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AN EXACTLY SOLVABLE MODEL FOR FERMIONIC GENERATIONS AND POINCARÉ STRESSES: PART TWO

By W. Królikowski*

III Physikalisches Institut, RWTH Aachen, West Germany

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An exactly solvable quantum mechanical model is further discussed, where leptons and quarks of higher generations are excited states with respect to small oscillations around an equilibrium realized within leptons and quarks of the first generation (considered as composite systems). The model predicts elements of Kobayashi–Maskawa matrix in consistency with recent experimental data, if the Cabibbo angle is used as an input. The heavier the top quark is, the smaller are the mixing angles of higher quark generations. The hypothetical next charged lepton is predicted unambiguously at 28.5 GeV.

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1. Introduction

The negative result of search for toponium in the PETRA energy range shifts our expectations for finding the top quark to the higher energy region which, fortunately, has become available in the CERN collider. As this shift in energy has relevant consequences for the model of fermionic generations we considered recently [1], we turn back to its foundations in the present paper.

Essentially, there are two such consequences. (*i*) The rapidly rising character of fermion mass spectrum (being a main ingredient of the model) becomes now even more pronounced. It makes the smallness of mixing angles for higher generations (expected in the model) more and more natural (cf. also [2]). In fact, this smallness is recently confirmed by the direct data [3]. (*ii*) The conjecture (explored in the model) of simple proportionality of up and down fermion masses to corresponding electric charges squared cannot be valid if, as it seems now, the top quark mass has to be distinctly larger than 4×5 GeV $= 20$ GeV. On the other hand, the electric charge remains the only known quantum number which distinguishes up and down fermions, so it is still natural to expect that the Coulomb repulsion is a kind of driving force in producing mass differences between up and down fermions

* On leave of absence from the Institute of Theoretical Physics, Warsaw University, Warszawa, Poland.

within each generation of leptons and quarks. It reminds us of the classical concept of Poincaré stresses [4] representing all non-electromagnetic forces necessary to provide the internal equilibrium in an extended particle against the Coulomb repulsion.

So, we can now rephrase the assumptions of the model in the following form:

(*i*) Leptons and quarks of the first generation ν_e, e^- and u, d are ground states of a composite system.

(*ii*) There are small radial oscillations around the equilibrium realized in these ground states, leading to excitations observed by us as leptons and quarks of higher generations.

(*iii*) These small oscillations can be described in a reasonable approximation by the following one-dimensional hamiltonian or mass operator:

$$M = m_0 + \frac{p^2}{2\mu} + \frac{\mu\omega^2}{2}[q + lf(q, p)]^2 - \frac{\omega}{2}, \tag{1}$$

where

$$i[p, f(q, p)] = M - m_0 \tag{2}$$

and

$$f(-q, -p) = -f(q, p), \tag{3}$$

while $p = -i\partial/\partial q$, $\mu > 0$, $\omega > 0$ and $l > 0$.

Small oscillations having properties (1), (2) and (3) turned out to be much different from the harmonic oscillations and were called in Ref. [1] the quasiharmonic oscillations. Here, we will use the term *pseudoharmonic oscillations* which, perhaps, is more adequate. Note that the condition (3) provides the symmetry of M under the discrete transformation $q \to -q$ which seems to be very natural for small oscillations (q is a radially oriented coordinate). This condition will be relaxed in Section 3. Note also that the constant μ plays here the role of a mass scale Λ of some hyperstrong interactions responsible for binding our composite system.

In contrast to Ref. [1] we will not assume here that ω is proportional to the electric charge squared: $\omega \sim Q^2$.

