NON-RELATIVISTIC PARTICLE OF ARBITRARY SPIN IN THE COULOMB FIELD

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Exact solutions of the equations of motion for the non-relativistic arbitrary spin particle in the Coulomb field are obtained. Galilei-invariant two-particle equations for spin-$\frac{1}{2}$ particles are proposed.

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1. *Introduction*

It is well-known that relativistic wave equations run into serious difficulties when one tries to describe the interaction of a particle of spin $s > \frac{1}{2}$ with external electromagnetic field. There are paradoxes connected with causality violation [1], the absence of stable solutions of the Coulomb problem [2] and others (see e.g. [3]).

Therefore it is interesting to study an alternative possibility of describing the spinning particle, one that makes use of Galilei-invariant wave equations (GIWE). Such equations were first proposed by Levy-Leblond [4], and generalized by Hagen and Hurley [5, 6] to the case of arbitrary spin. An extensive discussion of the Galilei invariant approach in classical and quantum mechanics is given in [7].

In the papers [8–12] GIWE (Galilei-Invariant Wave Equations) for particles of any spin are obtained which unlike those of Levy-Leblond and Hagen-Hurley, describe the spin-orbit and Darwin couplings of a particle with a field. The equations in [8–12] do not pretend to a complete description of the interaction of a charged particle with electromagnetic field, but they describe adequately the main properties of such an interaction for non-relativistic energies and predict the same physical effects as the one-particle Dirac equation.

In the present paper the GIWE found in [12] are used to describe the motion of a charged particle of arbitrary spin in the Coulomb field. Exact solutions of the corresponding motion equations are found and the energy spectrum of coupled states is discussed.

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Together with one-particle equations the motion equations for interacting particle systems are of great interest for physicists and mathematicians. The interest in these equations has grown considerably in recent years owing to the successes of meson spectroscopy.

Describing two-particle system one usually uses either the covariant Bethe-Salpeter (BS) equation or a semirelativistic equation of the Breit type [13]. The solutions of the BS equation depend on an additional parameter, viz. proper time of the particle system, the physical meaning of which is not clear. As to the Breit equation, it is not invariant under either Lorentz or Galilei transformations and so does not satisfy any relativity principle accepted in physics.

In the present paper a two-particle equation is proposed which is invariant under the Galilei group and leads to the same fine and hyperfine energy spectrum structure as the Breit equation does.

2. *Equation for radial wave function*

GIWE for a charged particle with arbitrary spin s in an external electromagnetic field has the form [12]

$$\left[\beta_\mu \pi^\mu + (1-\beta_0)2m + \frac{ek}{4m} S_{\mu\nu}F^{\mu\nu}\right]\Psi = 0, \tag{1}$$

where

$$\pi_\mu = p_\mu - eA_\mu, \quad F_{\mu\nu} = -i[\pi_\mu, \pi_\nu], \quad \mu = 0, 1, 2, 3,$$

$$p_0 = i\frac{\partial}{\partial t}, \quad p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1, 2, 3;$$

$$S_{0a} = -S_{a0} = \tfrac{1}{2}\varepsilon_{abc}\beta_0\beta_b\beta_c,$$

$$S_{ab} = \varepsilon_{abc}[(1-2\beta_0)\beta_c + k'S_{0c}], \tag{2}$$

$$\beta_0 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta_a = \frac{1}{s}\begin{pmatrix} 0 & S_a & K_a^+ \\ S_a & 0 & 0 \\ K_a & 0 & 0 \end{pmatrix}$$

A_μ is the vector-potential of electromagnetic field, I is $(2s+1)$-row unit matrix, S_a are $(2s+1)\times(2s+1)$-dimensional generators of irreducible representation $D(s)$ of O(3) group, K_a are matrices of dimensionality $(2s-1)\times(2s+1)$ which are determined up to a phase by the relations

$$K_a S_b - S'_b K_a = i\varepsilon_{abc}K_c,$$

$$S_a S_b + K_a^+ K_b = is\varepsilon_{abc}S_c + s^2\delta_{ab}, \tag{3}$$

where S'_a are the generators of the representation $D(s-1)$ of the O(3) group. The explicit form of the matrices, S'_a, K_a and S'_a (which is not used here) is given e.g. in [6].

