INTERACTING MASSLESS SCALAR AND SOURCE-FREE ELECTROMAGNETIC FIELDS

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The relativistic field equations for interacting massless attractive scalar and source-free electromagnetic fields in a cylindrically symmetric spacetime of one degree of freedom with reflection symmetry have been reduced to a first order implicit differential equation depending upon time which enables one to generate a class of solution to the field equations. The nature of the scalar and electromagnetic fields is discussed. It is shown that the geometry of the spacetime admits of an irrotational stiff fluid distribution without prejudice to the interacting electromagnetic fields.

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1. Introduction

The gravitational effects of cylindrically symmetric interacting massless scalar fields are a subject of current interest because of their possible applications to nuclear physics. The case of coupled source-free electromagnetic fields and stiff fluid distributions equivalent to massless scalar fields was investigated in a recent paper [1], when the geometry is described by the Einstein-Rosen metric. An earlier paper [2] dealt with self-gravitating irrotational stiff fluids analogous to scalar fields in the space-time described by the metric

$$ds^2 = e^C(dt^2 - dr^2) - e^B(r^2 d\theta^2 + dz^2) \tag{1a}$$

$$C = C(r, t), \quad B = B(r, t), \tag{1b}$$

where the coordinates (r, θ, z, t) correspond to (x^1, x^2, x^3, x^4) respectively. This metric is a specialization of a general one [3]. It has been shown in [2] that the parameter B is a linear function of time. A Cauchy problem pertaining to the Klein-Gordon equation for the scalar fields has been subsequently studied under the assumption that this parameter is independent of time.

The aim of the present investigations is to extend the above mentioned work, in which source-free electromagnetic fields interact with the massless scalar fields. The field equations have been reduced to a single first-order and implicit differential equation in time only whose solution can be used to generate a set of exact solutions to the field equations for the case where only one component of the electromagnetic field tensor survives. The behaviour of the scalar and electromagnetic fields is discussed in Sections 4 and 5, respectively. Conditions under which the solutions reduce to those given in [2] are mentioned in Sec. 6. It is shown in Sec. 4 that the scalar fields are analogous to irrotational stiff perfect fluids notwithstanding the presence of the electromagnetic fields. Finally, the uniformity and nullity of the electromagnetic fields are studied.

2. *Field equations*

The Einstein field equations for linearly coupled scalar and electromagnetic fields are

$$G_{ij} \equiv R_{ij}-(\tfrac{1}{2})g_{ij}R = -T_{ij}-(\tfrac{1}{4}\pi)M_{ij}, \tag{2}$$

$$T_{ij} = V_{,i}V_{,j}-(\tfrac{1}{2})g_{ij}V_{,k}V^{,k}, \tag{2a}$$

$$M_{ij} = -F_{is}F^{s}_{j}+(\tfrac{1}{4})g_{ij}F^{kl}F_{kl}, \tag{2b}$$

$$\Box V \equiv \frac{1}{\sqrt{-g}}(\sqrt{-g}\, g^{ij}V_{,i})_{,j} = 0, \tag{3}$$

$$F^{ij}{}_{,j} \equiv \frac{1}{\sqrt{-g}}(\sqrt{-g}\, F^{ij})_{,j} = 0, \tag{4a}$$

$$F_{ij} = A_{i,j}-A_{j,i}, \tag{4b}$$

where the units are so chosen that the velocity of light $c = 1$ and $K = 1/8\,\pi$. Here and in what follows, commas represent ordinary differentiation and semicolons, covariant differentiation. T_{ij}, M_{ij}, V, F_{ij} and A_i represent respectively scalar stress-energy tensor, electromagnetic stress-energy tensor, the scalar fields, electromagnetic field tensor and the electromagnetic four-potential. Equation (3) is the Klein-Gordon equation of the scalar fields. Because of the geometry of the space-time imposed by Eq. (1), we have

$$A_{i,\theta} = A_{i,z} = 0,\ V_{,z} = V_{,\theta} = 0 \quad \text{and} \quad F_{23} = 0. \tag{5}$$

For convenience in setting up the field equations, we introduce the transformations

$$u = t-r, \qquad v = t+r. \tag{6}$$

The subscripts 1 and 4 in the following denote differentiation with respect to u and v respectively. Five of the field equations (2) are associated with the five non-vanishing components of the Einstein tensor G_{ij}. The rest of the field equations can be written in the compact form (in view of Eq. (5)).