As follows from Ref. [1], the model is exactly solvable in terms of "pseudoannihilation" and "pseudocreation" operators a and a^+, where

$$a = \frac{1}{\sqrt{2}}\left\{\sqrt{\mu\omega}\,[q + lf(q, p)] + \frac{i}{\sqrt{\mu\omega}}\,p\right\}. \tag{4}$$

In fact, we can write Eqs. (1) and (2) in the form

$$M = m_0 + \frac{\omega}{2}(\lambda^2 + 1)N \tag{5}$$

and

$$[a, a^+] = 1 + (\lambda^2 - 1)N, \tag{6}$$

where

$$N = a^+ a \tag{7}$$

is a "pseudooccupation number" operator and

$$\lambda^2 = \frac{1+\omega l/2}{1-\omega l/2}. \tag{8}$$

Note that for $l \to 0$ or $\lambda^2 \to 1$ we get a harmonic oscillator. Henceforth, we will assume $\lambda^2 > 0$, which implies $\lambda^2 > 1$ as $\omega l > 0$. The eigenvalue equation

$$N|n\rangle = N_n|n\rangle, \quad \langle n|n\rangle = 1 \quad (n = 0, 1, 2, \ldots) \tag{9}$$

and the commutation relation (6) give

$$Na^+|n\rangle = (\lambda^2 N_n + 1)a^+|n\rangle, \quad Na|n\rangle = \frac{1}{\lambda^2}(N_n - 1)a|n\rangle. \tag{10}$$

Hence

$$a^+|n\rangle = \sqrt{\lambda^2 N_n + 1}\,|n+1\rangle, \quad a|n\rangle = \sqrt{N_n}\,|n-1\rangle \tag{11}$$

and

$$N_{n+1} = \lambda^2 N_n + 1. \tag{12}$$

Defining the ground state $|0\rangle$ through the condition $a|0\rangle = 0$ which gives $N_0 = 0$ and solving the recurrence formula (12) we obtain the spectrum for N:

$$N_n = \frac{\lambda^{2n}-1}{\lambda^2 - 1} = \begin{cases} 0 & \text{for} \quad n = 0, \\ 1 + \lambda^2 + \ldots + \lambda^{2n-2} & \text{for} \quad n \geqslant 1. \end{cases} \tag{13}$$

Then Eq. (5) gives the recurrence formula and spectrum for M:

$$m_{n+1} - m_0 = \lambda^2(m_n - m_0) + \frac{\omega}{2}(\lambda^2 + 1) \tag{14}$$

and

$$m_n = m_0 + \frac{\omega}{2}(\lambda^2 + 1)\frac{\lambda^{2n}-1}{\lambda^2 - 1}. \tag{15}$$

Since $\lambda^2 > 1$, Eq. (15) gives an exponentially growing spectrum $\sim \exp(2n \ln \lambda) + \text{const}$. It is bounded from below by m_0. In the case of $\lambda^2 \to 1$ we get a harmonic oscillator where $N_n = n$ and $m_n = m_0 + \omega n$. Formulae similar to (14) and (15) were found and discussed previously on a phenomenological ground [5].

2. *Fermion masses*

The spectrum (15) leads to the mass relation

$$m_{n+1} - m_n = \frac{\omega}{2}(\lambda^2 + 1)\lambda^{2n} \quad (n = 0, 1, 2, \ldots) \tag{16}$$

which in turn implies that

$$\frac{m_{n+2}-m_{n+1}}{m_{n+1}-m_n} = \lambda^2. \tag{17}$$

In particular, $m_1 - m_0 = \dfrac{\omega}{2}(\lambda^2+1)$ and $m_2 - m_0 = \dfrac{\omega}{2}(\lambda^2+1)^2$. If $m_1 \gg m_0$, we get $m_1 \simeq \dfrac{\omega}{2}(\lambda^2+1)$ and $m_2 \simeq \dfrac{\omega}{2}(\lambda^2+1)^2$.

In the case of charged leptons, identifying m_0, m_1, m_2 with m_e, m_μ, m_τ, respectively, we obtain

$$\lambda_e = 3.99, \quad \omega_e = 12.4 \text{ MeV}. \tag{18}$$

Then, the hypothetical next charged lepton (call it "ω") is predicted at the mass

$$m_\omega = m_3 = 28.5 \text{ GeV}. \tag{19}$$

In the case of neutrinos the degeneracy $m_{\nu_e} = m_{\nu_\mu}$, if it appeared, would imply $\omega_\nu = 0$ and hence the mass degeneracy of all neutrinos (and so no neutrino oscillations).