Eq. (1) may be considered as a generalization of the Levy-Leblond-Hagen-Hurley equations to the case of a particle with anomalous moment [12]. If $k = 0$, Eq. (1) reduces to the one obtained in [6].

Here we consider Eq. (1) for the case of the Coulomb field, i.e. where $A_0 = -\frac{Ze}{x}$, $A_a = 0$. It is convenient to go on from (1) to a second-order equation for the $(2s+1)$--component wave function. Multiplying (1) by β_0 and $1-\beta_0$ and expressing $(1-\beta_0)\Psi$ by $\beta_0\Psi$, one obtains from (1) the following equivalent system [12]

$$\left[\pi_0 + \frac{ek}{4m}\vec{\beta}\times\vec{\beta}\cdot\vec{E} - \frac{1}{2m}(\vec{\beta}\cdot\vec{\pi})^2\right]\beta_0\Psi = 0, \tag{4a}$$

$$(1-\beta_0)\Psi = \frac{1}{2m}\vec{\beta}\cdot\vec{\pi}\beta_0\Psi, \tag{4b}$$

where $\vec{E} = -i[\pi_0, \vec{\pi}] = -\frac{Ze\vec{x}}{x}$ is the vector of electric field strength.

We see that Eq. (1) reduces to the Eq. (4a) for the function $\Phi_s = \beta_0\Psi$, which has $2s+1$ non-zero components. The remaining $4s$ components (i.e. $(1-\beta_0)\Psi$) are expressed via Φ_s according to (4b).

Using the explicit form of matrices (2) and taking into account relations (3), one can write equation (4a) in the form

$$i\frac{\partial}{\partial t}\Phi_s = \left(\frac{p^2}{2m} - \frac{\alpha}{x} - \frac{ik\alpha}{2sm}\frac{\vec{S}\cdot\vec{x}}{x^3}\right)\Phi_s, \quad \alpha = Ze^2. \tag{5}$$

The solutions of Eq. (5) which correspond to the states with energy ε have the form $\Phi_s = \exp(-i\varepsilon t)\Phi_s(\vec{x})$. Taking into account the symmetry of Eq. (5) under the O(3) group, it is convenient to represent $\Phi_s(\vec{x})$ as a linear combination of spherical spinors

$$\Phi_s(\vec{x}) = \varphi_\nu(x)\Omega^s_{jj+s-\nu m}, \tag{6}$$

$$\nu = 0, 1, 2, \ldots, 2n_{sj}, \qquad n_{sj} = \min(s, j), \tag{7}$$

where $\{\Omega^s_{jj+s-\nu m}\}$ is the complete set of eigenfunctions of operators $\vec{J}^2$, $\vec{L}^2$, and J^3 ($\vec{J} = \vec{x}\times\vec{p}+\vec{S} \equiv \vec{L}+\vec{S}$), corresponding to the eigenvalues $j(j+1)$, $l(l+1)$ ($l = s+j-\nu$) and m.

Substituting (6) into (5), one obtains the following equations for radial functions

$$D\varphi_\nu(x) = x^{-2}b_{\nu\nu'}\varphi_{\nu'}, \tag{8}$$

where

$$b_{\nu\nu'} = [s(s+1)+2js-2\nu(j+s)+\nu(\nu-1)]\delta_{\nu\nu'} + \frac{ik\alpha}{s}a_{\nu\nu'},$$

$$D = \varepsilon + \frac{\alpha}{x} + \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{j(j+1)}{x^2}, \tag{9}$$

