TRANSVERSE MOMENTUM IN $e^+ + e^- \to A+B+X$*

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We review the predictions of Quantum Chromodynamics for the inclusive production of two hadrons in electron-positron annihilation at high energy in the "back-to-back" region. This cross section is sensitive to gluon bremsstrahlung effects. We discuss, in particular, the energy-energy correlation function and the two particle correlation function using higher moments in the particle energies. New results for the higher moments and for higher twist effects in the energy-energy correlation function are reported.

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1. Introduction

In this paper, we discuss the predictions of Quantum Chromodynamics for the process $e^+ + e^- \to$ hadron A + hadron B + anything, where the momenta P_A and P_B of the hadrons are in nearly opposite directions (the nearly "back-to-back" configuration). The physical picture for this process in QCD is roughly as follows. The virtual photon that was produced in the electron-positron annihilation decays into a quark and an antiquark that have large, exactly back-to-back momenta. The quarks are, however, substantially off shell. Each quark therefore begins to emit bremsstrahlung gluons. These gluons can have large momenta directed in roughly the initial quark direction (we will call these collinear gluons), or they can have small momenta (we will call these soft gluons). The quarks can also absorb gluons, and the gluons can combine with each other, or split up into more gluons, or split up into collinear quark-antiquark pairs. One thus obtains two back to back jets of quarks and gluons. Finally, the quark and gluon jets decay into hadron jets, in which are found hadrons A and B.

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The two observed hadrons will not usually be exactly back to back. Each will have some transverse momentum relative to the original quark momentum because of the transverse momentum transfers in the bremsstrahlung process. That is to say, the gluon radiation broadens the transverse momentum distribution of the quarks and gluons, and thus of the observed hadrons. As the total c.m. energy Q of the annihilation is increased, the original quarks are further and further off shell. More gluons are emitted, and the gluons carry more transverse momentum. Thus the width of the hadron transverse momentum distribution should increase as Q is increased.

The ease of emission of bremsstrahlung gluons in QCD is characteristic of gauge theories. In a gauge theory, both collinear and soft gluons are produced with a big amplitude in the high energy limit. One can contrast QCD with the all scalar ϕ^3 theory in $5+1$ dimensions, which, like QCD, is asymptotically free and behaves like the parton model in the high energy limit. In this theory, only collinear particles are produced with a big matrix element at high energy. (One thus finds only one factor of $\ln (Q_T/Q)$ for each order of perturbation theory, instead of the two factors of this logarithm found in QCD). In ϕ^3 theory, the only effect of increasing Q is that the maximum value of transverse momentum P_T allowed by the kinematics increases in proportion to Q. Near this maximum P_T the P_T distribution turns over and goes to zero. But the shape of the P_T distribution in the region $P_T \ll Q$ is independent of Q [1]. That is, in ϕ^3 theory, in contrast to QCD, the transverse momentum distribution does not become broader as Q is increased. It is therefore of great interest to compare the theoretical predictions of P_T broadening in QCD to experiment.

The prospects for experimental tests are encouraging. The recent upgrading of the PETRA accelerator to provide over 40 GeV of collision energy, Q, should allow a wide enough range of Q to test the theory quantitatively, albeit with rather low precision. In the not too distant future, results at $Q \sim 100$ GeV should be available from the Stanford Linear Collider and from LEP, making more stringent quantitative tests possible.

2. *P_T broadening and the two particle inclusive cross section*

What is a good way to look for the P_T broadening effect described above? One rather obvious method is to measure, at different energies, the distribution dN/dP_T^2, where P_T is the component of a hadron momentum transverse to the jet axis of an event. One must, of course, choose an algorithm for determining the jet axis for each event. For instance, the jet axis can be the so called "thrust axis" of the event. This method has a disadvantage, however: it is difficult to incorporate the algorithm for choosing the jet axis into the theoretical description of the process. Of course, if the theoretical description is a Monte Carlo program that generates complete events, there is no problem with choosing the jet axis. However, at present such Monte Carlo programs make use of rather crude approximations to QCD, and the approximations are generally not under complete control. No systematic method is yet known for improving these approximations. Thus we seek a measurable quantity that does not require a complete specification of the final state for its description.

