{\partial y^i}\right\}$ and the cotangent space by $\{dy^i\}$; therefore a metric can be written as $h = h_{ij}dy^i \otimes dy^j$ (and the inverse metric as $h^{-1} = h^{ij}\frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial y^j}$. However, it is in most cases easier to use a non coordinate basis composed of invariant vector fields. Let us give a simple example and consider in SU(2) the following metric:

$$h = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3.$$

It can be easily proved that h can also be written as

$$\begin{aligned} h &= \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3 \\ &= d\theta \otimes d\theta + d\varphi \otimes d\varphi + d\psi \otimes d\psi + \cos\theta (d\psi \otimes d\varphi + d\varphi \otimes d\psi). \end{aligned}$$

In order to study isometries, it is convenient to introduce the notion of Lie derivatives: if u, v, w are three vector fields, one defines

$$\mathscr{L}_u(v) = [u, v] \quad \text{and} \quad \mathscr{L}_u(v \otimes w) = \mathscr{L}_u(v) \otimes w + v \otimes \mathscr{L}_u(w).$$

In the previous example, the inverse metric is

$$h^{-1} = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 = \varepsilon_1 \otimes \varepsilon_1 + \varepsilon_2 \otimes \varepsilon_2 + \varepsilon_3 \otimes \varepsilon_3.$$

It can be easily shown that

$$\mathscr{L}_{\varepsilon_1} h^{-1} = \mathscr{L}_{e_1} h^{-1} = 0.$$

For this reason, we say that the isometry group Iso (h) of the metric h is $SU(2) \times SU(2)$; it can be shown to coincide — up to scale — with the usual Killing metric and corresponds to the usual "round" three sphere S^3 (as already recalled, SU(2) has the same topology as S^3). Let us consider now a less symmetric metric on SU(2):

$$h = \lambda_1^2 \omega^1 \otimes \omega^1 + \lambda_2^2 \omega^2 \otimes \omega^2 + \lambda_3^2 \omega^3 \otimes \omega^3,$$

where λ_1^2, λ_2^2, λ_3^2 are positive real numbers; it is clear that the inverse metric is:

$$h^{-1} = \frac{1}{\lambda_1^2}\varepsilon_1 \otimes \varepsilon_1 + \frac{1}{\lambda_2^2}\varepsilon_2 + \frac{1}{\lambda_3^2}\varepsilon_3 \otimes \varepsilon_3$$

and, using this last expression, it is also clear that the isometry group of this metric is SU(2), indeed $\mathscr{L}_{e_i}(h^{-1}) = 0$ since $[e_i, \varepsilon_j] = 0$. This property would not be obvious if we used the following equivalent expression for h!

$$\begin{aligned} h = {} & [\lambda_1^2 \cos^2\varphi + \lambda_2^2 \sin^2\varphi] d\theta \otimes d\theta + [[\lambda_1^2 \sin^2\varphi + \lambda_2^2 \cos^2\varphi] \sin^2\theta \\ & + \lambda_3^2 \cos^2\theta] d\psi \otimes d\psi + \lambda_3^2 d\varphi \otimes d\varphi + \sin\varphi \cos\varphi \sin\theta (\lambda_1^2 - \lambda_2^2) \\ & \times [d\theta \otimes d\psi - d\psi \otimes d\theta] + \lambda_2^2 \cos\theta [d\psi \otimes d\varphi + d\varphi \otimes d\psi]. \end{aligned}$$

The above metric is called left invariant (it can be written in terms of left invariant vector fields) and corresponds to a kind of hyper ellipsoid if $\lambda_1 \neq \lambda_2 \neq \lambda_3$, the first example was bi-invariant. Let us end these examples with the following $SU(2) \times U(1)$ invariant metric:

$$h = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \lambda^2(\omega^3 \otimes \omega^3),$$

$$h^{-1} = \varepsilon_1 \otimes \varepsilon_1 + \varepsilon_2 \otimes \varepsilon_2 + \frac{1}{\lambda^2}(\varepsilon_3 \otimes \varepsilon_3).$$

Indeed, $\mathscr{L}_{e_i}(h^{-1}) = 0$ for $i = 1, 2, 3$ and $\mathscr{L}_{\varepsilon_3}(h^{-1}) = 0$ and we will say that $e_1, e_2, e_3, \varepsilon_3$ are Killing vector fields.

3.3.2. General results

More generally, a left invariant metric on a Lie group G is entirely characterized by its expression at the origin e of G (i.e. it is a bilinear symmetric form on the Lie algebra $\mathscr{G}$); if one knows the scalar product $\langle v_1, v_2 \rangle$ of two vectors v_1, v_2 at $q \in G$, one also knows the scalar product of the left-transported vectors pv_1 and pv_2 at the point pq: $\langle v_1, v_2 \rangle_q = \langle pv_1, pv_2 \rangle_{pq}$. If $\{\varepsilon_\alpha(g)\}$ is a base of left invariant vector fields, any left invariant metric can be written as $h = \Lambda^{\alpha\beta} \varepsilon_\alpha \otimes \varepsilon_\beta$ where the $\Lambda^{\alpha\beta}$ is symmetric but independent of the point $g \in G$ (independent of ψ, θ, φ in the previous examples). More general metrics can be constructed by allowing the $\Lambda^{\alpha\beta}$ to depend upon the point (make $\lambda_1, \lambda_2, \lambda_3$ functions of ψ, θ, φ in the previous example). The space of all possible metrics on a Lie group is clearly infinite dimensional; however the space of left invariant metrics on a Lie group of dimension n is itself a finite dimensional of dimension $n(n+1)/2$ (Proof: choose a bilinear symmetric form in the Lie algebra). From time to time, questions of volume are irrelevant: one does not want to make the difference between a given "shape" and another one which is just "bigger": the manifold of left invariant metrics with fixed volume is of course of dimension $n(n+1)/2-1$ and one can easily show that it is homeomorphic with $SL(n)/SO(n)$. A metric can be bi-invariant, therefore $\mathrm{Iso}(h) = G \times G$ and there is only one such metric (up to scale) if G is simple, it may have no invariance at all, therefore $\mathrm{Iso}(h) =$ *Identity*, or it may have few symmetries (then $\mathrm{Iso}(h) = H \times K \subset G \times G$): in particular it may be left invariant.