$a_{\nu\nu'}$ are matrix elements of the operator $\vec{S}\cdot\hat{\vec{x}} = \frac{\vec{S}\cdot\vec{x}}{x}$ in spherical spinor basis, determined by the relation

$$\vec{S}\cdot\hat{\vec{x}}\Omega^{s}_{jj+s-\nu m} = a_{\nu\nu'}\Omega^{s}_{jj+s-\nu' m}.$$

The values of $a_{\nu\nu'}$ for arbitrary s and j are [14]

$$a_{\nu\nu'} = -\tfrac{1}{2}(\delta_{\nu\nu'+1}a_{\nu}+\delta_{\nu\nu'-1}a_{\nu+1}),$$

$$a_{\nu} = \left[\frac{\nu(d_j-\nu)(d_s-\nu)(d_{js}-\nu)}{(d_{js}-2\nu-1)(d_{js}-2\nu+1)}\right]^{1/2},$$

$$d_j = 2j+1, \quad d_s = 2s+1, \quad d_{js} = d_j+d_s. \tag{10}$$

So the problem of describing the motion of any spin charged particle in the Coulomb field is reduced to the solution of Eqs. (8)–(10) for the radial function.

3. *Energy spectrum of an arbitrary spin particle in the Coulomb field*

The matrix $||b_{\nu\nu'}||$ commutes with operator D (9) and is diagonalizable. So the system (8) can be reduced to noncoupled equations of the form

$$D\varphi = x^{-2}b^{sj}\varphi, \tag{11}$$

where D is the operator (9), b^{sj} are the matrix $||b_{\nu\nu'}||$ eigenvalues. Each of the equations (12) in its turn reduces to the well-known [15]

$$z\frac{d^2y}{dz^2}+\frac{dy}{dz}+\left(\beta-\frac{z}{4}-\frac{q^2}{4z}\right)y = 0, \tag{12}$$

where

$$y = \sqrt{z}\,\varphi, \quad z = 2\sqrt{-2m\varepsilon}\,x,$$

$$\beta = \sqrt{\frac{m}{-2\varepsilon}}\,\alpha^2, \quad q^2 = d_j^2+4b_{sj}. \tag{13}$$

The solutions of Eq. (12) for coupled states ($\varepsilon < 0$) are expressed in terms of Laguerre polynomials, and parameter β takes the values [15]

$$\beta = \tfrac{1}{2}(q+1)+n', \quad n' = 0, 1, 2, \ldots. \tag{14}$$

According to (13), (14)

$$\varepsilon = -\frac{m\alpha^2}{(\sqrt{(j+\frac{1}{2})^2+b^{sj}}+n'+\frac{1}{2})^2}. \tag{15}$$

As a result one obtains formula (15), which determines the energy levels of an arbitrary spin particle in the Coulomb field. Parameter b^{sj} in (15) takes values which coincide with the roots of the characteristic equation for matrix (9)

$$\det \|b_{\nu\nu'} - b^{sj}\delta_{\nu\nu'}\| \equiv \det \left\| [s(s+1)+2js-2\nu(j+s)+\nu(\nu-1) - b^{sj}]\delta_{\nu\nu'} + \frac{ik\alpha}{s} a_{\nu\nu'} \right\| = 0 \tag{16}$$

where $a_{\nu\nu'}$ are coefficients (10).

Formula (16) determines an algebraic equation of order $2n_{sj}+1$ which can be resolved in radicals for $s \leqslant \frac{3}{2}$ or $j \leqslant \frac{3}{2}$ only. To analyse the spectrum (15) for any s and j it is convenient to represent the solutions of Eq. (16) in the form

$$b^{sj} = s(s+1)+2js-2\nu(j+s)+\nu(\nu-1)-(k\alpha)^2 b_\nu^{sj} + o(k\alpha)^4 \tag{17}$$

where b_ν^{sj} are unknown functions. Then using (10) and neglecting terms of order α^4 one obtains

$$b_\nu^{sj} = \frac{1}{4s^2}(B_\nu - B_{\nu+1}), \quad B_\nu = \frac{a_\nu}{d_{sj}-2\nu}. \tag{18}$$