$$F_{14}F_{12} = F_{14}F_{13} = F_{14}F_{24} = F_{14}F_{34} = 0, \tag{7a}$$

$$F_{12}F_{34} = F_{13}F_{24} = 0. \tag{7b}$$

Equation (7a) suggests that either $F_{14} \neq 0$ or $F_{14} = 0$. Here we consider the former case for which Eq. (7b) is identically satisfied. The latter case is under investigation and will be published later. Under this condition, the field equations may be written in the equivalent forms,

$$B_{11}+(\tfrac{1}{2})B_1^2-C_1B_1+(\tfrac{1}{2}r)(B_1-C_1) = -V_1^2, \tag{8a}$$

$$B_{44}+(\tfrac{1}{2})B_4^2-C_4B_4-(\tfrac{1}{2}r)(B_4-C_4) = -V_4^2, \tag{8b}$$

$$2C_{14}+4B_{14}+3B_1B_4 = -2V_1V_4, \tag{8c}$$

$$B_1 = B_4, \tag{8d}$$

$$B_{14}+B_1B_4 = \frac{e^{-C}}{4\pi}F_{14}^2 \tag{8e}$$

in which the scalar and the electromagnetic fields are delinked.

Besides these equations three others are obtained from Eq. (3) and (4a):

$$2V_{41}+\left(\frac{1}{2r}+B_1\right)V_4+\left(-\frac{1}{2r}+B_4\right)V_1 = 0, \tag{9}$$

$$rF_{41,4} = -r(B_4-C_4)F_{41}+\tfrac{1}{2}F_{41}, \tag{10a}$$

$$rF_{14,1} = -r(B_1-C_1)F_{14}-\tfrac{1}{2}F_{14}. \tag{10b}$$

3. *Solutions*

As a first step, we integrate Eqs (8d) and (10a, b) and obtain

$$B = B(t), \tag{11}$$

$$F_{14} = \frac{a}{2r}e^{-B+C}. \tag{12}$$

Here and in what follows, all lower case latin letters except r, t, z, u, v and w represent arbitrary constants. The relation (12) is analogous to the relation (3) in Ref. [4]. But this relation may or may not generate a class of solutions to the field equations.

Substituting Eqs. (11) and (12) in the remaining field equations and reverting to the original coordinates, we have

$$\ddot{B}+\tfrac{1}{2}\dot{B}^2-(C_t+C_r)\dot{B}+\frac{1}{r}(\dot{B}-C_t-C_r) = -(V_t+V_r)^2, \tag{13a}$$

$$\ddot{B}+\tfrac{1}{2}\dot{B}^2-(C_t-C_r)\dot{B}-\frac{1}{r}(\dot{B}-C_t+C_r) = -(V_t-V_r)^2, \tag{13b}$$

$$2\ddot{B}+\tfrac{3}{2}\dot{B}^2+C_{tt}-C_{rr} = V_r^2-V_t^2, \tag{13c}$$

$$\ddot{B}+\tfrac{1}{2}\dot{B}^2 = \frac{a^2}{4\pi r^2}e^{-2B+C}, \tag{13d}$$

$$V_{tt}-V_{rr}-\frac{1}{r}V_r+\dot{B}V_t = 0, \tag{13e}$$

where $(\cdot)$ represents total differentiation with respect to time, and the subscripts r and t represent partial differentiation with respect to r and t respectively.

To solve these equations, Eq. (13d) may be written in the form

$$e^C = \frac{4\pi}{a^2}r^2\Phi\ddot{\Phi}, \tag{14}$$

where

$$\Phi \equiv \Phi(t) = e^B. \tag{14a}$$

Substituting Eq. (14) in the difference between Eqs. (13a) and (13b) we get

$$V_tV_r = \frac{W}{2r}, \tag{15}$$

where

$$W \equiv W(t) = 2\dot{B}+(\dddot{\Phi}/\ddot{\Phi}). \tag{15a}$$

Using Eqs. (14) and (15a) in the sum of Eqs. (13a) and (13b) and also in Eq. (13c), we reduce them to the forms

$$\ddot{B}+\tfrac{3}{2}\dot{B}^2-\dot{B}W-\frac{2}{r^2} = -V_t^2-V_r^2, \tag{16a}$$

$$\ddot{B}+\tfrac{3}{2}\dot{B}^2+\dot{W}+\frac{2}{r^2} = V_r^2-V_t^2. \tag{16b}$$