3.4. Metrics on homogeneous spaces

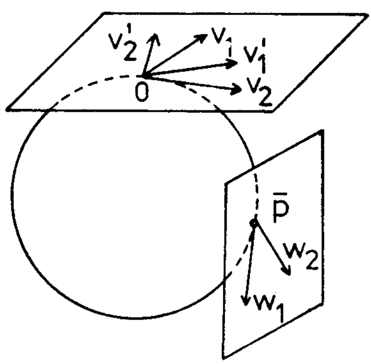
3.4.1. Technological remarks

When an equivalence relation is defined on a space E, the set of equivalence classes is called a coset space; when G is a group and H a subgroup, the coset space G/H of left classes along H is called a homogeneous space; when the homogeneous space G/H is not homeomorphic with a product of homogeneous spaces, it is called irreducible; if $\mathscr{G}$ is the Lie algebra of G and $\mathscr{H}$ the Lie algebra of H, the subspace $\mathscr{S}$ appearing in the decomposition $\mathscr{G} = \mathscr{H} \oplus \mathscr{S}$ can be interpreted as the tangent space at the origin of G/H and is the support of a representation of H (indeed $[\mathscr{H}, \mathscr{S}] \subset \mathscr{S}$) called the isotropy representation; when the isotropy representation is irreducible (on the real), the homogeneous space G/H is called isotropy irreducible. If a homogeneous space is irreducible symmetric and is such that $[\mathcal{S}, \mathcal{S}] \subset \mathcal{H}$, it is called irreducible symmetric (it is in particular isotropy irreducible). Example: SO(8)/SO(7) is a symmetric space, Spin $7/G_2$ is isotropy irreducible not symmetric, SU(4)/SU(3) is an irreducible homogeneous space which is neither symmetric nor isotropy irreducible; notice however that the three previous spaces are actually the same differential manifold: the seven-sphere S^7; one should therefore associate the previous qualifiers ("symmetric" etc...) to the pair (G, H) rather than to the underlying topological space.

To make the link with the previous section, it should be noticed that a homogeneous space G/H (with H a closed subgroup to avoid pathologies) can always be written as a fiber bundle with base G/N and typical fiber N/H, where N is the normaliser of H into G. Also one can prove that when G/H is isotropy irreducible (in particular if it is irreducible symmetric), the group N/H is discrete.

3.4.2. G-Invariant metrics on homogeneous spaces G/H

The group G acts on G/H by left multiplication (if $q \in G$ and $\bar{p} = pH \in G/H$, one has a map $\bar{q} \in G \to Lq(\bar{p}) = \bar{q}\bar{p} = qpH$), therefore one can try to do here what we did previously in the group case: we first go to the origin θ ($= eH = H$) of G/H and choose an arbitrary scalar product g_θ in the tangent space $\mathcal{S}$ at θ; we then go to another point $\bar{p}$ and consider two vectors W_1, W_2 in the tangent space at $\bar{p}$; one has to define what is the value of $g_{\bar{p}}(W_1, W_2)$. To do that we use the map L_p and its tangent map, the linear map DL_p and write $\bar{p} = L_p(e)$, $W_1 = DL_p(V_1)$, $W_2 = DL_p(V_2)$ for some V_1, V_2 in the tangent space $\mathcal{S}$ at θ. We then define $g_p(W_1, W_2) = g(V_1, V_2)$. But this definition is ambiguous, indeed if $\bar{p} = L_p(e)$ we have also $\bar{p} = L_{ph}(e)$ for $h \in H$, then for some V_1', V_2' in $\mathcal{S}$ we obtain $W_1 = DL_{ph}(V_1')$, $W_2 = DL_{ph}(V_2')$. But $g_\theta(V_1, V_2) \neq g_\theta(V_1', V_2')$ in general! (see Fig. 7).



Those (special) scalar products in $\mathcal{S}$ for which the last equality is true are called Ad H invariant and for them the previous method works; actually one proves that there is a one to one correspondence between G-invariant metrics on G/H and Ad H invariant bilinear scalar products in the tangent space at the origin. Let us give a few results and comments:

- There is always at least one G invariant metric on G/H, up to scale, when G is simple, it is called the "normal" metric and is obtained by taking the restriction of the Killing metric of G to the subspace $\mathcal{S}$($\mathcal{G} = \mathcal{H} + \mathcal{S}$ and $\mathcal{S} = \mathcal{H}^\perp$ for the Killing form).

● The normal metric on G/H associated to the pair (G, H) is not necessarily the most symmetric one; also the full group of isometries of the normal metric may be bigger than G.
● If G/H is irreducible symmetric, the normal metric h is the only G invariant metric (up to scale). Example $S^7 = SO(8)/SO(7)$. $\mathrm{Iso}(h) = O(8)$.
● If G/H is isotropy irreducible, the normal metric is also the only G invariant metric (up to scale). Example $S^7 = \mathrm{Spin}\, 7/G$. $\mathrm{Iso}(h) = O(8)$.
● If G/H is not isotropy irreducible, the dimension of the manifold of G-invariant metrics is $d = \sum_i r_i(r_i+1)/2$ where $\mathscr{S} = \bigoplus_i (V_i \otimes R^{r_i})$, V_i being the space of a real irreducible representation of Ad H.

Examples $S^7 = SU(4)/SU(3)$; $7 = 1+1[3+\bar{3}]$, $\quad d = 2$,

$$S^7 = USp(4)/USp(2);\ 7 = 3[1]+1[4], \qquad d = \frac{3\times 4}{2} + \frac{1\times 2}{2} = 7.$$

....

Basic references for this section are [1], [4]. Many comments, examples and other references can be found in [2, 5].

4. *Riemannian geometry almost without any formulae*

The specialist in general relativity may probably skip this section (see however Section 4.5) which is devoted to the definition of a few basic quantities like the scalar curvature. Starting with an analogy in the 2-dimensional case, we will define these objects almost without using the usual artillery of connexion coefficients, Riemann tensor, etc...

4.1. Gaussian curvature of a surface

4.1.1. First definition

The Gaussian curvature K_P at any point P of the surface S is the product of the curvatures of the curves through P cut out by normal planes. Of course the result is independent of the couple of normal planes.

