CLASSICAL YANG MILLS THEORY IN PRESENCE OF ELECTRIC AND MAGNETIC CHARGES*

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The classical Yang-Mills field equations in presence of static electric and magnetic sources have been studied. By means of a second potential the problem of obtaining the solutions has been reduced to the initial value problem in the temporal gauge. The solutions to the field equations have been described in terms of the interaction energy and total isotopic charge.

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1. *Introduction*

Because of the important role that Yang-Mills theories are playing today, there is a current interest in the classical solutions of the Yang-Mills field equations. One of the reasons for such an interest is the belief that some hint of confinement should be seen in the classical sector of exact Yang-Mills theories [1]. The investigations on the classical theories have already brought forth the existence of Wu-Yang monopole [2] and the Coleman's [3] non-Abelian plane wave. Based on the purely classical treatments, the topological properties of non-Abelian gauge theories have come through the discovery of t'Hooft-Polyakov monopole [4] and the instanton [5] and meron [6] solutions of Yang-Mills theory. Recently, an attempt [7] to provide a classical formulation for a quantum theory of the gauge fields interacting with external sources has been made and smooth solutions associated with a smooth non-Abelian source have been obtained [8]. Jacobs and Wuduka [9] have obtained the solutions to the gauge field equation in the presence of classical time independent electric and magnetic sources and classical static solutions of the Yang-Mills equations have also been studied [10] for spherically symmetric solutions.

An important contribution to such theories has been made by Sikivie and Weiss [11] through the study of the solutions of classical Yang-Mills field equations in presence of static external sources and discussing the Yang-Mills equations as an initial value problem in the temporal gauge. The convenience of the temporal gauge in the initial value problem

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has also been indicated by Joe Kiskis [8]. However, the external sources in such theories have been purely electric ones. In the present paper we consider the static sources as both electric and magnetic. We study the classical Yang-Mills field equations and attempt to formulate them as an initial value problem in the temporal gauge. In order to avoid the string variables associated with the magnetic charges, a two potential approach has been adopted. The initial value problem has been applied to discuss the Y-M fields produced by a system of point dyons and it has been observed that solutions differing in their total energy and isospin exist.

2. *The field equations*

We consider the following non-Abelian field tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ef^{abc}A^b_\mu A^c_\nu - \tfrac{1}{2}\,\delta_{\mu\nu\varrho\sigma}(\partial^\varrho B^{\sigma a} - \partial^\sigma B^{\varrho a} + gf^{abc}B^{\varrho b}B^{\sigma c}), \tag{1}$$

where A^a_μ and B^a_μ are the "two" non-Abelian potentials, which we require to include both the non-Abelian electric and magnetic sources without the use of string variables, e and g are the corresponding gauge coupling parameters, f^{abc} are the structure constants of the gauge group SU(2) and $\delta_{\mu\nu\varrho\sigma}$ is the antisymmetric tensor. The dual of equation (1) is

$$\tilde{F}^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + gf^{abc}B^b_\mu B^c_\nu + \tfrac{1}{2}\,\delta_{\mu\nu\varrho\sigma}(\partial^\varrho A^{\sigma a} - \partial^\sigma A^{\varrho a} + ef^{abc}A^{\varrho b}A^{\sigma c}). \tag{2}$$

In terms of field tensors (1) and (2) the Lagrangian density for the system of non-Abelian dyon may be written as

$$\mathscr{L} = -\tfrac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu a} - \tfrac{1}{4}\,\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu a} + j^a_\mu A^a_\mu + k^a_\mu B^a_\mu, \tag{3}$$

where j^a_μ and k^a_μ are respectively the electric and magnetic source densities. It can be observed that the field strength tensors (1) and (2) are covariant under the usual non-Abelian gauge transformations of A^a_μ and B^a_μ and correspondingly the free field part of the Lagrangian density is gauge covariant. The Euler-Lagrange variation of the Lagrangian density (3) gives the following field equations

$$D_\mu F^{\mu\nu a} = j^{\nu a} \tag{4a}$$

and

$$D'_\mu \tilde{F}^{\mu\nu a} = k^{\nu a}, \tag{4b}$$

where

$$D_\mu = \partial_\mu + ef^{abc}A^b_\mu \tag{5a}$$

and

$$D'_\mu = \partial_\mu + gf^{abc}B^b_\mu \tag{5b}$$

are the covariant derivatives. The gauge covariance of these derivatives as well as those of $j^{\nu a}$ and $k^{\nu a}$ makes the interaction part of the Lagrangian density also gauge covariant.