Consider, for this purpose, the two particle inclusive cross section for $e^+ + e^- \rightarrow A+B+X$:

$$d\sigma/dx_A dx_B d\cos\theta. \tag{2.1}$$

Here x_A and x_B are the momentum fractions carried by the two hadrons:

$$x_A = 2E_A/Q, \quad x_B = 2E_B/Q. \tag{2.2}$$

The angle θ is the angle between P_A and $-P_B$. Thus $\theta = 0$ is the back to back direction.

When θ is small, it is related in a simple fashion to the hadron transverse momenta measured relative to any jet axis that makes a small angle with P_A and P_B:

$$\left(\frac{1}{x_A} P_A^T + \frac{1}{x_B} P_B^T\right)^2 = Q^2 \sin^2(\theta/2). \tag{2.3}$$

(The hadron masses have been neglected here.) Thus by measuring θ, one measures a linear combination of hadron transverse momenta without having to know where the jet axis was.

What does QCD predict for the cross section (2.1)? We will discuss the detailed prediction later, but the qualitative features should already be clear. Let us change variables from $\cos(\theta)$ to

$$k_T = Q\sin(\theta/2) \tag{2.4}$$

which, as we have just seen, is a measure of the transverse momenta of the observed hadrons. Consider the k_T^2 distribution

$$\frac{d\sigma/dx_A dx_B dk_T^2}{d\sigma/dx_A dx_B}. \tag{2.5}$$

Because of the transverse momentum transfers caused by increased gluon bremsstrahlung at high Q in QCD, we expect the k_T^2 distribution (2.5) to be broader at high Q than it is at lower Q.

This effect should be visible in the k_T^2 distribution at values of k_T^2 that are small compared to Q^2. It is also possible to look at the average of k_T^2. However, if one looks only at $\langle k_T^2\rangle$, much valuable experimental information is thrown away. Furthermore, the QCD prediction for $\langle k_T^2\rangle$ is dominated by the behavior of the k_T^2 distribution at large values of k_T^2, $k_T^2 \sim Q^2$. But the region $k_T^2 \sim Q^2$ is not the region of interest for seeing the distinctive effects of gluon bremsstrahlung.

3. The energy-energy correlation function

For some purposes is useful to consider the moments of the cross section (2.1):

$$\frac{d\sigma_{AB}(M,N)}{d\cos\theta} = \int_0^1 dx_A x_A^M \int_0^1 dx_B x_B^N \frac{d\sigma(e^+e^- \rightarrow ABX)}{dx_A dx_B d\cos\theta}. \tag{3.1}$$

The $M = 1$, $N = 1$ moment, summed over all hadron flavors A and B and multiplied by a conventional normalization factor 1/4, is of special interest. It is the energy-energy correlation function [2, 3]:

$$\frac{d\Sigma}{d\cos\theta} = \frac{1}{4}\sum_{A,B}\int_0^1 dx_A x_A \int_0^1 dx_B x_B \, \frac{d\sigma(e^+e^- \to ABX)}{dx_A dx_B d\cos\theta}. \tag{3.2}$$

The theory of $d\Sigma/d\cos\theta$ is simpler than that of the differential cross section. In the QCD prediction for the differential cross section, parton decay functions

$$d_{A/a}(x; \mu) \tag{3.3}$$

appear. These decay functions tell the probability that a parton of type a will decay into a hadron of type A carrying a fraction x of the parton's momentum. The decay functions obey the sum rule [4]:

$$\sum_A \int_0^1 dx x d_{A/a}(x; \mu) = 1 \tag{3.4}$$

which says that the sum of the energies of the decay products of the parton equals the energy of the parton. The advantage of the energy-energy correlation function is that, because of the sum rule (3.4), the parton distribution functions drop out of the QCD formula for this quantity. If one works instead with the differential cross section (2.1) or one of the other moments (3.1) of the cross section, then one must insert into the QCD formula the parton decay functions as measured experimentally (in, say, $e^+ + e^- \to A + X$).