Examples:

(*i*) a cylinder of radius R; then $k_1 = 0$, $k_2 = \dfrac{1}{R}$, $K = k_1 k_2 = 0$ everywhere.

(*ii*) a sphere of radius R; then $k_1 = \dfrac{1}{R}$, $k_2 = \dfrac{1}{R}$, $K = k_1 k_2 = \dfrac{1}{R^2}$ everywhere.

(*iii*) The graph of the surface $s = \dfrac{x^2}{2R} - \dfrac{y^2}{2R}$ looks like a saddle at the origin $P = (0, 0)$.

At this point P, $k_1 = \dfrac{1}{R}$, $k_2 = -\dfrac{1}{R}$, $K_P = k_1 k_2 = -\dfrac{1}{R^2}$.

This definition using oscillating circles makes an explicit reference to the three dimensional embedding and cannot be generalised easily to higher dimensions.

4.1.2. The formula of Puiseux-Bertrand

Consider a 2-dimensional sphere S of radius R; in the 2-dimensional tangent plane at $P \in S$, draw a circle of radius ε (length $2\pi\varepsilon$); let $c(P, \varepsilon)$ be the curve obtained by drawing out of P the set of geodesics of length ε with tangent vector in the tangent plane; the set of the endpoints $c(P, \varepsilon)$ is therefore a circle of radius $R \sin \varepsilon/R$, its length $l(P, \varepsilon)$ is $2\pi R \sin \varepsilon/R$.

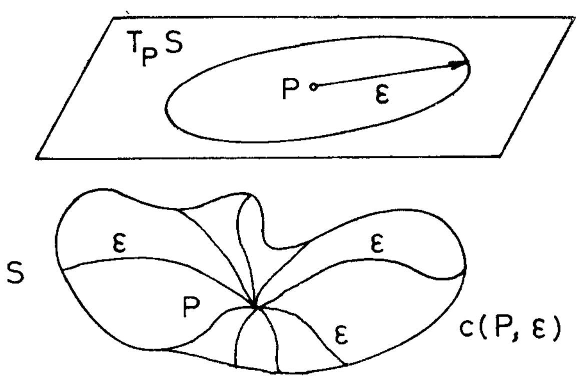
Notice that $l(P, \varepsilon) = 2\pi R \sin \dfrac{\varepsilon}{R} = 2\pi R - \dfrac{\pi}{3R^2}\varepsilon^3 + \ldots$ Therefore

$$\lim_{\varepsilon \to 0} \frac{3}{\pi} \frac{2\pi\varepsilon - l(P, \varepsilon)}{\varepsilon^3} = \frac{1}{R^2}.$$

The value of the above limit turns out to coincide with the value of the Gaussian curvature at P; a very old result tells us that it is always the case... hence the following definition of the Gaussian curvature (which does not make any reference to the embedding).

4.1.3. Second definition

In the 2-dimensional tangent plane at $P \in S$, draw a circle of radius ε, let $c(P, \varepsilon)$ be the curve obtained by drawing, out of P, the set of geodesics of length ε (with tangent vector in the tangent plane), call $l(P, \varepsilon)$ the length of $c(P, \varepsilon)$. Then *define* Gaussian curvature at $P: K_P = \lim\limits_{\varepsilon \to 0} \dfrac{3}{\pi} \dfrac{2\pi\varepsilon - l(P, \varepsilon)}{\varepsilon^3}$.



4.2. Sectional curvature associated to a 2-plane in n dimensions

If S is a n dimensional Riemannian manifold, $P \in S$ a point and $W \subset T_P S$, a 2-plane included in the tangent space $T_P S$, then, in W we can draw a circle of radius ε and draw on S the curve $c(P, \varepsilon)$. The sectional curvature of S at P associated to W is defined by $K_P(W)$. Sectional curvature is therefore a direct generalisation of the usual Gaussian curvature, but here, we have to specify which 2-plane W we choose (in the case of a 2-dimensional surface, such a precision is of course unnecessary!). The sectional curvature at P associated to a 2-plane W can also be written $K_P(W) = K_P(u, v)$ where u and v are two orthonormal vectors in W.

It can be proved that the knowledge of all the sectional curvatures at P characterizes completely the curvature at P (it is equivalent to the knowledge of the Riemann tensor at P, that we do not introduce).

4.3. Ricci curvature and scalar curvature

Let u_1 be a tangent vector at P of norm 1 and $u_1, u_2, \ldots, u_n$ an orthonormal basis. Then the Ricci curvature at P associated to the vector u_1 is $\varrho_P(u_1) = \sum_{i=2}^{n} K(u_1, u_i)$ and the scalar curvature at P is $\tau = \sum_{i=1}^{n} \varrho(u_i)$.

Notice that:
the sectional curvature associates a real number to a couple of vectors at a point;
the Ricci curvature associates a real number to one vector at a point;
the scalar curvature associates a real number to a point.

One should distinguish between:

- spaces of constant sectional curvature (they are simply called spaces of constant curvature)
- spaces of constant Ricci curvature (they are called Einstein spaces)
- spaces of constant scalar curvature.

The usual definition of an Einstein space makes use of the Ricci tensor (that we do not introduce), it is a space where (Ricci) is proportional to the metric; this definition is equivalent to the one given above.

Notice finally that there are spaces endowed with a homogeneous metric, they constitute a special case of spaces of constant scalar curvature (examples: any Lie group with a left invariant metric, any homogeneous space G/H with a G-invariant metric; see also the example below).

4.4. Ricci and scalar curvature of S^3 with a SU(2)×U(1) invariant metric

We already mentioned in Section 3.3 the following metric on S^3

$$h^{-1} = \varepsilon_1 \otimes \varepsilon_1 + \varepsilon_2 \otimes \varepsilon_2 + \frac{1}{\lambda^2} \varepsilon_3 \otimes \varepsilon_3$$

which admits a group of isometries $SU(2) \times U(1)$. The actual computation of the quantities K, ϱ, τ can be made by using the usual formulae of Riemannian geometry (that we did not introduce), with the following results: the functions K, ϱ, τ are independent of the point $P \in S^3$ (it is a homogeneous metric) *but* K and ϱ are not constant (their value depends upon the direction(s); the space is homogeneous but not isotropic. More precisely, one introduces an orthonormal basis $E_1 = \varepsilon_1$, $E_2 = \varepsilon_2$, $E_3 = \varepsilon_3/\lambda$ and finds the following values: Ricci curvatures $\varrho(E_1) = \varrho(E_2) = 4-2\lambda^2$, $\varrho(E_3) = 2\lambda^2$; Scalar curvature $\tau = 2(4-\lambda^2)$.