In the case of up quarks, putting $m_0 = m_u \simeq 0$ and $m_1 = m_c \simeq 1.5$ GeV we get

$$\frac{\omega_u}{2}(\lambda_u^2+1) \simeq 1.5 \text{ GeV}. \tag{20}$$

Hence for the top quark

$$m_t = m_2 \simeq (1.5 \text{ GeV})(\lambda_u^2+1). \tag{21}$$

Thus

$$\lambda_u \simeq \sqrt{\frac{m_t}{1.5 \text{ GeV}} - 1}, \qquad \omega_u \simeq \frac{2\cdot(1.5 \text{ GeV})^2}{m_t}. \tag{22}$$

In particular for

$$m_t \simeq 20 \text{ GeV}, \quad 25 \text{ GeV}, \quad 35 \text{ GeV}, \quad 40 \text{ GeV} \tag{23}$$

we obtain respectively:

$$\lambda_u \simeq 3.5, 4, 4.7, 5.1 \tag{24}$$

$$\omega_u \simeq 230 \text{ MeV}, 180 \text{ MeV}, 130 \text{ MeV}, 110 \text{ MeV}. \tag{25}$$

So, if the constant λ appearing in the commutation relation (6) was a universal constant i.e. if

$$\lambda_u = \lambda_d = \lambda_e = \lambda_\nu \simeq 4, \tag{26}$$

there would be

$$m_t = m_2 \simeq 25 \text{ GeV}. \tag{27}$$

The hypothetical next up quark (call it "h" for "harmony") is predicted at

$$m_h = m_3 \simeq 250 \text{ GeV},\ 400 \text{ GeV},\ 780 \text{ GeV},\ 1000 \text{ GeV}, \tag{28}$$

respectively.

Analogically, in the case of down quarks, putting $m_0 = m_d \simeq 0$ and $m_2 = m_b \simeq 5$ GeV we get

$$\frac{\omega_d}{2}(\lambda_d^2+1)^2 \simeq 5 \text{ GeV}. \tag{29}$$

Thus for

$$\lambda_d \simeq 3.5,\ 4,\ 4.7,\ 5.1 \tag{30}$$

we have

$$\omega_d \simeq 57 \text{ MeV},\ 35 \text{ MeV},\ 19 \text{ MeV},\ 14 \text{ MeV}, \tag{31}$$

respectively. Then, the strange quark has the mass

$$m_s = m_1 \simeq 0.38 \text{ GeV},\ 0.29 \text{ GeV},\ 0.22 \text{ GeV},\ 0.19 \text{ GeV}, \tag{32}$$

while the hypothetical next down quark (call it "f" for "fun") is predicted at

$$m_f = m_3 \simeq 62 \text{ GeV},\ 80 \text{ GeV},\ 110 \text{ GeV},\ 130 \text{ GeV}. \tag{33}$$

Note that for $\lambda_u = \lambda_d \simeq 3.5$ there would be $\omega_u : \omega_d \simeq Q_u^2 : Q_d^2$ and $m_t \simeq 20$ GeV, what seems to be experimentally excluded. Of course, the possibility of $\lambda_u = \lambda_d$ with a higher value (e.g. $\lambda_u = \lambda_d \simeq 4$ corresponding to $m_t \simeq 25$ GeV) is experimentally open.