Eqs. (15) and (16a, b) may be considered in place of Eqs. (13a), (13b) and (13c). Adding and subtracting Eqs. (16a) and (16b), we get

$$2\ddot{B}+3\dot{B}^2+\dot{W}-\dot{B}W = -2V_t^2, \tag{17a}$$

$$\dot{W}+\dot{B}W+\frac{4}{r^2} = 2V_r^2. \tag{17b}$$

It is evident from Eq. (17a) that V_t is a function of time alone. Using this fact along with Eq. (15), we reduce Eq. (13e) to the equivalent forms

$$V_{tt} = -\dot{B}V_t \quad \text{and} \quad V_{rr} = -\frac{V_r}{r}, \tag{18}$$

which immediately yield

$$V_t = be^{-B} \tag{19a}$$

and

$$V_r = \frac{m}{r}. \tag{19b}$$

The constant of integration m may be evaluated to be $\pm\sqrt{2}$ by substituting Eqs. (19) and (15) in Eq. (17b) so that the latter is identically satisfied. Thus we obtain from Eqs (15), (19a) and (19b)

$$V_r = \pm\sqrt{2}\,\frac{1}{r}, \tag{19c}$$

$$W = \pm 2\sqrt{2}\, be^{-B}. \tag{20}$$

In view of these two equations, Eq. (17a) takes the form

$$2\ddot{B}+3\dot{B}^2 \mp 4\sqrt{2}\, b\dot{B}e^{-B}+2b^2e^{-2B} = 0. \tag{21}$$

A first integral of this differential equation obtained by standard methods has the form

$$e^B = n\frac{(\dot{B}e^B \mp 2\sqrt{2}\,b-\sqrt{6}\,b)^{\pm\frac{2}{\sqrt{3}}-1}}{(\dot{B}e^B \mp \sqrt{2}\,b+\sqrt{6}\,b)^{\pm\frac{2}{\sqrt{3}}+1}}, \tag{22}$$

where n is an arbitrary constant.

Any solution to this first-order implicit differential equation may generate a class of solutions to the field equations. The solution obtained from Eqs. (14), (19a), (19c) using Eqs. (15a) and (20) appears in the form

$$V = b\int e^{-B}dt \pm\sqrt{2}\log r+a', \tag{23}$$

$$e^C = \frac{4\pi b'}{a^2}\, r^2 \exp\left(\pm 2\sqrt{2}\, b\int e^{-B}dt-B\right), \tag{24}$$

$$F_{14} = \frac{2\pi b'}{a}\, r \exp\left(\pm 2\sqrt{2}\, b\int e^{-B}dt-2B\right), \tag{25}$$

where B is a solution of (22). The constant b' appears when Eq. (20) and Eq. (15a) are integrated to obtain $\ddot{\Phi}$. The last equation is obtained from Eq. (12).

4. *Nature of the scalar field*

Eq. (23) shows that on the t = constant hypersurface, the scalar field behaves logarithmically with respect to r. If the negative sign in that equation is taken, V falls from $+\infty$ on the axis to $-\infty$ at $r = +\infty$. In physically valid situations any scalar field should fall off to zero at very large distances. This is achieved in the present case, by choosing the constant a' appropriately.

The lagrangian of the scalar fields given by

$$L = \tfrac{1}{2} V_{,s} V^{,s} \tag{26}$$

may be found to be related to the curvature invariant R by the formula

$$L = -\tfrac{1}{2} R \tag{26a}$$

as can be seen by contracting both sides of Eq. (2). The scalar invariant has the form

$$R = \frac{a^2}{4\pi b'} \frac{1}{r^2} \left(\frac{2}{r^2} - b^2 e^{-2B} \right) \exp \left(\mp 2\sqrt{2}\, b \int e^{-B} dt + B \right). \tag{27}$$

It has a singularity on the axis. According to [2], an irrotational stiff perfect fluid characterized by $p = \varrho = V_{,s} V^{,s}$ (p and ϱ refer respectively to pressure and density) indicates the same behaviour as R, even in the presence of the electromagnetic fields.

With the help of Eq. (20), Eq. (15) can be written as,

$$V_t V_r = \pm \sqrt{2} \frac{b}{r} e^{-B}, \tag{28}$$

which has a geometrical significance. The left hand side is equal to the scalar product of the tangents to the curves $r = r_0$ and $t = t_0$ through the point $(r_0, t_0, V_0 \equiv V(r_0, t_0))$ on the integral surface [5] of Eq. (13e). The angle between the two tangents is given by [6]

$$\cos\alpha = \frac{V_r V_t}{\sqrt{(1+V_r^2)(1+V_t^2)}} = \frac{\pm\sqrt{2} \frac{b}{r} e^{-B}}{\sqrt{\left(1+\frac{2}{r^2}\right)(1+b^2 e^{-2B})}}. \tag{29}$$

