# ON THE ONE-ELECTRON ATOM IN AN EXTERNAL GRAVITATIONAL FIELD

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It is argued that an external gravitational field may substantially influence atomic spectra. As an example, a hydrogen atom falling freely in Schwarzschild's space-time is investigated.

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This paper is devoted to the following question: Does the gravitational field have an influence on atomic spectra which gives rise to observable effects? This question is of particular importance in astrophysics, since most of the information on celestial bodies is available from spectral investigations. Almost all observations, so far, are based on the assumption that the external gravitational field has no influence on the atomic spectra, apart from an overall redshift.

A theoretical estimate [1] reveals that, even on the surface of a neutron star with $M \sim 0.4\, M_{\odot}$, the characteristic energies of the line splitting of a hydrogen atom are of the order of magnitude of $3.5 \cdot 10^{-12}$ eV. The recent investigation by Parker [2] of the spectrum of a hydrogen-like atom in an arbitrary curved space-time confirmed this result. In general, space-time must be highly curved (characteristic curvature radius about $10^{-3}$ cm) if gravitational corrections are to become of the order of magnitude of the Lamb shift ($4.4 \cdot 10^{-6}$ eV). Calculations similar to those by Parker have been performed by the author [3, 4]. Further references on the subject can be found in the papers by Audretsch and Schäfer [5], which deal with an extensive study of the hydrogen-like atom in the expanding universe.

However, we will show in the following that there are situations in which an external gravitational field may lead to corrections of the order of magnitude of the fine structure, even if the field is much weaker than that mentioned above (e.g., the field outside a Schwartzschild black hole).

The following classical considerations illustrate the idea. The equation of geodesic deviation implies that, besides the Coulomb force (caused by the charge of the nucleus), an additional "tidal force" (see, e.g. [6]) acts on the electron, which is proportional to $R^i_{jkl}\, U^j\, U^k\, \xi^l$ ($\xi^l$ is the distance vector between electron and nucleus, $U^i$ is the four-velocity of the nucleus). Because $\xi^k$ is very small ($\xi^k \sim 10^{-8}$ cm), the "tidal force" can essentially contribute to the electron energy only if either $R^i_{jkl}$ or $U^i$ take large values. This appears to be the case in the vicinity of singularities (this case was discussed in the paper of Parker [2]) or under ultrarelativistic velocities of the atom (when $v \to c$ some components of $U^i$ can take infinitely large values). The latter possibility is of much more interest for an external observer.

Let us now check this conjecture on the basis of quantum mechanics. Because the velocity of the atom may be very large, a comoving system of reference (i.e., a system in which the nucleus is at rest) is conveniently used. Moreover, the influence of the electron on the motion of the nucleus is neglected. The problem can be treated by using the quantum mechanics in an external gravitational field, which has been developed by the author [4, 7] (for details see the book [8]). The equations derived there enable us to express the Hamiltonian in terms of position, momentum and spin operators in a covariant manner. However, its explicit form depends on the choice of the family $f(\tau)$ of hypersurfaces (i.e., on the choice of the reference frame; see [9]). In the framework of quantum mechanics the reference frame of a single observer [10] is of particular importance. In the following we will use the explicit expression for the Hamiltonian in this frame of reference [9] and its two--component approximation [7]. For a hydrogen-like atom moving in external electromagnetic field $F_{ij}$ we obtain within the two-component theory, in a very good approximation, the following form of the Hamiltonian in the comoving frame:

