Vol. B16 (1985) ACTA PHYSICA POLONICA No 10

ON THE RELATION BETWEEN GALILEAN, POINCARÉ AND EUCLIDEAN FIELD EQUATIONS

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(Received March 6, 1985)

A new kind of relation between Galilean, Poincaré and Euclidean field equations is demonstrated.

PACS numbers: 03.50.Kk

1. Introduction

The aim of the present paper is to show that there exists a new kind of relation between the Galilean and relativistic field equations which is different from the relation discussed in Ref. [1]. The material presented definitely contradicts the standard understanding of the non-relativistic physics as the limiting case of the relativistic one. In our procedure the Galilean covariant field equations are the primary objects and form a universal basis for all other field equations. In particular, from the Galilean field equations we obtain both the Poincaré and Euclidean field equations and this is true not only for free fields but also for the coupled fields in the abelian gauge theory.

Needless to say, the considered problem is very important for a deeper understanding of the meaning of the relativity principles in physics. After presenting the main results we try to interpret them in a more general way using the modern language of fibre bundles. In our formulation, various relativity principles are implemented by different maps from a universal bundle space to the physical space-times and these maps in turn are determined by particular choices of solutions of the Galilean field equations.

2. The "relativistic" properties of the Galilean field equations

In Ref. [2] we have introduced the most general Galilean invariant field equation in the form

$$\left[2\frac{\partial}{\partial\theta}\frac{\partial}{\partial t}-\Delta+\alpha\frac{\partial^2}{\partial\theta^2}+i\beta\frac{\partial}{\partial\theta}+\gamma\right]\psi(\vec{x},t,\theta)=0, \tag{2.1}$$

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where $\vec{x}$, t, θ are the five coordinates of the extended Galilean space-time and α, β, γ are arbitrary dimensional constants which should be expressed in terms of some primary parameters of the particle described by the field ψ. In Ref. [2] we admitted only the rest energy Ω and the Planck constant $\hbar$ as such primary constants and this led to the choice

$$\alpha = \gamma = 0$$

$$\beta = -2\frac{\Omega}{\hbar}. \tag{2.2}$$

It turns out, that more interesting results may be derived if we include into the set of primary constants also the mass m of the particle. With such set of primary constants instead of (2.2) we may then make the choice

$$\alpha = -\frac{\Omega}{m}$$

$$\beta = 0$$

$$\gamma = \frac{m\Omega}{\hbar^2}, \tag{2.3}$$

which improves the behaviour of (2.1) with respect to time inversion and also leads to the usual Schrödinger equation when we pass from the one-parameter extension of the Galilei group to the usual Galilei group. Indeed, taking in (2.1)

$$\psi(\vec{x}, t, \theta) = \exp\left(\frac{im\theta}{\hbar}\right)\varphi(\vec{x}, t) \tag{2.4}$$

we get for the wave function $\varphi(\vec{x}, t)$ the Schrödinger equation with rest energy Ω and mass m.

The field equation (2.1) with the choice (2.3) besides the solutions of the type (2.4) possesses many other solutions. In particular, let us consider the solutions of the type

$$\psi(\vec{x}, t, \theta) = u\left(\vec{x}, t + \frac{1}{c^2}\theta\right), \tag{2.5}$$

where $u(\vec{x}, \tau)$ are twice differentiable functions. Substituting (2.5) into (2.1) and taking the relativistic relation

$$\Omega = mc^2 \tag{2.6}$$

we get the Klein-Gordon wave equation

$$\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]u = 0, \tag{2.7}$$

where

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} - \Delta. \tag{2.8}$$

Similarly, considering solutions of (2.1) of the type

$$\psi(\vec{x}, t, \theta) = Z\left(\vec{x}, \frac{1}{c^2}\theta\right) \tag{2.9}$$

and taking (2.6) we get the Euclidean field equation

$$\left[\Delta_4 - \left(\frac{mc}{\hbar}\right)^2\right] Z = 0, \tag{2.10}$$

where Δ_4 is the four-dimensional Laplacé operator

$$\Delta_4 = \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} + \Delta. \tag{2.11}$$

We therefore see that equation (2.1) leads to all free field equations used in physics for scalar particles and we should somehow explain the physical meaning of this situation.