Warning:
* Although the space S^3 is compact, the scalar curvature τ can be negative (this is the case for $\lambda^2 > 4$)!

The most complete book(s) on Riemannian geometry is probably [6]; the previous class of metrics on S^3 is studied in more details in [7] in relation with the spectrum of the Dirac operator.

5. *Multidimensional universes*

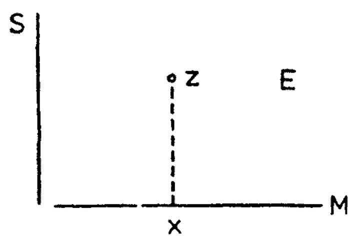
5.1. Intuitive aspects

The idea which is at the root of all this is the following: in a given physical situation, the "real world" can be described by a universe E which has more than 4 dimensions but, "classically" we may have the feeling that "we live in 4 dimensions". This is a rather vague (philosophical) statement and the aim of this section is to precise its meaning. Usually, in order to describe some physical systems, one starts with a Lagrangian, or a Lagrangian density $\mathscr{L}$; here, we have to assume that $\mathscr{L}(z \in E)$ can be written as a function of four variables $x \in M$ (M being spacetime) (at this point, the reader may think of the extra-coordinates as the analogue of cyclic variables in classical mechanics). The real valued function $\mathscr{L}(z)$ which will be interpreted as a Lagrangian is chosen to be the scalar curvature of E at the point z, i.e., the Einstein action. Two problems come immediately to the mind: what is the topology of E? (It will be a "kind" of product "spacetime $\times$ internal spaces") and what is the geometry of E? (What are the metrics of E for which $\mathscr{L}(z)$ depends only upon M?).

5.2. The topological structure of E

This question has already been answered in Section 2 and we just restate the result.

Let E be a manifold with a right action of a compact Lie group G, and suppose that all isotropy groups H_u ($u \in E$) are conjugate to a standard one, say $H_{u_0} = H$. Let M be the set of all orbits, $S = H\backslash G$ the coset space of right classes along H and let N be the normaliser of H in G. Then M is a manifold and E is an associated bundle with structure group N/H and typical fiber S. Then we can draw the following picture:



Of course M is interpreted as space-time, S is the typical internal space (the fiber above $x \in M$ being the internal space at x). Notice that if E is such that all isotropy groups H are not conjugate to a standard one, we can decompose E into strata parametrised by set of conjugacy classes of stabilizers and the previous theorem applies to each individual stratum.

5.3. The Riemannian structure of E

Endowed with the above structure, our multidimensional universe E may have a lot different "shapes"; however, we are interested here only in those shapes for which the scalar curvature τ^E is constant along the fibers: for those metrics only, we will obtain "dimensional reduction", i.e., $\tau^E(z)$ will be a function of $x \in M$. Actually, this class of metric is even too big and we limit ourselves to the class of G-invariant metrics on E (it is a subset of the set of all metrics on E for which τ^E is constant along the fibers). In Section 5.3.1, we discuss the case where S is a Lie group G and in Section 5.3.2 the more general case $S = H\backslash G$.

5.3.1. Let P be a principal bundle with base M and structure group G, then a G-invariant metric g on P determines and is determined by

i) an arbitrary metric γ on M
ii) a G-invariant metric h_x on each fiber G_x of E above $x \in M$.
iii) a connection in P (the connection form is valued in Lie(G)).

Moreover, in some coordinate system, the metric g can be written as

$$g = \gamma_{\mu\nu} dx^\mu \otimes dx^\nu + h_{\alpha\beta}(x)\,[(\sigma^\alpha + A^\alpha_\mu(x)dx^\mu) \otimes (\sigma^\beta + A^\beta_\nu(x)dx^\nu)]$$

where $\{\sigma^\alpha\}$ can be thought of as the dual of right invariant vector fields in the copy of G above x; of course the inverse metric is

$$g^{-1} = \gamma^{\mu\nu}(\partial_\mu - A^\alpha_\mu(u)e)\otimes(\partial_\nu - A^\beta_\nu(x)e_\beta) + h^{\alpha\beta} e_\alpha \otimes e_\beta.$$

At this point it is maybe necessary to draw the attention of the reader on the following: on a Lie group, one can define right and left multiplication, hence left invariant vector fields ε_α (dual ω^α) and right invariant vector fields e_α (dual σ^α) — see Section 3.2 — ; however, on a principal bundle P, there is only a global right action of G on P — see Section 2.3.1, the associated vector fields are usually called fundamental vector fields and we denote them by ε_α, if we make a local gauge choice in a neighborhood of $x \in M$, we have really a copy of G above x and we can define locally the "right invariant" vector fields e_α; these vector fields enter in the expression of the metric g given above, for this reason, the previous expression is not obviously gauge invariant (although it is!) but exhibits clearly G-invariance (the Killing fields being the ε_α).

Finally, the scalar curvature $\tau^E(z)$ is only a function of $x \in M$ and can be written, up to a total divergency, as

$$\tau^E(z) = \tau^M(x) + \tau^S(x) - \tfrac{1}{4} F^\alpha_{\mu\nu}(x) F^{\mu\nu\beta}(x) h_{\alpha\beta}(x)$$
$$- \tfrac{1}{4} h^{\alpha\beta}(x) h^{\gamma\delta}(x)\,(D_\mu h_{\alpha\beta} D^\mu h_{\gamma\delta} + D_\mu h_{\alpha\gamma} D^\mu h_{\beta\delta}),$$

where $\tau^M(x)$ is the scalar curvature of M at x, $\tau^S(x)$ is the scalar curvature of the internal space $S_x = G_x$, $F^\alpha_{\mu\nu}$ is the usual Yang-Mills strength and

$$D_\mu h_{\alpha\beta} = \partial_\mu h_{\alpha\beta} - A^\gamma_\mu C^\delta_{\gamma\alpha} h_{\delta\beta} - A^\gamma_\mu C^\delta_{\beta\gamma} h_{\delta\alpha}.$$