Thus the Lagrangian density (3) is gauge covariant. The conserved but not gauge covariant Noether currents can be given by

$$J^{\nu a} = j^{\nu a} - ef^{abc}A^b_\mu F^{\mu\nu c} \tag{6a}$$

and

$$K^{\nu a} = k^{\nu a} - gf^{abc}B^b_\mu \tilde{F}^{\mu\nu c}, \tag{6b}$$

where $j^{\nu a}$ and $k^{\nu a}$ are gauge covariant but not conserved.

From the Lagrangian density (3), we can calculate the energy momentum tensor and hence the total energy as

$$H = \int d^3x T^{00} \tag{7}$$

$$= \int [\tfrac{1}{2}(F^a_{0i}F^{0ia} + \tilde{F}^a_{0i}\tilde{F}^{0ia}) + j^a_i A^a_i + k^a_i B^a_i] d^3x, \tag{8}$$

where the surface constraints

$$\int d\vec{s} F^a_{0i} A^a_0 = 0 = \int d\vec{s} \tilde{F}^a_{0i} B^a_0 \tag{9}$$

have been used. The F^a_{0i} and $\tilde{F}^a_{0i}$ have the following forms:

$$F^a_{0i} = \partial_0 A^a_i - \partial_i A^a_0 + ef^{abc}A^b_0 A^c_i - (\vec{\nabla}\times\vec{B}^a)_i - g/2 f^{abc}(\vec{B}^b\times\vec{B}^c)_i \tag{10}$$

and

$$\tilde{F}^a_{0i} = \partial_0 B^a_i - \partial_i B^a_0 + gf^{abc}B^b_0 B^c_i + (\vec{\nabla}\times\vec{A}^a)_i + e/2 f^{abc}(\vec{A}^b\times\vec{A}^c)_i. \tag{11}$$

In these equations the cross product of potentials is in gauge group [SU(2)] space and $i = 1, 2, 3$.

3. The static sources

The static electric and magnetic sources may be described by

$$j^{\mu a} = \delta^{\mu o} j^{0a} \tag{12}$$

and

$$k^{\mu a} = \delta^{\mu o} k^{0a} \tag{13}$$

where $a = 1, 2, 3$. The gauge covariant sources $j^{\mu a}$ and $k^{\mu a}$ satisfy

$$D_\mu j^{\mu a} = 0 \tag{14}$$

and

$$D'_\mu k^{\mu a} = 0, \tag{15}$$

from which it may be implied that

$$\partial_0 j^{0a} = -f^{abc}A^b_0 j^{0c} \tag{16}$$

and

$$\partial_0 k^{0a} = -f^{abc} B_0^b k^{0c}. \tag{17}$$

The energy of the static source comprising of both electric and magnetic charges will be given by

$$H = \tfrac{1}{2} \int (F_{0i}^a F^{0ia} + \tilde{F}_{0i}^a \tilde{F}^{0ia}) d^3\vec{x}. \tag{18}$$

4. *The initial value problem in the temporal gauge*

We assume that

$$A_0 = A_0^a T^a = 0 \tag{19a}$$

and

$$B_0 = B_0^a T^a = 0, \tag{19b}$$

where T^a are the gauge group generators of SU(2) algebra, constitute the temporal gauge condition for the system of both electric and magnetic charges. Then considering equation (4a) in the gauge $A_0 = 0$, we obtain

$$\partial_0 F_{0i}^a = \partial_j F_{ji}^a + e(A_j \times F_{ji})^a \tag{20}$$

$$\partial_i F_{0i}^a + e(A_i \times F_{0i})^a = j^{0a}(\vec{x}) \tag{21}$$

$$\partial_0 A_i^a(\vec{x}, t) = F_{0i}^a(\vec{x}) + \{\vec{\nabla} \times B^a(\vec{x}, t)\}_i + g/2 f^{abc} \{B^b(\vec{x}, t) \times B^c(\vec{x}, t)\}_i. \tag{22}$$