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1, \tag{1}$$

where $\mathscr{H}_0$ denotes the unperturbed Hamiltonian for a free hydrogen-like atom (including fine structure terms). The second term is given by

$$\begin{aligned}\mathscr{H}_1 = {} & \frac{e}{2m_0 c}(L_{(\kappa)}+2S_{(\kappa)})H^{(\kappa)} - \omega^{(\kappa)}(L_{(\kappa)}+S_{(\kappa)}) \\ & + m_0\left(W_{(\alpha)} + \frac{e}{m_0}E_{(\alpha)}\right)q^{(\alpha)} + \left[\frac{e^2}{8m_0c^2}(H^2\delta_{(\kappa)(\tau)} - H_{(\kappa)}H_{(\tau)}) \right. \\ & \left. + \frac{m_0c^2}{2}R_{(4)(\kappa)(4)(\tau)}\right]q^{(\kappa)}q^{(\tau)},\end{aligned} \tag{2}$$

where $q^{(\alpha)}$, $S^{(\alpha)}$ and $L_{(\alpha)}$ are position, spin and angular momentum operators, respectively. $e$ and $m_0$ denote charge and mass of the electron, respectively. $W^{(\alpha)}$ $\left(W^{(\alpha)} = h^{(\alpha)}_i \dfrac{\delta}{\delta\tau} U^i\right)$ and $\omega^{(\alpha)}$ $\left((\omega^{(\alpha)} = 1/2\, e^{(\alpha)(\kappa)(\tau)}\, h_{(\tau)i} \dfrac{\delta}{\delta\tau} h^i_{(\kappa)}\right)$ are acceleration and angular velocity of the reference frame. (The tetrad $h^i_{(\kappa)}$ is defined in terms of the four velocity $U^i = \dfrac{d\xi^i}{d\tau}$ of the atom along the observer's world line according to $h^i_{(4)} = 1/c\, U^i$.) The quantities $H_{(\alpha)}$ and $E_{(\alpha)}$ are related to the tetrad components $F_{(n)(m)}$ of the electromagnetic field tensor on the world line of the atom as follows:

$$E_{(\alpha)} = F_{(\alpha)(4)}, \quad H_{(\alpha)} = e_{(\alpha)(\kappa)(\tau)} F^{(\kappa)(\tau)}, \tag{3}$$

where $e_{(\alpha)(\kappa)(\tau)}$ denotes the three-dimensional Levi-Civita symbol. The tetrad components of the curvature tensor are also to be evaluated on the world line of the observer (atom). The Hamiltonian (1), (2) enables us to determine the energy levels of an atom moving arbitrarily in an external gravitational field.

As an example, consider the Schwarzschild field described by the line element

$$ds^2 = e^{-\nu} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - e^{\nu}(dx^4)^2, \tag{4}$$

where we have used the usual abbreviation $e^{\nu} = 1 - r_g/r$ with $r_g = Mc^2/4\pi$. The equation of motion of the atom has the form $\frac{\delta}{\delta\tau} U^i = 0$. As is well known, the general solution of this equations reads (see, e.g., [11])

$$U^i = \left\{ \pm \sqrt{c^2 A^2 - e^{\nu}\left(c^2 + \frac{D^2}{r^2}\right)},\, 0,\, \frac{D}{r^2},\, cAe^{-\nu} \right\}, \tag{5}$$

where $A$ and $D$ are constants of integration; $mc^2 A$ and $mD$ can be interpreted as energy and angular momentum of the atom, respectively ($m$ denotes the mass of the nucleus). The sign (+ or −) appearing in the last formula corresponds to an atom moving off the gravitational centre or towards it, respectively. After a simple but tedious calculation (for details see [4, 7]) we obtain

$$\mathscr{H}_1 = -m_0 c^2 \frac{r_g}{2r^3} \left\{ \frac{1}{\Lambda^2}\left(A^2 + \frac{D^2 e^{\nu}}{2r^2 c^2}\right) \hat{X}^2 + \frac{3AD\Sigma}{cr\Lambda} \hat{X}\hat{Y} \right.$$
$$\left. + \left[\frac{3D^2}{2c^2 r^2} - \frac{1}{2\Lambda^2}\left(A^2 + \frac{2D^2 e^{\nu}}{c^2 r^2}\right)\right] \hat{Y}^2 - \tfrac{1}{2}\left(1 + \frac{3D^2}{c^2 r^2}\right) \hat{Z}^2 \right\}, \tag{6}$$

where the following abbreviations are used:

$$\Lambda = \sqrt{A^2 - \frac{e^{\nu} D^2}{r^2 c^2}}, \quad \Sigma = \sqrt{1 - \frac{e^{\nu}}{\Lambda^2}},$$