We have just seen that there is some advantage in theoretical simplicity at high Q in looking at the energy-energy correlation function, the (1,1) moment of the cross section. However, there is also a problem with the (1,1) moment. Low energy particles in the final state, say those with energies less than some energy $E_0 \sim 1$ GeV, are not reliably described by the theory, which is designed to describe the production of particles that carry a non--zero fraction of the c.m. energy Q. The energy weighting in the energy-energy correlation function suppresses the contribution of low energy particles at large Q: the contribution of the low energy particles at a given angle θ is proportional to E_0/Q. However, when Q is not very large, this suppression is not very effective. For instance, when $Q = 17$ GeV, a simple parton model estimate based on $\langle P_T^2\rangle^{1/2} = 300$ MeV suggests that the energy--energy correlation function at small angles is dominated by final state particles with energies less than 1 GeV.

One can still use QCD to describe the energy-energy correlation function at modest values of Q, but extra care and some reliance on the parton model are necessary. We discuss these issues in Sect. 5.

4. *The QCD formula*

In this section we will consider the formula obtained in QCD for the cross section $d\sigma/dx_A dx_B d\cos\theta$ in the high energy limit, with $0 \leqslant \theta \leqslant \pi/2$ [5].

Let us consider the standard factorization result that one obtains when θ is not small [6]:

$$\frac{1}{\sigma_T}\frac{d\sigma(e^+e^-\to ABX)}{dx_A dx_B d\cos\theta} \sim \sum_a \int_{x_A}^{1} \frac{d\xi_A}{\xi_A} d_{A/a}(\xi_A;Q) \sum_b \int_{x_B}^{1} \frac{d\xi_B}{\xi_B} d_{B/b}(\xi_B;Q)$$

$$\times T_{ab}\left(\frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}, \theta, Q; g(Q), Q\right). \tag{4.1}$$

Here the d's are the parton decay functions that we have met already. The function T (not to be confused with T in the DDT formula) is called the hard scattering function. It describes the cross section for $e^+ + e^-$ going to partons a and b plus anything, but with infrared singularities removed and put into the parton decay functions.

The hard scattering function can be calculated perturbatively. It has the form

$$T \sim \sum_{N=0}^{\infty} T^{(N)} \left[\frac{\alpha_s(Q)}{\pi}\right]^N. \tag{4.2}$$

This expansion is useful when θ is not small, since $\alpha_s(Q)$ is small when Q^2 is large. However, the expansion is not useful when θ is small. The functions $T^{(N)}$ have the small angle behavior

$$T^{(N)} \propto \frac{[\ln(\theta)]^{2N-1}}{\theta^2}. \tag{4.3}$$

Since $\ln(\theta)$ is large when θ is small, the high order terms in Eq. (4.2) are not small compared to the low order terms when θ is small.

Thus, at small θ, one needs a different form of factorization. The first step in obtaining a useful factorization while keeping the good features of the standard factorization in the finite θ region is to write

$$\frac{1}{\sigma_T}\frac{d\sigma(e^+ + e^-\to ABX)}{dx_A dx_B d\cos\theta} \sim \frac{Q^2}{8\pi}\left(\tfrac{1}{2}\sum_j e_j^2\right)^{-1} W + Y, \tag{4.4}$$

where the sum runs over flavors j of quarks and antiquarks, $j = u, \bar{u}, d, \bar{d}, \ldots$. Here W contains the dominant part of the cross section at small θ and Y is a finite θ correction. (We have changed normalization conventions for W and Y from Ref. [5]). The function Y has an ordinary factorized form,

$$Y \sim \sum_a \int_{x_A}^{1} \frac{d\xi_A}{\xi_A} d_{A/a}(\xi_A;Q) \sum_b \int_{x_B}^{1} \frac{d\xi_B}{\xi_B} d_{B/b}(\xi_B;Q)$$

$$\times R_{ab}\left(\frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}, \theta, Q; g(Q), Q\right). \tag{4.5}$$

Here R is just T with the

$$\frac{1}{\sin^2(\theta/2)}(\text{const. or } \ln^N\theta)$$

pieces removed. These pieces are put into W. The function W is a large Q, small θ approximation to the cross section, with "higher twist" terms, which are suppressed by powers of mass/Q or $k_T/Q = \sin(\theta/2)$, thrown away.