Similarly, in the gauge $B_0 = 0$, the following equations may be obtained from Eq. (4b)

$$\partial_0 \tilde{F}_{0i}^a = \partial_j \tilde{F}_{jk}^a + g(B_j \times \tilde{F}_{ji})^a \tag{23}$$

$$\partial_i \tilde{F}_{0i}^a + g(B_i \times \tilde{F}_{0i})^a = k^{0a}(\vec{x}) \tag{24}$$

$$\partial_0 B_i^a(\vec{x}, t) = \tilde{F}_{0i}^a(\vec{x}) - \{\vec{\nabla} \times A^a(\vec{x}, t)\}_i - e/2 f^{abc} \{A^b(\vec{x}, t) \times A^c(\vec{x}, t)\}_i. \tag{25}$$

Thus in the gauge $A_0 = 0$, $B_0 = 0$ Eqs. (21) and (24), with j^{0a} and k^{0a} as static, are constraints for finding the minimum of energy (Eq. 18). If the constraints (21) and (24) are satisfied at the initial time they will be preserved in the time evolution, just as the value of energy (Eq. 18) is preserved. It may be observed from equations (4a) and (14) that in the gauge $A_0 = 0$

$$\partial_0 (D_i F^{0ia}) = 0. \tag{26}$$

Similarly in the gauge $B_0 = 0$ equations (4b) and (15) give

$$\partial_0 (D'_i \tilde{F}^{0ia}) = 0. \tag{27}$$

These equations suggest that equations (21) and (24) are the $t = t_0$ (initial time) constraints. It will therefore be sufficient for our purpose to study the energy of a certain initial configuration. Substitution of F_{0i}^a from Eq. (22) into (20) tells us that Eq. (20) is a second order equation for the time evolution of $A_i^a(\vec{x}, t)$. Similarly, using Eq. (25), a second order time evolution equation for $B_i^a(\vec{x}, t)$ may be obtained from Eq. (23). Thus, when potentials $A_i^a(\vec{x}, t)$, $B_i^a(\vec{x}, t)$ and the fields $F_{0i}^a(\vec{x}, t)$, $\tilde{F}_{0i}^a(\vec{x}, t)$ are given at some initial time $t = t_0$, the Yang-Mills fields with electric and magnetic sources are specified for all times through Eqs. (20), (22) and (23), (25). The initial values of the potentials $A_i^a(\vec{x}, t)$ and $B_i^a(\vec{x}, t)$ at the initial time $t = t_0$ may be obtained from Eqs. (22) and (25) respectively. For this purpose we consider that [12]

$$A_i^a(\vec{x}, t) = A_i^a(\vec{x}) + a_i^a(\vec{x})\, e^{s\omega_a t} \tag{28}$$

and

$$B_i^a(\vec{x}, t) = B_i^a(\vec{x}) + b_i^a(\vec{x})\, e^{s\omega_b t}, \qquad s = \sqrt{-1}, \tag{29}$$

where $a_i^a(\vec{x}, t)$ and $b_i^a(\vec{x}, t)$, which may be complex, are the small perturbations on the potentials; ω_a and ω_b are the frequencies of eigenmodes which determine [13] the Abelian and non-Abelian character of potentials $A_i^a(\vec{x}, t)$ and $B_i^a(\vec{x}, t)$ respectively. Substituting equation (29) into equation (22) and then integrating it from the initial time t_0 to the time t, we obtain (see the Appendix)

$$A_i^{Ia} = A_i^a(\vec{x}, t)|_{t=t_0} = 0. \tag{30}$$

Similarly, the substitution of equation (28) into equation (25) gives the initial value

$$B_i^{Ia} = B_i^a(\vec{x}, t)|_{t=t_0} = 0. \tag{31}$$

That is the initial values of both potentials employed to describe the fields associated with electric and magnetic charges are vanishing. These zero initial values will help to eliminate the non-linearity from the Yang-Mills field equations.