Let us consider now a few subspaces:

a) $S = G = \mathrm{U}(1)$, then τ^S vanishes; if $h_x = l^2 d\theta^2$ with constant l then the fourth term vanishes as well but if $h_x = l^2(x)d\theta^2$ then the fourth term gives a contribution $-1/2 - \frac{\partial_\mu \sigma \partial^\mu \sigma}{\sigma^2}$, with $\sigma = l^2$.

b) $S = G = \mathrm{U}(1)$, then τ^S does not vanish. If $h_x = \delta_{\alpha\beta}\sigma^\alpha \otimes \sigma^\beta$ that is if the restriction of the metric g to the copy of the group G above x is bi-invariant, the term τ^S can be interpreted as a cosmological constant and the fourth term vanishes; then, we recover the usual Yang-Mills Lagrangian coupled to gravity. But in the general situation $h_x = h_{\alpha\beta}(x)\sigma^\alpha \otimes \sigma^\beta$, $-\tau^S$ can be interpreted as a potential term for the scalar fields $h_{\alpha\beta}(x)$ and those scalar fields are coupled covariantly to the gauge field in a way which is reminiscent of the σ-models.

The emergence of these scalars $h_{\alpha\beta}(x)$ — they are scalar fields from the space time point of view —, which measure in a very precise sense the shape of the internal space at x, is quite intriguing, in particular because they are very natural but do not seem to appear in the today's description of particle physics phenomenology.

As an explicit example, let us consider the case where $G = \mathrm{SU}(2)$ and, in some gauge, in a neighborhood of $x \in M$, the metric h_x can be written as

$$h_x^{-1} = \frac{e_1 \otimes e_1}{\mu_1(x)} + \frac{e_2 \otimes e_2}{\mu_2(x)} + \frac{e_3 \otimes e_3}{\mu_3(x)} \text{ i.e.} \quad \begin{aligned} h_{11} &= \mu_1(x) > 0 \\ h_{22} &= \mu_2(x) > 0 \\ h_{33} &= \mu_3(x) > 0. \end{aligned}$$

Then

$$\tau^E = \tau^M + \tau^G - \tfrac{1}{4} F^1_{\mu\nu} F^{\mu\nu^1} \mu_1(x) + F^2_{\mu\nu} F^{\mu\nu^2} \mu_2(x) + F^3_{\mu\nu} F^{\mu\nu^3} \mu_3(x)$$
$$-\tfrac{1}{4}\left\{ 2 \sum_{l} \frac{\partial_\mu \mu_i \partial^\mu \mu_i}{\mu_i^2} + 2 \sum_{i<j} \frac{\partial_\mu \mu_i \partial^\mu \mu_j}{\mu_i \mu_j} + 8 \sum_{\substack{i<j \\ l \neq i \\ l \neq j}} \frac{(\mu_i - \mu_j)^2}{\mu_i \mu_j} A^l_\mu A^{\mu l} \right\}$$

(See also examples in Section 3.3.1.)

Moreover, $\tau^G = \frac{2}{\sigma_3}(4\sigma_2 - \sigma_1^2)$ where $\begin{cases} \sigma_1 = \mu_1 + \mu_2 + \mu_3 \\ \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 \\ \sigma_3 = \mu_1\mu_2\mu_3 \end{cases}$

It is clear, on this example, that $h_{\alpha\beta}(x)$ has to be a field (transforming according to $\mathrm{Ad} \otimes \mathrm{Ad}$) and not a constant; in the opposite case the gauge symmetry would be broken — unless $\mu_1 = \mu_2 = \mu_3 = \text{cte}$.

5.3.2. We now discuss the general case $S = H \backslash G$

Let E be a fiber bundle with base M, typical fiber $H \backslash G = S$ and structure group $N|H$, then a G-invariant metric g on E determines and is determined by

(*i*) an arbitrary metric γ on M

(*ii*) a G-invariant metric h_x on each fiber S_x of E above $x \in M$
(*iii*) a connection in the principal bundle $P \subset E$ (the connection form is valued in Lie $(N\backslash H)$, $N\backslash H$ being the typical fiber of P).

Of course the theorem given in Section 5.3.1 is a consequence of this one for $H = E$. The above result is proved and discussed in full details in [2] and this discussion will not be repeated here. However, after the study of Sections 2 and 3, this theorem should appear as quite "natural". It is interesting to notice that, in this case, the "global" symmetry group G and the "local" symmetry group $N|H$ do not coincide. Let us mention only how the previous relations where the internal space S is a group generalise to the case where S is a homogeneous space $H\backslash G$: call $\mathscr{S}$ the tangent space at the origin of $S = H\backslash G$ and $\{T_i\}$ a basis in the Lie algebra of G which is orthonormal for the Killing metric, then write the decomposition $\mathrm{Lie}(G) = \mathrm{Lie}(H)+\mathscr{S}$, $\mathscr{S} = \mathrm{Lie}(N|H)+\mathscr{L}$ we assume that the basis $\{T_i\}$ is adapted to this decomposition and write:

$$\{T_i\} = \{T_{\hat{\alpha}}, T_\alpha\},$$
$$\{T_\alpha\} = \{T_{\hat{a}}, T_a\}.$$

Let us call also C_{ij}^k the structure constants of G associated to the basis $\{T_i\}$. Then we obtain:

$$\tau^E(Z) = \tau^M(x)+\tau^S(x)-\tfrac{1}{4}F^{\hat{a}}_{\mu\nu}(x)F^{\mu\nu\hat{l}}(x)h_{\hat{a}\hat{l}}(x)-\tfrac{1}{4}h^{\alpha\beta}(x)h^{\gamma\delta}(x)$$
$$(D_\mu h_{\alpha\beta}D^\mu h_{\gamma\delta}+D_\mu h_{\alpha\gamma}D^\mu h_{\beta\delta})+(\text{total divergency}),$$

with

$$\tau^S = h^{\beta\beta'}(\tfrac{1}{2}C^\gamma_{\alpha\beta}C^\alpha_{\beta'\gamma}-\tfrac{1}{4}h^{\alpha\alpha'}h_{\gamma\gamma'}C^\gamma_{\alpha\beta}C^{\gamma'}_{\alpha'\beta'}+C^{\hat{\alpha}}_{\alpha\beta}C^\alpha_{\beta'\hat{\alpha}}-C^\alpha_{\alpha\beta}C^\gamma_{\gamma\beta'})$$

and

$$D_\mu h_{\alpha\beta} = \partial_\mu h_{\alpha\beta}-A^{\hat{a}}_\mu C^\delta_{\hat{a}\alpha}h_{\delta\beta}-A^{\hat{l}}_\mu C^\delta_{\beta\hat{l}}h_{\delta\alpha}.$$