Representing (16) as a power series in α^2 and using (17), (18) one obtains the relations

$$\varepsilon = -\frac{m\alpha^2}{n^2} - \frac{mk^2\alpha^4 b_\nu^{sj}}{n^3(l+\frac{1}{2})} + o(\alpha^6),$$

$$n = n'+j+s+1-\nu = 1, 2, \ldots, \quad l = j+s-\nu = 0, 1, \ldots, n-1. \tag{19}$$

Formula (19) determines the fine structure of the spectrum of an arbitrary spin non-relativistic particle in the Coulombic field. Besides Balmer's term $-\frac{m\alpha^2}{n^2}$ formula (19) contains an additional one, which is caused by the existence of particle spin. It follows from the results of paper [12] that this additional term may be interpreted as the contribution of spin-orbit, Darwin and quadrupole interactions.

According to (19), any energy level corresponding to a fixed value of the main quantum number n, is split into $n-1$ sublevels with different values of l. Each of these sublevels in its turn is split into $2n_{sj}+1$ sublevels corresponding to possible values of ν, $n_{sj} = \min(s, j)$.

For $s = \frac{1}{2}$ spectrum (19) is not degenerated, contrary to the relativistic case. According to (13), (18) one has

$$s = \tfrac{1}{2}: \quad \varepsilon = -\frac{m\alpha^2}{n^2} - \frac{\lambda m k^2 \alpha^4}{n^3(l+\frac{1}{2})(l+\lambda+\frac{1}{2})}, \quad \lambda = \nu - \tfrac{1}{2} = \pm\tfrac{1}{2} \tag{20}$$

It is interesting to compare spectrum (20) with the corresponding one for the Dirac electron. Denoting Dirac energy levels by ε_D it is not difficult to show that formula (20) may be rewritten in the form

$$\varepsilon = \varepsilon_D - m - \left\langle \frac{p^4}{8m^3} \right\rangle \tag{21}$$

where averaging is made with Schrödinger wave functions. So spectrum (20) contains contributions predicted by the relativistic Dirac equation (neglecting the mass term and relativistic correction to a kinetic energy) — i.e. spin-orbit and Darwin interaction.

For $s \leqslant \frac{3}{2}$ or $j \leqslant \frac{3}{2}$ Eq. (16) can be solved in radicals. The corresponding solutions for $s \leqslant 1$ and $j \leqslant 1$ are

$$b^{0j} = 0; \qquad b^{\frac{1}{2}j} = \tfrac{1}{4} \pm \tfrac{1}{2}\sqrt{d_j^2 - 4(k\alpha)^2};$$

$$b^{1\frac{1}{2}} = \frac{c}{3} + 2\sqrt{-c}\cos\left[\tfrac{1}{3}\left(\gamma + \lambda\frac{\pi}{2}\right)\right],$$

$$\lambda = 0, \quad \pm\tfrac{1}{2}, \quad j \neq 0;$$

$$b^{s0} = 0, \qquad b^{s\frac{1}{2}} = \tfrac{1}{4}(d_s^2 - 3) \pm \tfrac{1}{2}\sqrt{d_s^2 - \left(\frac{k\alpha}{s}\right)^2};$$

$$b^{s1} = s(s+1) - 2 + \tfrac{1}{3}d + 2\sqrt{-d}\cos\left[\tfrac{1}{3}\left(\xi + \mu\frac{\pi}{2}\right)\right],$$

$$\mu = 0, \quad \pm 1, \quad s \neq 0, \tag{22}$$

where

$$\cos\gamma = \frac{b}{\sqrt{-c^3}}, \qquad b = \tfrac{2}{3}(k\alpha)^2 + \tfrac{1}{3}d_j^2 - \tfrac{1}{27}, \qquad c = (k\alpha)^2 - \tfrac{4}{27} - b,$$

$$\cos\xi = \frac{f}{\sqrt{-d^3}}, \qquad f = \tfrac{2}{3}\left(\frac{k\alpha}{s}\right)^2 + \tfrac{1}{3}d_s^2 - \tfrac{1}{27}, \qquad f = \left(\frac{k\alpha}{s}\right)^2 - \tfrac{4}{27} - f.$$

Formulae (15), (22) give the exact spectrum of nonrelativistic particles with spins $s = \frac{1}{2}$ and $s = 1$ (and with any spin for $j \leqslant 1$) in the Coulomb field.