5. The point dyons

Now, we consider the Yang-Mills fields produced by a system of two point dyons. Let $\hat{n}_1^a$ and $\hat{n}_2^a$ be the unit vectors giving the orientation of two dyons separated in the gauge space through a distance $r = |\vec{a}_1 - \vec{a}_2|$ and let $(z_1 e, z_1 g)$ be the electric and magnetic charges of one dyon and $(z_2 e, z_2 g)$ be those of the other, e and g being the usual electric and magnetic coupling parameters. The electric and magnetic gauge source distributions of the system may be written as

$$j^{0a}(\vec{x}) = z_1 e\hat{n}_i^a \delta(\vec{x} - \vec{a}_1) + z_2 e\hat{n}_2^a \delta(\vec{x} - \vec{a}_2) \tag{32}$$

and

$$k^{0a}(\vec{x}) = z_1 g\hat{n}_1^a \delta(\vec{x} - \vec{a}_1) + z_2 g\hat{n}_2^a \delta(\vec{x} - \vec{a}_2). \tag{33}$$

The time independent conserved isocharge may be written from equation (6) as

$$I^a = \int (J^{0a} + K^{0a}) d^3\vec{x} = \int (\partial_i F_{0i} + \partial_i \tilde{F}^{0i})^a d^3\vec{x} = \int\limits_{\substack{\text{surface} \\ \text{at } \infty}} (F_{0i} + \tilde{F}_{0i})^a d\vec{s}. \tag{34}$$

Considering the gauge transformations $U(\vec{x})$ obeying the condition [14]

$$U(\vec{x}) \xrightarrow[|\vec{x}|\to\infty]{} 1 \tag{35}$$

the gauge covariance of isocharge may be observed. Our aim is now to characterize solutions of equations (4) with electric and magnetic gauge sources (32) and (33) respectively in terms of their energy and total isotropic charge, as these quantities are gauge invariant as well as conserved. The total isocharge means the isocharge of source plus the isocharge carried by Yang-Mills fields. Since equations (21) and (24) are the constraint equations obtained from equations (4) in the temporal gauge, we solve these constraint equations at $t = t_0$ for the source distributions (32) and (33) respectively. At $t = t_0$, the constraint equations may then be written as

$$\partial_i \chi_i^a + f^{abc} A_i^{lb} \chi_i^c = z_1 e \hat{n}_1^a \delta(\vec{x}-\vec{a}_1) + z_2 e \hat{n}_2^a \delta(\vec{x}-\vec{a}_2) \tag{36a}$$

and

$$\partial_i \tilde{\chi}_i^a + f^{bac} B_i^{lb} \tilde{\chi}_i^c = z_1 g \hat{n}_1^a \delta(\vec{x}-\vec{a}_1) + z_2 g \hat{n}_2^a \delta(\vec{x}-\vec{a}_2), \tag{36b}$$

where χ_i^a and $\tilde{\chi}_i^a$ are the $t = t_0$ values of F_{0i}^a and $\tilde{F}_{0i}^a$ respectively. Using equations (30) and (31) the non-linear terms in these equations vanish and equations (36) may then be solved to obtain

$$\chi_i^a = \frac{z_1 e}{4\pi} \hat{n}_1^a \frac{(x_i - a_{1i})}{|\vec{x}-\vec{a}_1|^3} + \frac{z_2 e}{4\pi} \frac{\hat{n}_2^a (x_i - a_{2i})}{|\vec{x}-\vec{a}_2|^3} \tag{37a}$$

and

$$\tilde{\chi}_i^a = \frac{z_1 g}{4\pi} \hat{n}_1^a \frac{(x_i - a_{1i})}{|\vec{x}-\vec{a}_1|^3} + \frac{z_2 g}{4\pi} \frac{\hat{n}_2^a (x_i - a_{2i})}{|\vec{x}-\vec{a}_2|^3}, \tag{37b}$$

where

$$\nabla^2 \frac{1}{|\vec{x}-\vec{a}|} = -4\pi\delta(\vec{x}-\vec{a}) \tag{38a}$$

and

$$\frac{\vec{x}-\vec{a}}{|\vec{x}-\vec{a}|^3} = -\vec{\nabla} \frac{1}{|\vec{x}-\vec{a}|} \tag{38b}$$

have been used.