It proves very useful to work with the Fourier transform of W [7],

$$\tilde{W}(b, Q, x_A, x_B) = (2\pi)^{-2} \int d^2k_T \exp(ik_T \cdot b) W(k, Q, x_A, x_B), \tag{4.6}$$

where $k_T = Q\sin(\theta/2)$. Now, of course, $\tilde{W}$ has an ordinary factorized form that is useful when Q and $1/b$ are large, with Qb finite. But at order α_s^N, the hard scattering factor in this factorization contains $\ln(Qb)$ to the power $2N$. Thus this factorization is not useful when Qb is large. What is needed now is to write $\tilde{W}$ in a factorized form that is useful when Q is large and, in addition, $Qb \gg 1$.

We will not repeat the derivation [5] of the required result, but will be content to describe it and to point out some of its important features. We write the result in a form that is somewhat more convenient than that used in [5]. This form, along with some discussion of the derivation, can be found in [8] (for the case of the Drell-Yan process instead of electron-positron annihilation).

Notice that a function $\tilde{W}$ of four variables x_A, x_B, b, Q and of the particle types A and B could, in general, have quite a complicated structure. In QCD, however, the form of $\tilde{W}$ is quite simple. One finds that

$$\begin{aligned}\tilde{W}(b; Q; x_A, x_B) \sim \exp\Bigg\{ - \int_{C_1^2/b^2}^{C_2^2Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\Bigg[\ln\left(\frac{C_2^2Q^2}{\bar{\mu}^2}\right) A(g(\bar{\mu}), m_q/\bar{\mu}; C_1) \\ + B(g(\bar{\mu}), m_q/\bar{\mu}; C_1, C_2)\Bigg]\Bigg\} \\ \times \sum_j e_j^2 \tilde{P}_{A/j}(x_A, b; C_1/C_2)\tilde{P}_{B/\bar{j}}(x_B, b; C_1/C_2).\end{aligned} \tag{4.7}$$

Here C_1 and C_2 are constants of order 1 that are inserted in order to allow the optimization of perturbation theory — that is, to allow one to keep high order terms in perturbation theory, which will contain logarithms of C_1 and C_2, reasonably small by adjusting these constants. If the reader wishes, he may just set C_1 and C_2 equal to 1. The functions A and B have conventional perturbative expansions. They may depend on quark masses m_q. When $b \ll 1/\Lambda$ one can neglect the dependence of A and B on light quark masses. Then these functions are simply power series in the coupling $g(\bar{\mu})$ with numerical coefficients:

$$A = A(g(\bar{\mu}); C_1) = \sum_{N=1}^{\infty} A^{(N)}(C_1)\,[\alpha_s(\bar{\mu})/\pi]^N, \tag{4.8}$$

$$B = B(g(\bar{\mu}); C_1, C_2) = \sum_{N=1}^{\infty} B^{(N)}(C_1, C_2)\,[\alpha_s(\bar{\mu})/\pi]^N. \tag{4.9}$$

The functions $\tilde{P}$ may be thought of as being the Fourier transforms of the "intrinsic" P_T distributions of the hadrons produced in the parton decay. There is a further factorization of these functions that is useful when the detected hadron has large transverse momentum (that is, when $b \ll 1/\Lambda$):

$$P_{A/j}(x, b; C_1/C_2) = \sum_a \int_x^1 \frac{d\xi}{\xi} C_{aj}(x/\xi, b; g(C_1/b), C_1/b; C_1/C_2)$$

$$\times d_{A/a}(\xi; C_1/b) + O(\text{mass} \times b), \tag{4.10}$$

where $d(\xi)$ is the parton decay function discussed earlier, evaluated at a renormalization scale $\mu = C_1/b$. The hard scattering function C has a conventional perturbative expansion in powers of $\alpha_s(\mu)$, $\mu = C_1/b$:

$$C_{aj}(\xi/x, b; g(\mu), \mu; C_1/C_2) = \delta(\xi/x - 1)\delta_{aj}$$

$$+ \sum C_{aj}^{(N)}(\xi/x, b, \mu, C_1/C_2) [\alpha_s(\mu)/\pi]^N. \tag{4.11}$$

The first term in this expansion is a delta function, so that

$$\tilde{P}_{A/j}(x, b; C_1/C_2) = d_{A/j}(x; C_1/b) + O(\alpha_s(C_1/b)). \tag{4.12}$$

We now discuss how the factorization (4.7) can be used when Q^2 is large. We consider two different regions of b, $b \ll 1/\Lambda$ and $b \gtrsim 1/\Lambda$.