4. *Two-particle equations*

One-particle GIWE admit an immediate generalization to the case of a system of two arbitrary spin particles. Consider the example of such equations for two spin-$\frac{1}{2}$ particles.

Let us denote by $\Psi(t, \vec{x}_1, \vec{x}_2)$ a 12-component wave function, where $\vec{x}_1$ and $\vec{x}_2$ are the sets of spatial variables (coordinates of first and second particle), and by L a first-order differential operator of the form

$$L = L(p_0, \vec{p}_1, \vec{p}_2, V)$$
$$= \beta_0 p_0 - \vec{\beta}_1 \cdot \vec{p}_1 - \vec{\beta}_2 \cdot \vec{p}_2 + 2\beta_4 m_1 + 2\beta_5 m_2$$
$$\equiv \begin{pmatrix} (p_0+V)I & \vec{\sigma}^1 \cdot \vec{p}_1 & \vec{\sigma}^2 \cdot \vec{p}_2 \\ \vec{\sigma}^1 \cdot \vec{p}_1 & 2m_1 I & 0 \\ \vec{\sigma}^2 \cdot \vec{p}_2 & 0 & 2m_2 I \end{pmatrix}, \quad (23)$$

where $\vec{\sigma}^1$ and $\vec{\sigma}^2$ are two commuting sets of Pauli matrices, i.e. matrices of dimensionality 4×4, satisfying the relations

$$[\sigma_a^\alpha, \sigma_b^\beta] = 2i\delta_{\alpha\beta}\varepsilon_{abc}\sigma_c^\alpha, \quad (\sigma_a^\alpha)^2 = I, \quad \alpha = 1, 2 \quad (24)$$

I and 0 are 4×4 unit and zero matrices, $\vec{p}_\alpha = -i\dfrac{\partial}{\partial \vec{x}_\alpha}$, V is scalar function of $\vec{r} = \vec{x}_1 - \vec{x}_2$, $\dfrac{\partial}{\partial \vec{r}}$. Then the two-particle Galilei-invariant equation can be written in the form

$$L(p_0, \vec{p}_1, \vec{p}_2, V)\Psi(t, \vec{x}_1, \vec{x}_2) = 0. \quad (25)$$

Galilei invariance of Eq. (25) follows from the fact that for any Galilei transformation

$$\vec{x}_\alpha \to \vec{x}'_\alpha = R\vec{x}_\alpha + \vec{V}t + \vec{a}, \quad t \to t' = t+b \quad (26)$$

one may collate the following transformation of a function $\Psi(t, \vec{x}_1, \vec{x}_2)$

$$\Psi(t, \vec{x}_1, \vec{x}_2) \to \Psi'(t', \vec{x}'_1, \vec{x}'_2) = U^{-1}\Psi(t, \vec{x}_1, \vec{x}_2),$$
$$U = \exp\left[if(t, \vec{x})\right](1 + i\vec{\lambda}\cdot\vec{V})\exp(i\vec{S}\cdot\vec{\theta}), \quad (27)$$

where $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ are spatial rotation parameters,

$$f(t, \vec{x}) = m\vec{V}\cdot\vec{x} + \tfrac{1}{2}m\vec{V}^2 t, \quad m = m_1 + m_2,$$
$$\vec{x} = \frac{1}{m}(m_1\vec{x}_1 + m_2\vec{x}_2), \quad \vec{S} = \hat{I}\otimes\hat{\vec{S}},$$
$$\hat{\vec{S}} = \tfrac{1}{2}(\vec{\sigma}^1 + \vec{\sigma}^2), \quad \vec{\lambda} = \tfrac{1}{2}(1-\beta_0)(\vec{\beta}_1 + \vec{\beta}_2),$$

I is the three-row unit matrix.