The families of curves r = constant, t = constant divide the integral surface into meshes, each having the area [6]

$$S = \iint \sqrt{1 + \frac{2}{r^2} + b^2 e^{-2B}}\, dr dt. \tag{30}$$

5. *Behaviour of the electromagnetic fields*

The electromagnetic fields, given by Eq. (15) becomes in the rt-coordinates

$$F_{rt} = \frac{4\pi b'}{a} r \exp\left(\pm 2\sqrt{2}\, b \int e^{-B} dt + B \right) \tag{31}$$

which vanishes on the axis and tends to ∞ as $r \to \infty$. This is a non-radiating electric field because the component of the energy-momentum tensor M_{rt} corresponding to the Poynting vector, obtained from Eq. (2b) using Eq. (31), vanishes. The energy-density of the electromagnetic field given by

$$M_{tt} = 2\pi ab \ \exp\left(\pm 2\sqrt{2}\, b \int e^{-B} dt - 3B\right) \tag{32}$$

is independent of the space coordinates, confirming non-existence of energy flow.

The dual [7] of the electric field considered above, defined by

$$F^{*ij} = -\tfrac{1}{2}(-g)^{-1/2}\varepsilon_{ijkl}F_{kl} \tag{33}$$

has only one component i.e., F^{*}_{23} which is equal to the const. a and represents a magnetostatic field. In the above equation, ε_{ijkl} is the Levi-Civita tensor density. In the present case, the nullity function [8] defined by

$$w = \left[(F_{ij}F^{ij})^2 + (F_{ij}F^{*ij})^2\right]^{1/2} \tag{34}$$

takes the form

$$w = -\frac{a^2}{2r^2}\, e^{-2B} \tag{35}$$

which has a singularity on the axis and vanishes at infinity. Thus the field is non-null at finite distances. That it is also non-uniform appears from the fact that the tensor $F_{ij;k}$ has only two components given by

$$F_{14;1} = -\frac{a}{4r^2}(1 + r\dot{B})e^{-B+C}, \qquad F_{14;4} = \frac{a}{4r^2}(1 - r\dot{B})e^{-B+C}. \tag{36}$$

which are not in general zero.

6. *Conclusion*

The geometry assumed in Eq. (1) describes a cylindrically symmetric spacetime with reflexion symmetry which is capable of sustaining gravitational waves [9]. It has a single degree of freedom represented by the metric potential B. The corresponding field equations pertaining to irrotational stiff fluid distributions have solutions that are equivalent to those of massless scalar fields [2]. The resulting fluid pressure equal to energy density is physically viable and therefore corresponds to the entropy level of the initial state of the universe indicating that it is not chaotic but isotropic and queiscent [10]. It has been shown in the present investigations that this pressure depends only upon the Ricci Scalar of the spacetime but is not affected by the presence of the non-radiating electric field considered here. This electric field described by only one nonvanishing component of the field tensor, is one of the two types admitted by the spacetime assumed here. The main purpose of our investigations is to reduce the field equations to a single differential equation in B whose solutions may generate all possible solutions of the field equations, and to relate all the physical quantities involved, including the scalar and electromagnetic fields, directly to the single degree of freedom of the spacetime represented by B.

It turns out that the single differential equation referred to above is implicit and of first order in B depending upon time alone. The electromagnetic fields, the scalar fields and the equivalent irrotational stiff fluid pressure can be directly computed from B, once the equivalent equation is solved. The electromagnetic field is non-null and non-uniform. Moreover, it is satisfying to note that all the physical quantities involved are fully inherent in the spacetime.

For an electromagnetic field, the parameter B satisfies the inequality (see Eq. (13d))

$$\ddot{B}+\dot{B}^2 \geqq 0. \tag{37}$$

Consequently, Eq. (21) shows that $b = 0$, $W = 0$ and $V_t = 0$ if and only if B is constant.

When the electromagnetic field is set equal to zero, relation (37) becomes an equality. Equation (21) then reduces to

$$\dot{B}^2 \mp 4\sqrt{2}\, b\dot{B}e^{-B}+2b^2e^{-2B} = 0 \tag{38}$$

which has the solution

$$e^B = 1+kt, \tag{39}$$

where

$$k = (\pm 2\sqrt{2} \pm 2\sqrt{6})b. \tag{39a}$$

Under this condition, Eq. (8c) reduces to the one that was considered redundant in [2]. Solution (39) also corresponds to the solution for B in [2]. Thus, the case reduces to that considered in that work.

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