In all the cases covered by this theorem, dimensional reduction is automatic i.e. $\tau(Z)$, $Z \in E$ is only a function of $x = \pi(Z) \in M$ and we may have the feeling that the "real world" is only 4 dimensional. The following table gives a few examples:

G	H	$H\backslash G$	$N\|H$
G	e	G	G
SO(n+1)	SO(n)	S^n	Z_2
SU(n+1)	SU(n)	S^{2n+1}	U(1)
USp(2n)	USp(2n−2)	S^{4n-1}	SU(2)
E_8	E_6	E_8/E_6	SU(3)/Z_3

5.4. Symmetric gauge fields

Suppose that E is locally $M\times G/H$ and that G is not simple ($G = G_1G_2$ say). In many cases E can also be written locally as $U\times G_1$ where U is a manifold with a dimension bigger than dim (M). We can therefore proceed to an intermediate dimensional reduction (from

E to U); however the obtained metric and Yang Mills fields defined on U will possess symmetries (under G_2), a reflection of the fact that the dimensional reduction was not complete. One can also go backwards, start with metric and Yang Mills field on a manifold U, assuming symmetries under a group G_2 and asking about the outcome of dimensional reduction [8]; using the theorem of reduction (Section 5.3.2) as a tool, it is not so difficult to answer precisely this question in the more general case [9, 10].

The theorem of reduction, in the case $S = G$, has been floating around, in the mathematical folklore for a long time (but I do not know any precise reference); in the non principal case, $S = H\backslash G$, it has been proved and discussed in [2] where many examples are also worked out.

6. *Einstein spaces and symmetry breaking*

6.1. Intuitive aspects

The motivation for this section is the following: from the one hand, we saw in the previous section that the potential for scalar fields (they measure the "shape" of the internal spaces at $x \in M$) can be interpreted as the scalar curvature of the internal space at x (which is a space of constant *scalar* curvature), from the other hand, there is an old theorem (due to Hilbert) which says that saddle points of the total scalar curvature — considered as a functional on the space of metrics — for fixed volume, coincide with the Einstein metrics — putting these two facts together and remembering that the scalars $h_{\alpha\beta}(x)$ are covariantly coupled to the gauge field, one may look for an analogue of the usual Higgs mechanism.

6.2. Einstein metrics on groups and homogeneous spaces

6.2.1. Standard examples

The Killing metric on a Lie group G is an Einstein metric; the (unique, up to scale) G-invariant metric on an irreducible symmetric space $H\backslash G$ is Einstein; the (unique, up to scale) G-invariant metric on an isotropy irreducible space $H\backslash G$ is Einstein.

Warning:

a G-invariant metric on a non-isotropy irreducible space $H\backslash G$ is usually not Einstein, even if it is a normal metric (obtained from a bi-invariant metric on G by going to the quotient).

6.2.2. Einstein metrics as saddle points

As stated in 6.1, a very old theorem assures that for a given manifold, the total scalar curvature defined as $\int_S \tau^S d\,\mathrm{vol}$, considered as a functional on the space of metrics (with fixed volume) admits saddle points for all Einstein metrics. The constraint of fixed volume can be easily understood; consider for instance a standard two-sphere of radius R, its scalar curvature can be made arbitrarily small or large by increasing or decreasing its radius ($\tau = 2/R^2$); this kind of variation is not interesting: we have to fix the volume and study the variation of the total scalar curvature ("squashing deformations"). In what follows, we will not study Einstein metrics in general but only the homogeneous Einstein metrics (more precisely the G-invariant Einstein metrics on S, S being G or G/H); in this last case the total scalar curvature is equal to the product $\tau \times \mathrm{vol}(S)$ and we have only to look at variation of τ, τ being considered as a functional on the space of G-invariant metrics (of course, with such a restriction, we only get a necessary condition but, for all the cases treated here, it can be shown to be sufficient).

6.2.3. A general method

The basic strategy is more or less always the same: one first chooses some fibering of the manifold S under consideration (for example, the group G as a K-bundle over G/K, G/H as a $N|H$ bundle over G/N, etc...), then chooses a particular "natural" metric on S (for example the Killing metric on G or the normal metric on G/H...) and begins to "distort" it in a way appropriate to the fibering; in the obtained family of new metrics, one looks for those where the Einstein condition is satisfied, either by computing directly the Ricci tensor or by looking at saddle points of some functional.

6.2.4. Non standard Einstein metrics on groups

Let us do it explicitly in the case where S is a simple compact Lie group G; following the above recipe, we choose K, a subgroup of G and write G as a K bundle over G/K (to ease the following, assume that (G, K) is a symmetric pair), then we start from the following bi-invariant (and Einstein) metric on $G: g = -$ (Killing form). Notice that we can, at this stage, apply the decomposition formulae of Section 5.3.1 and write $\tau^G = \tau^{G/K} + \tau^K - \frac{1}{4} C^c_{ab} C^{ab}_c$ or

$$\frac{n}{4} = \frac{l}{2} + c\,\frac{k}{4} - \tfrac{1}{4}\,k(1-c)$$

where $n = \dim G$, $k = \dim K$, $l = n-k = \dim G/K$ and c, the index of K in G being defined as follows: Killing metric of G restricted to $K = g_{\hat{a}\hat{b}} = -C^{\alpha}_{\hat{a}\beta} C^{\beta}_{\hat{b}\alpha}$, Killing metric of $K = \overset{K}{g}_{\hat{a}\hat{b}} = -C^{\hat{d}}_{\hat{a}\hat{c}} C^{\hat{c}}_{\hat{b}\hat{d}}$ and calling c the coefficient $\overset{K}{g}_{\hat{a}\hat{b}} = c g_{\hat{a}\hat{b}}$. The previous decomposition of τ^G is a "Kaluza Klein reduction" (before Kaluza Klein) where the "external" space is G/K, the "internal" space is K and the "field strength" is $C^{\hat{c}}_{ab}$. Notice that when (G, K) is a symmetric pair we have indeed $c = 1 - l/2k$ but this would not be true in the general case. Following the general recipe, we now write $\mathrm{Lie}(G) = \mathrm{Lie}(K) \oplus P$ — orthogonal decomposition for g — and consider the following family of metrics (t is a real parameter) on G: $h = g/P + t^2 g/K$. These metrics, obtained by a scaling of g in the direction K are no longer $G \times G$ invariant but only $G \times K$ invariant; the scalar curvature of G is now $\tau^G = \dfrac{l}{2} + \left(c\,\dfrac{k}{4}\right)\dfrac{1}{t^2} - \tfrac{1}{4}\,k(1-c)t^2$.