The energy of the configuration may be obtained as

$$W = \frac{1}{8\pi} \int (\chi_i^a \chi_i^a + \tilde{\chi}_i^a \tilde{\chi}_i^a) d^3\vec{x} \tag{39}$$

from which the interaction energy may be obtained after subtracting the infinite self energy of each dyon. That is

$$W_{\text{int}} = \frac{z_1 z_2 (e^2 + g^2) \hat{n}_1^a \hat{n}_2^a}{16\pi^2 r}. \tag{40}$$

The total isocharge may be obtained from equation (34) as

$$I^a = (e+g)(z_1\hat{n}_1^a + z_2\hat{n}_2^a). \tag{41}$$

Therefore, the solutions for the system of point dyons may be characterized by the interaction energy (40) and the isocharge (41). The solutions will be static when

$$\partial_0\chi_i^a = \partial_0\tilde{\chi}_i^a = 0 \tag{42}$$

which requires

$$z_1\hat{n}_1^a = \pm z_2\hat{n}_2^a. \tag{43}$$

Using equation (43) the following solutions corresponding to the Yang-Mills field equations in terms of the interaction energy and the total isocharge may be obtained

$$I^a = 2z_1(e+g)\hat{n}_1^a, \quad W_{\text{int}} = \frac{z_1^2(e^2+g^2)}{16\pi^2 r}, \tag{44a}$$

$$I^a = 0, \quad W_{\text{int}} = \frac{-z_1^2(e^2+g^2)}{16\pi^2 r}, \tag{44b}$$

$$I^a = 2z_1 e\hat{n}_1^a, \quad W_{\text{int}} = \frac{z_1^2 e^2}{16\pi^2 r}, \tag{44c}$$

$$I^a = 2z_1 g\hat{n}_1^a, \quad W_{\text{int}} = \frac{z_1^2 g^2}{16\pi^2 r}. \tag{44d}$$

Due to the presence of both electric and magnetic charges the solutions (44a) and (44b) are admissible.

6. *Discussion*

We have obtained the solutions to the static Yang–Mills fields in presence of both electric and magnetic charges. Unlike the magnetic charges of topological origin [4], we have assumed that the point particles carry both electric and magnetic charges. This assumption is similar to the one [15] used to formulate non-Abelian classical Lagrangian for point electric and magnetic charges, where however, the string variables have been used to describe the magnetic charges. We have instead introduced a non-Abelian field tensor $F_{\mu\nu}^a$ in terms of two potentials. A similar two potential quantum mechanical Abelian formulation of electron monopole interaction without string has been given by Barker and Graziani [16].

APPENDIX

In this appendix, we derive the equations (30) and (31). Substituting equation (29) into (22) we may obtain

$$\partial_0 A_i^a(\vec{x}, t) = P(\vec{x}) + Q(\vec{x})e^{s\omega_b t} + R(\vec{x})e^{2s\omega_b t}, \tag{A1}$$

where

$$P(\vec{x}) = F^a_{0i}(\vec{x})+\{\vec{\nabla}\times B^a(\vec{x})\}_i+\frac{g}{2}f^{abc}\{B^b(\vec{x})\times B^c(\vec{x})\}_i, \tag{A2}$$

$$Q(\vec{x}) = \{\vec{\nabla}\times b^a(\vec{x})\}_i+\frac{g}{2}\{B^b(\vec{x})\times b^c(\vec{x})+b^b(\vec{x})\times B^c(\vec{x})\}_i \tag{A3}$$

and

$$R(\vec{x}) = \{b^b(\vec{x})\times b^c(\vec{x})\}_i. \tag{A4}$$

Integration of Eq. (A1) from the initial time t_0 to the time t gives

$$A^a_i(\vec{x},t) = P(\vec{x})(t-t_0)-\frac{sQ(\vec{x})}{\omega_b}(e^{s\omega_b t}-e^{s\omega_b t_0}),$$

$$-\frac{sR(\vec{x})}{2\omega_b}(e^{2s\omega_b t}-e^{2s\omega_b t_0}) \tag{A5}$$

which for $t = t_0$ gives the equation (30) viz.

$$A^a_i(\vec{x},t)|_{t=t_0} = 0.$$

It may be noted that even if equation (28) is time dependent similar to equation (29) or just $F^a_{0i}(\vec{x},t) = \vec{F}^a_{0i}(\vec{x})e^{s\omega t}$ equation (30) shall still be valid.

Similarly, the integration of equation (25) after substitution of equation (28) will lead to equation (31).

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