Before doing this let us show that the same situation is valid for other Galilean field equations. As the first example, let us take the Dirac equation which in the extended Galilean space-time has the form (see Appendix)

$$\left(\vec{\gamma}\cdot\vec{\nabla} + \sqrt{\frac{m}{\Omega}}(\gamma^0 - \gamma_5)\frac{\partial}{\partial t} + \sqrt{\frac{\Omega}{m}}\gamma_5\frac{\partial}{\partial\theta} + i\frac{\sqrt{m\Omega}}{\hbar}\right)\psi(\vec{x}, t, \theta) = 0 \tag{2.12}$$

where γ^0, γ^k and γ_5 are the usual Dirac matrices. Taking in this equation ψ in the form (2.4) we get the non-relativistic Dirac equation for the spinor $\varphi(\vec{x}, t)$:

$$\left[\vec{\gamma}\cdot\vec{\nabla} + \sqrt{\frac{m}{\Omega}}(\gamma^0 - \gamma_5)\frac{\partial}{\partial t} + i\frac{\sqrt{m\Omega}}{\hbar}(I + \gamma_5)\right]\varphi(\vec{x}, t) = 0 \tag{2.13}$$

which, however, is different from the non-relativistic Dirac equation derived in Ref. [3]. The reason for such a discrepancy is that in Ref. [3] only the singular case $\Omega = 0$, $m \neq 0$ was considered. We shall not consider this singular case here because it has no relativistic counterpart.

It is easily seen that taking in (2.12) ψ in the form of (2.5) we get for the spinor $u(\vec{x}, \tau)$ the usual relativistic Dirac equation

$$\left(\gamma^\mu \partial_\mu + \frac{imc}{\hbar}\right) u = 0, \tag{2.14}$$

where

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial\tau}, \tag{2.15}$$

while taking ψ in the form of (2.8) we get for the spinor $Z(\vec{x}, \tau)$ the Euclidean Dirac equation

$$\left(\vec{\gamma}\cdot\vec{\nabla} + \frac{1}{c}\gamma^5\frac{\partial}{\partial\tau} + \frac{imc}{\hbar}\right) Z = 0. \tag{2.16}$$

This shows that the relativistic wave equations follow from the Galilean equations not only for the case of scalar fields but also for the case of spinor fields. It is the choice of the solutions of the basic Galilean field theory which determines the character of the wave equation!

It is easy to see that the same is also true for interacting fields, provided the particular choice of solutions of the Galilean field equations does not introduce constraints for the fields. An example of the theory where such constraints arise is the Galilean gauge field theory and we shall discuss this case in detail.

The field equations for the Galilean gauge theory are of the form [2]:

$$2\frac{\partial^2 \vec{A}}{\partial t\partial\theta} - \Delta\vec{A} - c^2\frac{\partial^2\vec{A}}{\partial\theta^2} + \operatorname{grad}\left(\operatorname{div}\vec{A} + \frac{\partial V}{\partial\theta}\right) - \operatorname{grad}\left(\frac{\partial W}{\partial t} - c^2\frac{\partial W}{\partial\theta}\right) = g\bar{\psi}\vec{\gamma}\psi, \tag{2.17a}$$

$$\frac{\partial^2 V}{\partial t\partial\theta} - \Delta V - \frac{\partial^2 W}{\partial t^2} + c^2\Delta W + \frac{\partial}{\partial t}\operatorname{div}\vec{A} - c^2\frac{\partial}{\partial\theta}\operatorname{div}\vec{A} = \sqrt{\frac{\Omega}{m}}\, g\bar{\psi}\gamma_5\psi, \tag{2.17b}$$