However, when t varies, the volume of G varies; in order to keep it fixed we just have to make a conformal rescaling and consider the family of metric $\bar{h} = \left(\dfrac{1}{t^2}\right)^{k/n} h$, then

$\det \bar{h} = \left(\frac{1}{t^2}\right)^k \cdot (t^2)^k = \text{const}$. The associated scalar curvature reads

$$\tau^G = (t^2)^{k/n}\left[\frac{l}{2} + \frac{ck}{4t^2} + \frac{k}{4}(c-1)t^2\right].$$

We now vary this expression with respect to t and find

$$\left(\frac{d\tau^G}{dt}\right) = -\frac{l}{4}\frac{(2k+l)}{(k+l)}t^{2k/n-3}(t^2-1)\left(t^2-\frac{2k-l}{2k+l}\right),$$

where we used the property $c = 1-l/2k$, valid for a symmetric pair. We find therefore two Einstein metrics: corresponding to the values $t^2 = 1$ and $t^2 = \frac{2k-l}{2k+l}$. The first value corresponds of course to the bi-invariant metric g used in the beginning, the other value corresponding to a non standard $G\times K$ invariant metric on G $\Bigg($for example, studying $G = SU(3)$, if we choose $K = SO(3)$, we obtain an $SU(3)\times SO(3)$ non standard Einstein metric on $SU(3)$ for the value $t^2 = \frac{2\times 3-5}{2\times 3+5} = \frac{1}{11}\Bigg)$. Using the same recipe in a more general case leads to the following results: Let G be a simple Lie group, $g = -$(Killing form on G), K a connected subgroup of G (not necessarily simple) and write the following decompositions: $\text{Lie}(G) = \text{Lie}(K)\oplus P$ and $\text{Lie}(K) = K_0\oplus K_1 \ldots K_s$, where K_0 is the center of $\text{Lie}(K)$ and K_i $(i = 0)$ are simple components, than calling $n = \dim G$, $k = \dim K$, $k_i = \dim K_i$, $l = n-k$ and c_i the index of K_i in $\text{Lie}(G)$, we obtain a family of $G\times K_i$ invariant metrics of $G: h = g/P+\alpha_0 g/K_0+\alpha_1 g/K_1+ \ldots +\alpha_s g/K_s$; ($\alpha_i$ being positive real number) and a new family $h = \frac{1}{f}h$, where $f = (\prod_{i=1}^{s}\alpha_i^{k_i})^{1/n}$ of metrics of fixed volume. For $\bar{h}$ to be an Einstein metric, the following set of equations have to be satisfied:

$$\alpha_0 = \frac{1}{l+2k_0}\left\{l+2k_0-\sum_{i=1}^{s}2k_i(1-c_i)(\alpha_i-1)\right\}$$

$$(l+2k_0+2k_j)(1-c_j)\alpha_j^2+\sum_{\substack{i=1\\i\neq j}}^{s}2k_i(1-c_i)\alpha_i\alpha_j-(l+2k_0+\sum_{i=1}^{s}2k_i(1-c_i))\alpha_j$$
$$+(l+2k_0)c_j = 0,$$

when P is not Ad K irreducible, we have to check that the found solution $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ satisfies a third condition.

6.2.5. Non standard Einstein metrics on homogeneous spaces

As mentioned previously, the general method of Section 6.2.3 applies here also; we will not enter into the details but just mention a few examples: by writing G/H as a N/H bundle over G/N (N being the normaliser of H into G) and scaling the normal metric on

G/H (which is not necessarily Einstein, although it is induced by a bi-invariant metric on G) along the N/H direction, one can obtain new Einstein metrics for a wide class of spaces, for special values of the scaling parameter (this is the case, for example, of S^{4n+3} spheres written as SU(2) bundles over HP^n); other kinds of fibering may be used (for example S^{15} as a S^7 fibering over S^8 etc.).

6.3. Symmetry breaking

6.3.1. A word of warning

In the Higgs setting, Higgs fields are section of some associated vector bundles and one looks for a non zero local minimum v_0 of a suitable fourth degree G-invariant polynomial; the shifted Higgs field $\varphi' = \varphi - v_0$ is not equivariant under G since v_0 is only invariant under the subgroup G_0 which fixes it; consequently, φ' does not yield a well defined section of the associated bundle. This problem (very often overlooked) can be handled in several ways (see [11]) and we will not go into it but just mention it because there is, in our setting, an analogous difficulty (which has not been studied yet). Let us nevertheless proceed and write formally $h_{\alpha\beta}(x) = h^0_{\alpha\beta} + h'_{\alpha\beta}(x)$ in the Lagrangian (i.e., the expression for $\tau^E(z)$ given in Section 5.3.1 (or 5.3.2)), $h^0_{\alpha\beta}$ being an Einstein metric in the "internal space" S.