Transformations (27) realise representation of Galilei group and retain Eq. (25) invariant, in as much as

$$U^+ L(p_0, \vec{p}_1, \vec{p}_2, V)U = L(p'_0, \vec{p}'_1, \vec{p}'_2, V),$$

where

$$p'_0 = i\frac{\partial}{\partial t'}, \quad \vec{p}'_\alpha = -i\frac{\partial}{\partial \vec{x}'_\alpha}.$$

Denoting Ψ = coulomb (Ψ_1, Ψ_2, Ψ_3), where Ψ_a $(a = 1, 2, 3)$ are four component functions, one obtains from (23)–(25) the following equation for Ψ_1

$$i\frac{\partial}{\partial t}\Psi_1 = \left(\frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + V\right)\Psi_1, \tag{28}$$

which coincides with the two-particle Schrödinger equation. It is not difficult to demonstrate that Eq. (25), (23) is the simplest first order GIWE which leads to Eq. (28) for four-component wave function.

Choosing interaction potential V in the form

$$V = -\frac{Ze^2}{r} - \frac{ikZe^2}{2ms}\frac{\vec{S}\cdot\vec{r}}{r^3}$$

and turning in (28) to c.m. frame, we arrive at an exactly solvable equation of type (5). The choice

$$V = -Ze^2\left[\frac{1}{r} + \frac{1}{2m_1m_2}\left(k_1\vec{p}\cdot\frac{1}{r}\vec{p} + k_2\vec{p}\cdot\vec{r}\frac{1}{r^3}\vec{r}\cdot\vec{p}\right)\right.$$
$$\left. + \frac{i}{2}\left(\frac{\vec{\sigma}^1}{m_1} - \frac{\vec{\sigma}^2}{m_2}\right)\cdot\frac{\vec{r}}{r^3} - \frac{1}{m_1m_2}\frac{\vec{S}\cdot\vec{L}}{r^3} - \frac{2\pi}{m_1m_2}(\vec{S}^2 - \tfrac{3}{2})\delta(\vec{r})\right] \tag{29}$$

where $\vec{p} = -i\frac{\partial}{\partial\vec{r}}$, $\vec{L} = \vec{r}\times\vec{p}$, $k_1 = 1-\frac{1}{4}\delta$, $k_2 = 1+\frac{3}{2}\delta$, $\delta = (m_1^3+m_2^3)\,[m_1m_2(m_1+m_2)]^{-1}$, leads to an equation which is Galilei invariant and predicts the same energy spectrum of coupled particles as the semirelativistic Breit equation [13] does. To verify this statement one may substitute (29) into (28) and calculate the Hamiltonian eigenvalues using the standard perturbation technique.

Equations (23), (25) can also be used for the description of a particle system in an external electromagnetic field. Indeed, the minimal substitution $\vec{p}_a \to \vec{p}_a - e_a\vec{A}(t, \vec{x}_a)$, $p_0 \to p_0 - e_1A_0(t, \vec{x}_1) - e_2A_0(t, \vec{x}_2)$ keeps these equations Galilei invariant and automatically leads to the appearance of Pauli terms $\frac{e_a\vec{\sigma}^a\cdot\vec{H}(t, \vec{x}^a)}{2m_a}$ in the Schrödinger equation (28). Another possibility is to introduce into (23), (25) anomalous interaction terms, as is done in [12] for the Levi-Leblond-Hagen-Hurley equation [6] (see also Section 2 of the present paper).

So two-particle GIWE can serve as a suitable mathematical model for the description of a system of interacting particles with spins $\frac{1}{2}$. Eqs. (23), (25) and two-particle GIWE for arbitrary spin particles will be considered in greater detail in future publications.

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