As is described in Ref. [1], the pseudoharmonic oscillations can be considered as small if

$$\frac{1}{\mu\omega}\tfrac{1}{2}[(\lambda^2+1)N_n+1] \ll r_{\text{exp}}^2, \tag{34}$$

where $r_{\text{exp}} \sim 10^{-16}$ cm is the experimental upper bound for radius of leptons and quarks. Thus the consistency requires that

$$\frac{1}{\mu} \ll \frac{10^{-20}}{17N_n+1} \text{ cm} \tag{35}$$

(where $N_n = 0, 1, 17, 273, \ldots$ for $n = 0, 1, 2, 3, \ldots$) if we put $\lambda = 4$ and $\omega \sim 10$ MeV (as for charged leptons). In particular, for the extreme value of μ equal to the Planck mass $M_{PL} \sim 10^{19}$ GeV the model is certainly consistent with r_{exp}, leaving plenty of margin for decrease of r_{exp} (in this case $1/\mu \sim 10^{-33}$ cm). Inversely, given the mass scale μ, the number of fermionic generations n (which can be described as pseudoharmonic excitations) is restricted by the condition $17N_n+1 \ll \mu/(10^6 \text{ GeV})$. In general, the larger n is, the worse becomes the description. But, the inequality sign here may be not so sharp.

As is noted in Ref. [1], the excited states with respect to our pseudoharmonic oscillations are radiatively stable as far as one-photon or one-gluon transitions are concerned.

3. *Cabibbo-like mixing*

Now, in order to allow for mixing of n states (defined as eigenstates of the mass operator (1)) we would like to relax a bit our model of small oscillations by introducing a tiny violation of its symmetry under the discrete transformation $q \to -q$.

To this end we make in Eq. (1) the substitution

$$f(q, p) \to f(q, p) + f_0, \tag{36}$$

where f_0 is a small constant, while $f(q, p)$ still satisfies the conditions (2) and (3). Then we obtain the new perturbed hamiltonian or mass operator

$$M^{\mathrm{P}} = m_0 + \frac{p^2}{2\mu} + \frac{\mu\omega^2}{2}[q + lf(q, p) + lf_0]^2 - \frac{\omega}{2} \tag{37}$$

which gives

$$M^{\mathrm{P}} = m_0 + \omega[\tfrac{1}{2}(\lambda^2+1)N + g(a+a^+) + g^2] \tag{38}$$

with

$$g = lf_0 \sqrt{\frac{\mu\omega}{2}}, \tag{39}$$

whilst the commutation relation (6) still holds for a and a^+. Here, only one new free parameter appears: g (beside two old: λ and ω). Note that the substitution (36) introduces to the mass operator a kind of initial stress, which is acting even when $q + lf(q, p) \to 0$.

In the representation defined by eigenstates $|n\rangle$ of the operator N (which are also eigenstates of the unperturbed mass operator $M = M^{\mathrm{P}}_{f_0=0}$) we get

$$\langle n'|M^{\mathrm{P}}|n\rangle = (m_n + g^2\omega)\delta_{n'n} + g\omega(\sqrt{N_n}\,\delta_{n'n-1} + \sqrt{N_{n+1}}\,\delta_{n'n+1}], \tag{40}$$

where N_n and m_n are given in Eqs. (13) and (15). Denoting the eigenvalues and eigenstates of perturbed mass operator M^{P} by m_n^{P} and $|n\rangle^{\mathrm{P}}$ we can write

$$\langle n'|U^{-1}M^{\mathrm{P}}U|n\rangle = m_n^{\mathrm{P}}\delta_{n'n} \tag{41}$$

and

$$|n\rangle^{\mathrm{P}} = U|n\rangle = \sum_{n'} |n'\rangle\langle n'|U|n\rangle, \tag{42}$$

where U is a unitary operator.

If only three generations $n = 0, 1, 2$ are relevant, the nondiagonal mass matrix defined by Eq. (40) takes the form

$$M^{\mathrm{P}} = \begin{pmatrix} m_0 + \frac{m_1^2}{\omega}s^2, & m_1 s, & 0 \\ m_1 s, & m_1 + \frac{m_1^2}{\omega}s^2, & m_1 s\sqrt{\lambda^2+1} \\ 0, & m_1 s\sqrt{\lambda^2+1}, & m_2 + \frac{m_1^2}{\omega}s^2 \end{pmatrix}, \tag{43}$$