$$\frac{\partial^2 W}{\partial t\partial\theta} - \Delta W + \frac{\partial}{\partial\theta}\left(\operatorname{div}\vec{A} + \frac{\partial V}{\partial\theta}\right) = \sqrt{\frac{m}{\Omega}}\, g\bar{\psi}(\gamma^0 - \gamma_5)\psi \tag{2.17c}$$

and

$$\left[\vec{\gamma}\cdot(\vec{\nabla} - ig\vec{A}) + \sqrt{\frac{m}{\Omega}}(\gamma^0 - \gamma_5)\left(\frac{\partial}{\partial t} - igV\right) + \sqrt{\frac{\Omega}{m}}\gamma_5\left(\frac{\partial}{\partial\theta} - igW\right) + \frac{i\sqrt{m\Omega}}{\hbar}\right]\psi = 0, \tag{2.18}$$

where c^2 is an arbitrary constant of the dimension of the square of velocity. In the relativistic case we shall identify this constant with the square of the velocity of light.

For the solutions of the field equation (2.18) of the type (2.5) we have the identity

$$\frac{\partial u}{\partial t} = c^2\frac{\partial u}{\partial\theta} \tag{2.19}$$

and in order to preserve this identity also for the gauge covariant derivatives we need the following relation between the gauge fields

$$V\left(\vec{x}, t + \frac{1}{c^2}\theta\right) = c^2 W\left(\vec{x}, t + \frac{1}{c^2}\theta\right). \tag{2.20}$$

With this relation the left-hand side of (2.17b) vanishes, while the right-hand side in general does not. We must therefore include the constraint relation (2.20) in the Lagrangian for the gauge fields by means of a Lagrange multiplier. We shall do this by introducing into the total Lagrangian of the gauge theory a gauge fixing term of the form

$$\lambda(\vec{x}, t, \theta)\,(n_t W + n_\theta V - \vec{n} \cdot \vec{A}) \tag{2.21}$$

where $\lambda(\vec{x}, t, \theta)$ is a Galilean scalar auxiliary field and $(\vec{n}, \vec{n}_t, n_\theta)$ is an arbitrary constant five-vector in the extended Galilean space-time. The expression (2.21) is then a Galilean scalar and the theory with this term in the Lagrangian remains to be Galilean covariant. The presence of (2.21) in the Lagrangian adds to the left-hand sides of (2.17a), (2.17b) and (2.17c) the terms $-\lambda\vec{n}$, λn_t and λn_θ, respectively, and the variation with respect to λ gives the gauge fixing condition

$$n_t W + n_\theta V - \vec{n} \cdot \vec{A} = 0. \tag{2.22}$$

Obviously, for the Galilean gauge theory we may always choose $\vec{n} = n_t = n_\theta = 0$.

Now, choosing $\vec{n} = 0$, $n_t = -c^2$, $n_\theta = 1$ we get the constraint (2.20) and from (2.17b) we get

$$\lambda = -\frac{g}{c}\,\bar{\psi}\gamma_5\psi. \tag{2.23}$$

Substituting this into (2.17c) we get the equation

$$\Box W - \frac{1}{c^2}\frac{\partial}{\partial\tau}\left(\operatorname{div}\vec{A} + \frac{\partial W}{\partial\tau}\right) = cg\bar{\psi}\gamma^0\psi, \tag{2.24}$$

which is the correct relativistic wave equation for the field $W(\vec{x}, \tau)$. The remaining equations (2.17a) and (2.18) also automatically take the relativistic form.

A similar discussion can be made for the Euclidean type of solutions. Here the fields do not depend on t and therefore we need in the gauge covariant derivative the relation

$$V = 0. \tag{2.25}$$

We shall take into account this relation by chosing in (2.22) $\vec{n} = n_t = 0$, $n_\theta = 1$. From (2.17b) and (2.17c) we then get

$$\lambda = \frac{g}{c}\,\bar{\psi}\gamma_0\psi \tag{2.26}$$

and the Euclidean gauge theory is obtained.