6.3.2. A mass spectrum for the gauge fields?

Let us analyse the situation when S is a group G (we suppose G unimodular, then $C^a_{ab} = 0$). Using the definition of D_μ, the term $L = -\frac{1}{4} h^{\alpha\beta} h^{\gamma\delta}(D_\mu h_{\alpha\beta} D^\mu h_{\gamma\delta} + D_\mu h_{\alpha\gamma} D^\mu h_{\beta\delta})$ can be expanded and we get

$$L = -\tfrac{1}{4} h^{\alpha\beta} h^{\gamma\delta}(\partial_\mu h_{\alpha\beta}\partial^\mu h_{\gamma\delta} + \partial_\mu h_{\alpha\gamma}\partial^\mu h_{\beta\delta}) - h^{\alpha\beta}(\partial_\mu h)_{\beta\delta} C^\delta_{\alpha\gamma} A^\gamma_\mu + \tfrac{1}{2}\mathcal{M}_{\alpha\beta} A^\alpha_\mu A^{\beta\mu},$$

where $\mathcal{M}_{\alpha\beta} = J_{\alpha\beta} + K_{\alpha\beta}$; $K_{\alpha\beta} = C^\delta_{\alpha\gamma} C^\gamma_{\beta\delta}$ is the Killing form and $\mathcal{J}_{\alpha\beta} = h^{\gamma\gamma'} h_{\delta\delta'},\ C^\delta_{\alpha\gamma} C^{\delta'}_{\beta\gamma'}$.

Let us now make formally the shift $h_{\alpha\beta}(x) = h^0_{\alpha\beta} + h'_{\alpha\beta}(x)$, $h^0_{\alpha\beta}$ being a homogeneous Einstein metric on G that we supposedly obtained by the method of Section 6.2.4. i.e. a $G \times K$ invariant Einstein metric, we even suppose that (G, K) is a symmetric pair $\left(\text{then the value of the scaling parameter is } t^2 = \dfrac{2k-l}{2k+l}\right)$. The calculation is straightforward, we find that

$$+\tfrac{1}{2}\mathcal{M}_{\alpha\beta} A^\alpha_\mu A^{\mu\beta} = +\tfrac{1}{2}\mathring{\mathcal{M}}_{\alpha\beta} A^\alpha_\mu A^{\mu\beta} + \text{Rest},$$

where

$$\mathring{\mathcal{M}}_{\hat{a}\hat{b}} = 0,$$

$$\mathring{\mathcal{M}}_{a\hat{a}} = 0,$$

$$\mathring{\mathcal{M}}_{ab} = \tfrac{1}{2}\left(t^2 + \frac{1}{t^2}\right) - 1.$$

In other words, when $t^2 = 1$ (i.e., we expand around the bi-invariant metric of G), the gauge field stays massless; however, when $t^2 = \frac{2k-l}{2k+l}$ (i.e. we expand around a non standard Einstein metric on G, $G\times K$ invariant, with the notations of Section 6.2.4) then, the components of the gauge field taking their value in Lie (K) stay massless but the components lying in the subspace P (Lie (G) = Lie $(K)+P$) acquire a mass $m^2 = 2l^2/4k^2-l^2$ in dimensionless units. Let us assume for example that our internal space is $G = \mathrm{SU}(4)$ and that the volume is fixed, then if we expand the scalar fields $h_{\alpha\beta}$ around the bi-invariant metric, we get 15 massless gauge fields but we can also expand $h_{\alpha\beta}$ around the following $G\times K$ invariant Einstein metrics:

(*i*) $K = \mathrm{S(U(2)\times U(2))} \simeq \mathrm{SU(2)\times SU(2)\times U(1)}$ for $t^2 = 3/11$

(*ii*) $K = \mathrm{USp(4)}$ for $t^2 = 3/5$

(*iii*) $K = \mathrm{SO(4)}$ for $t^2 = 1/7$

In case (*i*) we get $k = 7$ massless gauge fields and $= 8$ massive fields of mass $m^2 = 32/33$.
In case (*ii*) we get $h = 10$ massless gauge fields and $= 5$ massive fields of mass $m^2 = 2/15$.
In case (*iii*) we get $h = 6$ massless gauge fields and $= 9$ massive fields of mass $m^2 = 18/7$.

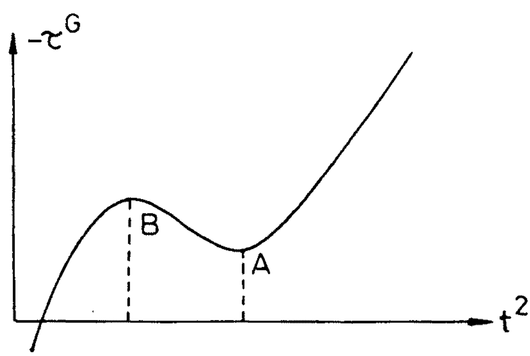
In the previous examples, the pair (SU(4), K) is symmetric but there exist other saddle points (other Einstein metrics) involving non symmetric pairs.

6.3.3. A problem of interpretation

Besides the problem already mentioned in 6.3.1, there is another difficulty which is better explained by looking at the following figure which is a graph of the negative of the scalar curvature for the one parameter family of metrics found in Section 6.2.4.

$$\tau^G = (t^2)^{k/n}\left[\frac{l}{2} + \frac{ck}{4t^2} + \frac{k}{4}(c-1)t^2\right].$$

For typical values of the parameters, we get the following curve:



The point A corresponds to the standard bi-invariant metric on G and B to a non standard Einstein metric. This "potential" is therefore not bounded from below — even with the fixed volume restriction —, moreover the non standard Einstein metric of this family corresponds to a local maximum of the curve; in a more general situation, Einstein metrics correspond to saddle points which are neither minima nor maxima of the total scalar curvature function. The "physical" interpretation of the results of Section 6.3.2 is therefore unclear: we explained what happens if we expand a non trivial saddle point but

1) are we allowed to "expand" around them?

2) even if we can, why do it and are these saddle points important in a quantum mechanical perspective?

The above ideas are certainly not conventional but seem to show a new direction in the study of what is usually called "spontaneous symmetry breaking". The potential for the scalar fields that we study here is actually not completely unrelated with the familiar Higgs potential: there exists a precise relation which is investigated in one of the sections of [10].

The most complete treatment of Einstein metrics on compact groups is [12]. Many other references (and comments) may be found in [5].

7. *Conclusion*

Rather than going through the several sections of these lectures, it is probably more important to stress the advantages of the multidimensional approach to the study of gauge fields; there are, at least, two (good) reasons to look at gauge fields in that way. The first one is philosophical: usual Yang Mills theories can be recast under the form of a metric theory in a universe which is "bigger" (there is a one to one equivalence of formalism, see Section 5.3) this is an important epistemological fact. Moreover, and maybe more important, this another way of thinking about Yang Mills theories gives new ideas in order to generalise them and to try to describe the geometry of our "real" world.

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