where

$$s = \frac{g\omega}{m_1} \tag{44}$$

and $m_1 = m_0 + \frac{\omega}{2}(\lambda^2+1)$ and $m_2 = m_0 + \frac{\omega}{2}(\lambda^2+1)^2$. The perturbative solution for eigenvalues and eigenstates of the mass matrix (43) leads to [6]

$$U \simeq \begin{bmatrix} 1-\frac{1}{2}s^2, & s, & 0.015s^2 \\ -s, & 1-\frac{1.066}{2}s^2, & 0.26s \\ 0.24s^2, & -0.26s, & 1-\frac{0.066}{2}s^2 \end{bmatrix} + O(s^3 \text{ or } s^4), \tag{45}$$

where $m_0 = 0$ and $\lambda \simeq 4$ are used. Then $s = 2g/(\lambda^2+1) \simeq 0.12g$.

As is shown in Ref. [1], the explicit form of the U matrix (which diagonalizes the mass matrix (43)) enables us to calculate in our model the Kobayashi-Maskawa matrix [7]. In fact, it was demonstrated in the case of $\lambda_u = \lambda_d$ that the generalized Cabibbo-Kobayashi--Maskawa unitary operator has the form

$$V = U_u^{-1}U_d, \tag{46}$$

where U_u and U_d are the U operators for up and down quark families, u, c, t, ... and d, s, b, ..., respectively. In the case of three generations $n = 0, 1, 2$ the V operator becomes the usual Kobayashi-Maskawa matrix:

$$V = \begin{pmatrix} V_{ud}, & V_{us}, & V_{ub} \\ V_{cd}, & V_{cs}, & V_{cb} \\ V_{td}, & V_{ts}, & V_{tb} \end{pmatrix} = \begin{pmatrix} c_1, & s_1c_3, & s_1s_3 \\ -s_1c_2, & c_1c_2c_3+s_2s_3e^{i\delta}, & c_1c_2s_3-s_2c_3e^{i\delta} \\ -s_1s_2, & c_1s_2c_3-c_2s_3e^{i\delta}, & c_1s_2s_3+c_2c_3e^{i\delta} \end{pmatrix}, \tag{47}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ ($i = 1, 2, 3$). In our case we can put $\delta = 0$ since the V matrix is real.

Using formulae (45) and (46) we calculate the matrix V. Then, comparing the result with Eq. (47) we get [6]

$$\begin{aligned} s_1 &= s_d - s_u, \\ s_3 - s_2 &= 0.26(s_d - s_u), \\ s_3 + s_2 &= -0.23(s_d + s_u). \end{aligned} \tag{48}$$

On the other hand, the perturbative solution for m_0^P, which in the case of matrix (43) is

$$m_0^P = m_0 + \frac{m_1^2}{\omega}s^2\frac{\lambda^2-1}{\lambda^2+1} + O(s^4), \tag{49}$$

gives

$$\frac{s_u^2}{s_d^2} \simeq \frac{m_u}{m_d}\frac{\omega_d}{\omega_u} \simeq \tfrac{1}{9}, \tag{50}$$

where $m_0 = 0$ and $\lambda \simeq 4$ are used and the popular current masses $m_u \simeq 4$ MeV and $m_d \simeq 7$ MeV are assumed for m_0^P (here, $\omega_u \simeq 180$ MeV and $\omega_d \simeq 35$ MeV as follows from Eqs. (25) and (30) for $\lambda_u = \lambda_d \simeq 4$). The set of four equations (48) and (50) can be solved for s_u, s_d and s_2, s_3 if the experimental value of

$$s_1 \simeq V_{us} = \cos\theta_C \tag{51}$$

is taken as an input. For the input $s_1 \simeq 0.22$ or $s_1 \simeq 0.23$ we obtain in this way

$$V \simeq \begin{bmatrix} 0.98, & 0.22, & \begin{Bmatrix} -0.0045 \\ 0.0035 \end{Bmatrix} \\ -0.22, & 0.97, & 0.057 \\ \begin{Bmatrix} 0.017 \\ 0.0089 \end{Bmatrix}, & -0.057, & 1.0 \end{bmatrix} \tag{52}$$