3. The interpretation of the results

We shall now interpret our previous results in the modern language of fibre bundles. As it is well-known [4], in order to define a fibre bundle we must indicate three objects: the total space of the bundle P, the base space M and the projection $\pi: P \to M$. For our purpose we shall define the total space P as the five dimensional vector space R_5 with coordinates $(y_1, \ldots y_5)$ previously denoted by $\vec{x}$, t and θ. We assume that each physical elementary event p is represented by a point in P and the field phenomena are described by some fields $\psi_\alpha(y_1, \dots y_5)$, which satisfy the equations of a given Galilean field theory. The space-time coordinates of the event p are represented by a point in the four-dimensional vector space M with coordinates (x^0, x^1, x^2, x^3) which are linear combinations of the coordinates y_k. The projection π is implemented as the linear relation

$$x^\mu = \sum_{k=1}^{5} \alpha^{\mu k} y_k + \alpha^\mu, \qquad (\mu = 0, 1, 2, 3). \tag{3.1}$$

Since the physical coordinates x^μ depend on the chosen reference system the projection π and therefore the coefficients $\alpha^{\mu k}$ and α^μ change with the reference system. We are therefore dealing not with a single fibre bundle (P, π, M) but with a collection of fibre bundles $(P, \pi(O), M)$ with fixed total and base spaces and variable projections, and the notation $\pi(O)$ indicates the dependence of the projections on the observers or reference systems. If the observer O' with coordinates x'^μ is related to the observer O with the coordinates x^μ by a given relativity principle described by a symmetry group of the base space, we must have

$$x'^\mu = L^\mu{}_\nu x^\nu + a^\mu, \tag{3.2}$$

where $L^\mu{}_\nu$ represent the homogenous part of the symmetry group and a^μ describe the translations. Combining (3.1) with (3.2) we see that the coefficients $\alpha'^{\mu k}$ and α'^μ of the projection $\pi(O')$ are related to the coefficients $\alpha^{\mu k}$ and α^μ of the projection $\pi(O)$ by the formulae:

$$\begin{aligned} \alpha'^{\mu k} &= L^\mu{}_\nu \alpha^{\nu k}, \\ \alpha'^\mu &= L^\mu{}_\nu \alpha^\nu + a^\mu \end{aligned} \tag{3.3}$$

and these formulae show that in order to specify the projections $\pi(O)$ it is sufficient to specify the coefficients $\alpha^{\mu k}$ and α^μ in one arbitrarily chosen reference system. The coefficients of the projection $\pi(O)$ are fixed by the choice of some special solutions of the field equations satisfied by the fields $\psi_\alpha(y_1, \dots y_5)$. We assume that these special solutions are of the form

$$\psi_\alpha(y_1, \dots y_5) = e^{i\beta(y_1, \dots y_5)} \varphi_\alpha(x^0, \dots x^3), \tag{3.4}$$

where the phase function β and the wave functions φ_α are chosen in such a way that the resulting equations for φ_α are invariant under the given symmetry group of the particular space-time. The examples considered in the previous Section show that for the Galilean symmetry group we must have

$$\begin{aligned} \beta &= \frac{m y_5}{h}, \\ \alpha^\nu &= 0, \\ \alpha^{ab} &= \delta^{ab}, \qquad (a, b = 1, 2, 3) \\ \alpha^{a4} &= \alpha^{a5} = 0, \\ \alpha^{04} &= 1, \\ \alpha^{05} &= \alpha^{Ca} = 0. \end{aligned} \tag{3.5}$$

For the Lorentz symmetry group we have

$$\begin{aligned}\beta &= 0,\\ \alpha^{\nu} &= 0,\\ \alpha^{ab} &= \delta^{ab},\\ \alpha^{a4} &= \alpha^{a5} = 0,\\ \alpha^{04} &= 1,\\ \alpha^{05} &= \frac{1}{c^2},\\ \alpha^{0a} &= 0,\end{aligned} \tag{3.6}$$

while for the Euclidean group we have

$$\alpha^{04} = 0 \tag{3.7}$$

with the remaining coefficients being the same as for the Lorentz group.