$q^{(1)} = \hat{X}$, $q^{(2)} = \hat{Y}$, $q^{(3)} = \hat{Z}$. It should be noted that the Schwarzschild coordinates $r, \theta, \varphi, x^4 = ct$ have nothing to do with the positions operator $q^{(\alpha)}$. They only fix the location of the atom and are consequently taken as functions of the proper time $\tau$ in the Hamiltonian: $r = r(\tau)$ etc.

A detailed examination of Eq. (6) reveals that the external gravitational field may substantially influence the atomic spectrum in the region $r \sim r_g$, if the constant $D$, which is proportional to the angular momentum of the atom, takes large values. It can be shown that the Hamiltonian can be regarded as quasistationary for all trajectories of the atom (i.e., that the gravitationally caused transitions have negligible probabilities). Thus one may speak of quasistationary energy levels.

Now we turn to the following simple examples: a) circular motion, b) free fall from infinity.

In the case of a circular orbit ($r = R$) we find that $\mathscr{H}_1$ leads to negligible corrections for all geodesics, if $R$ essentially differs from the value $3/2\, r_g$. Therefore, we consider trajectories with $R = 3/2\, r_g\, (1+\delta)$, where $0 < \delta \ll 1$. For these trajectories we obtain

$$\mathscr{H}_1 \cong -\tfrac{2}{27} \frac{m_0 c^2}{r_g^2 \delta} (\hat{X}^2 - \hat{Y}^2). \tag{7}$$

Consequently, the corrections have the order of magnitude of the fine structure only for orbits with $\delta \sim 10^{-18}$, where we have used the typical value $r_g \sim 10^5$ cm ($M \sim M_\odot = 1.9 \cdot 10^{38}$ g). Since no stable motion is possible in this region (see, e.g., [12]), these trajectories are only of theoretical interest.

Now we consider an atom which falls freely from infinity with the initial velocity $v$ and the impact parameter $l$ onto a black hole. It can easily be shown that, in the approximation $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \gg 1$, $\mathscr{H}_1$ is given by

$$\mathscr{H}_1 \approx \tfrac{3}{4} m_0 c^2 \frac{r_g l^2 \gamma^2}{r^5} (\hat{Z}^2 - \hat{Y}^2). \tag{8}$$

The calculation reveals the gravitational line splitting and line shift to be of the order of magnitude of the fine structure splitting in the vicinity of the horizon already for $\gamma^{-1} \lesssim 10^{-9}$, where we have used the values $r \sim r_g \sim l \sim 10^5$ cm. The energies of the atom which correspond to these values of $\gamma$ ($\sim 10^{12}$ erg) are possible for particles in cosmic rays.

Consequently, an observer at infinity will observe absorption and emission spectra of a gas cloud accreting into a black hole which have both a large and irregular red shift (proportional to $\gamma$) and a broadening and shift of the spectral lines due to the time dependence of $\mathscr{H}_1$ and different values of $A$ and $D$ for the atoms in the cloud, provided the values of $D$ for the atoms are large enough.

Finally, it should be noted that in the case of a charged black hole (the Reissner-Nordström solution) measurable effects arise even for essentially smaller values of $\gamma$ as a consequence of the external electromagnetic field. In this case quasistationary Zeeman and Stark effects (see Eq. (2)) occur. The effective electric and magnetic fields $E_{(\alpha)}$ and $H_{(\alpha)}$ depend essentially on the trajectories in the external gravitational field. Finally, it should be possible to generalize Eq. (6) to the case of the Kerr metric. In this case one has to replace Eq. (5) by the corresponding expression for the components of $U^i$ applying to a rotating black hole. These expressions can be found in [13].

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