or

$$V \simeq \begin{bmatrix} 0.97, & 0.23, & \begin{Bmatrix} -0.0054 \\ 0.0038 \end{Bmatrix} \\ -0.23, & 0.97, & 0.060 \\ \begin{Bmatrix} 0.018 \\ 0.010 \end{Bmatrix}, & -0.060, & 1.0 \end{bmatrix}, \tag{53}$$

respectively (for the meaning of matrix elements in Eqs. (52) and (53) compare Eq. (47)). Here, the upper or lower numbers for V_{ub} and V_{td} correspond to two possible cases of $s_u s_d > 0$ or < 0, respectively.

The results (52) and (53) are consistent with the recent experimental estimate

$$\left|\frac{V_{ub}}{V_{cb}}\right| < 0.15, \quad |V_{cb}| = 0.053^{+0.010}_{-0.009} \tag{54}$$

extracted from data for B-meson decays [3] (cf. also the experimental analysis in Ref. [8]). In fact, they give

$$\frac{V_{ub}}{V_{cb}} \simeq \begin{Bmatrix} -0.080 \\ 0.062 \end{Bmatrix}, \quad V_{cb} \simeq 0.057 \tag{55}$$

or

$$\frac{V_{ub}}{V_{cb}} \simeq \begin{Bmatrix} -0.090 \\ 0.063 \end{Bmatrix}, \quad V_{cb} \simeq 0.060. \tag{56}$$

We would like to stress that, in contrast to Ref. [1], in our present calculation no extra parameter appears (beside $\lambda_{u,d}$ and $\omega_{u,d}$ estimated from heavy quark masses and $s_{u,d}$ estimated from the ratio m_u/m_d and the Cabibbo angle θ_C). For values of $\lambda_u = \lambda_d$ larger than $\lambda_u = \lambda_d \simeq 4$ used here (corresponding to values of m_t larger than $m_t \simeq 25$ GeV) one gets the mixing angles θ_2 and θ_3 smaller and smaller.

4. *Classical analogy*

The operator $f(q, p)$ defined in Eqs. (2) and (3) satisfies with respect to q the nonlinear differential equation

$$\frac{\partial f(q,p)}{\partial q} = \frac{p^2}{2\mu} + \frac{\mu\omega^2}{2}[q+lf(q,p)]^2 - \frac{\omega}{2} \tag{57}$$

with $p = -i\partial/\partial q$. Fortunately, there was no need to try to solve this equation explicitly. It may be interesting, however, to find an analogy of $f(q, p)$ in the case of classical mechanics, where Eq. (57) depends parametrically on the momentum p and, therefore, can be easily solved. Then we obtain

$$q+lf(q,p) = \frac{C(p)}{\Omega}\tan\left[\Omega C(p)q + \arctan\frac{\Omega l f(0,p)}{C(p)}\right], \tag{58}$$

where

$$\Omega = \sqrt{\frac{\mu l}{2}}\,\omega, \quad C(p) = \sqrt{1-\frac{\omega l}{2}+l\frac{p^2}{2\mu}}. \tag{59}$$

Here, $C^2(p) > 0$ since $\omega l < 2$ for $\lambda^2 > 0$ as it follows from Eq. (8) (in particular $\omega l = 1.76$ corresponds to $\lambda = 4$). We can see that the momentum-dependent interaction in the classical pseudoharmonic oscillator,

$$V = \frac{\mu\omega^2}{2}[q+lf(q,p)]^2 - \frac{\omega}{2}, \tag{60}$$

has singularities (double poles) at the points

$$q_\nu = \frac{(2\nu+1)\pi}{2C(p)\Omega}\ (\nu = 0, \pm 1, \pm 2, \ldots). \tag{61}$$

Note that the condition (3) is fulfilled, if the initial value in Eq. (58) satisfies the condition $f(0, -p) = -f(0, p)$.