The fact that the combination $y_4 + \frac{1}{c^2} y_5$ is indeed the relativistic time immediately follows from the interpretation of the fifth coordinate of the extended Galilei group as the first relativistic correction to the non-relativistic time [5]. The present paper shows that an infinitesimal indication how to go from non-relativistic space-time to a relativistic one is sufficient for achieving the global passage.

4. *Conclusions*

Concluding our paper we should like to comment on two aspects of presented model of abelian gauge theory. The first one is connected with the classical version of the theory while the second with its possible quantization.

The classical theory discussed has solutions with a definite kind of transformation properties for the space-time coordinates and these properties are not assumed a priori but are determined by the field equations. The theory represents therefore a model which unifies all models of flat space-time used in physics. Each particular space-time may be considered as some four-dimensional subspace of the universal five-dimensional manifold.

The situation may change in the quantum version of our model. If the quantization is understood in the sense of a path integral formulation we must integrate our model over all possible field configurations. In this integration we should also take into account field configurations which do not satisfy the classical field equations. In particular, we may violate the constraint equations (2.20) and/or (2.25) which just choose one possible sector of the classical theory. Since the theory without equations (2.20) or (2.25) may have a different kind of space-time symmetry, we may expect that quantization may lead to new models of space-time with symmetry groups different from the Galilean, Poincaré or Euclidean ones. We enter therefore into the exciting problem of quantum corrections to classical relativity principles. We shall come back to this problem in the future.

APPENDIX

The Galilean Dirac equation

Since the equation (2.12) is quite new, we sketch here its short derivation. According to the general philosophy of Dirac equations we look for the equation of the type

$$\left(\vec{A}\cdot\vec{\nabla}+B\frac{\partial}{\partial t}+c\frac{\partial}{\partial\theta}+D\right)\psi=0, \tag{A.1}$$

where $\vec{A}=(A_1, A_2, A_3)$, B, C and D are matrices to be determined. For each type of the equation (A.1) we should find a differential matrix operator of the type

$$\vec{A}'\cdot\vec{\nabla}+B'\frac{\partial}{\partial t}+C'\frac{\partial}{\partial\theta}+D' \tag{A.2}$$

which being applied to (A.1) will give equation (2.1). It is then easy to see that the corresponding matrices should satisfy the relations

$$\begin{aligned}
A'_iA_j+A'_jA_i &= -2\delta_{ij}I,\\
A'_iB+B'A_i &= 0,\\
A'_iC+C'A_i &= 0,\\
A'_iD+D'A_i &= 0,\\
B'B &= 0,\\
B'C+C'B &= 2I,\\
B'D+D'B &= 0,\\
C'C &= -\frac{\Omega}{m}I,\\
C'D+D'C &= 0,\\
D'D &= \frac{m\Omega}{\hbar^2}I,
\end{aligned} \tag{A.3}$$

which have the following solution in terms of the Dirac matrices

$$A'_i=A_i=\gamma^i,$$

$$B'=B=\sqrt{\frac{m}{\Omega}}(\gamma^0-\gamma_5),$$

$$C' = C = \sqrt{\frac{\Omega}{m}}\,\gamma_5,$$

$$D' = -D = -i\,\frac{\sqrt{m\Omega}}{\hbar}\,I, \tag{A.4}$$

where

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$$

and this leads to equation (2.12).

I am sincerely indebted to Dr A. Horzela for many discussions and especially for suggesting the use of the gauge fixing term.

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