It happens that, if $f(0, p) = 0$ and if $p^2/2\mu$ is neglected in $C(p)$, the interaction V can be expressed by the Pöschl-Teller potential discussed several decades ago in the theory of multiatomic molecules [9]:

$$V = V_{PT}(q-q_0) - \frac{1}{l}, \tag{62}$$

where

$$V_{\mathrm{PT}}(q-q_0) = \tfrac{1}{2}\frac{\alpha^2}{\mu}\left[\frac{(\omega l/2)^{-2}}{\sin^2\alpha(q-q_0)} + \frac{(\omega l/2)^{-2}}{\cos^2\alpha(q-q_0)}\right] \tag{63}$$

with

$$C = \sqrt{1-\frac{\omega l}{2}}, \quad \alpha = \frac{C\Omega}{2}, \quad q_0 = \frac{\pi}{2C\Omega}. \tag{64}$$

Such a truncated V, if inserted back to the Schrödinger equation with $H = M$ and $m_0 = 0$, gives (exactly)

$$m_n = \frac{C^2}{l}\left(\frac{\lambda^2-1}{\lambda^2+1}\right)^2\left[n+\tfrac{1}{2}+\frac{\lambda^2+1}{\lambda^2-1}\sqrt{1+\tfrac{1}{4}\left(\frac{\lambda^2-1}{\lambda^2+1}\right)^2}\right]^2 - \frac{1}{l}, \tag{65}$$

whereas our previous exact solution is

$$m_n = \frac{\lambda^{2n}-1}{l} \quad (n = 0, 1, 2, \ldots) \tag{66}$$

(cf. Eq. (15)) because $m_0 = 0$ and

$$\frac{\omega}{2}(\lambda^2+1) = \frac{\lambda^2-1}{l} \tag{67}$$

(from Eq. (8)). We can see that neglecting the momentum dependence in the interaction part of M we deform drastically the mass spectrum, losing its exponential character. Nevertheless, the periodicity of V_{PT} $(q-q_0)$ reflects in some way the many-body nature of our composite system.

Finally, it should be mentioned that the operator $\tilde{q} = q+lf(q,p)$ appearing in the interaction part of the mass operator (1) satisfies, due to the condition (2), the commutation relation

$$[\tilde{q}, p] = i[1+l(M-m_0)]. \tag{68}$$

If the operator $\tilde{q}$ is formally replaced by the coordinate q and the operator $M-m_0$ by the general hamiltonian, Eq. (68) transits into the noncanonical commutation relation $[q,p] = i(1+lH)$ considered by Saavedra and Utreras [10] who boldly conjectured that inside hadrons the usual Heisenberg canonical commutation relation $[q,p] = i$ should be thus modified. (Note that in our case such a modification might be pertinent inside leptons and quarks rather than inside hadrons.)

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REFERENCES

[1] W. Królikowski, *Acta Phys. Pol.* **B14**, 689 (1983). The present paper is the Part Two of this Ref.
[2] P. Ginsparg, S. Glashow, M. Wise, *Phys. Rev. Lett.* **50**, 1415 (1983).
[3] F. Fernandez et al., *Phys. Rev. Lett.* **51**, 1022 (1983); N. S. Lockyer et al., *Phys. Rev. Lett.* **51**, 1316 (1983).
[4] H. Poincaré, *Rendiconti di Palermo* **21**, 129 (1906).
[5] W. Królikowski, *Acta Phys. Pol.* **B10**, 767 (1979); **B12**, 913 (1981).
[6] W. Królikowski, Aachen report PITHA 83/20 (October 1983).
[7] M. Kobayashi, T. Maskawa, *Prog. Theor. Phys.* **49**, 652 (1973).
[8] K. Kleinknecht, B. Renk, *Phys. Lett.* **130B**, 459 (1983).
[9] G. Pöschl, E. Teller, *Z. Phys.* **83**, 143 (1933).
[10] I. Saavedra, C. Utreras, *Phys. Lett.* **98B**, 74 (1981); cf. also H. S. Snyder, *Phys. Rev.* **71**, 38 (1947).