When $Q \gg \Lambda$ and $b \ll 1/\Lambda$, one can use the perturbative results for the functions A and B. In addition, one can use the perturbative relation between $\tilde{P}$ and d. This leaves the parton decay functions $d(x; C_1/b)$, which cannot be perturbatively evaluated and must be taken from experiment. Notice that there are no large logarithms, ln (Qb), to compromise the usefulness of the low order perturbative results, since each function that is evaluated perturbatively is either a function of the scale $\bar{\mu}$, or a function of b, but never a function of both $\bar{\mu}$ and b.

When $Q \gg \Lambda$ but $b \gtrsim 1/\Lambda$, one cannot use the small-b, perturbative form of $\tilde{W}$. Instead, one has available only the structural information contained in the large-b form of $\tilde{W}$, Eq. (4.7). However, this structural information is quite useful.

There is a convenient trick [5] for making a smooth interpolation between the perturbative, $b \ll 1/\Lambda$, region and the nonperturbative, $b \gtrsim 1/\Lambda$, region, while keeping the valuable structural information contained in the factorization formula (4.7). One defines a function $b_*(b)$ which approximately equals b when b is small, but which stays smaller than some specified value b_{max} when b is large. One chooses the value of b_{max} sufficiently small so that one can trust a perturbative expansion in powers of $\alpha_s(C_1/b_{max})$. We have used the function

$$b_*(b) = b/[1 + b^2/b_{max}^2]^{1/2}. \tag{4.13}$$

Now we use the trivial identity,

$$\tilde{W}_j(b) = \tilde{W}_j(b_*) \frac{\tilde{W}_j(b)}{\tilde{W}_j(b_*)}, \tag{4.14}$$

where $\tilde{W}_j$ is the contribution to $\tilde{W}$ in Eq. (4.7) from quark flavor j. In the first factor, $\tilde{W}_j(b_*)$, one can use a finite order of perturbation theory in the small-b form of W, since $\alpha_s(C_1/b_*)$ is always small. In the second factor, we use Eq. (4.7), which gives the structure of $\tilde{W}_j(b)$ for all b. Eq. (4.7) implies that the second factor has the form

$$\frac{\tilde{W}_j(b;Q;x_A,x_B)}{\tilde{W}_j(b_*;Q;x_A,x_B)} = \exp\{-\ln(Q^2/Q_0^2)f_1(b) - f_{A/j}(x_A,b) - f_{B/\bar{j}}(x_B,b)\}. \quad (4.15)$$

(The parameter Q_0 is inserted here to keep the dimensions consistent. Its value can be chosen arbitrarily.) We know that $\tilde{W}_j(b)/\tilde{W}_j(b_*) \to 1$ as $b \to 1$ as $b \to 0$, since $b_* \to b$ as $b \to 0$. Thus

$$\begin{aligned} &f_1(b) \to 0 \\ &f_{A/j}(x_A, b) \to 0 \quad \text{as} \quad b \to 0. \\ &f_{B/\bar{j}}(x_B, b) \to 0 \end{aligned} \quad (4.16)$$

While we know how the functions f should behave for small b, we can only speculate about how they behave in the non-perturbative regime, $b \gtrsim 1/\Lambda$. Their role is quite similar to that of the parton decay functions $d(x)$ or of the parton distribution functions $f(x)$ that are measured in deeply inelastic lepton scattering. We do not know how to compute these functions in QCD, so we measure them experimentally. Then we use the measured values to make predictions for different experiments or different energies. In the case of the functions $f(b)$, one can measure the functions in $e^+ + e^- \to A + B + X$ at two energies. One then has a prediction at any third energy.

In the case of the energy-energy correlation function, Eq. (4.15) is slightly modified to read

$$\tilde{W}(b, Q)/\tilde{W}(b_*, Q) = \exp(-\ln(Q^2/Q_0^2)f_1(b) - f_2(b)). \quad (4.17)$$

5. *Parton model correction for the energy-energy correlation function*

In this section, we examine the contribution to the energy-energy correlation function from particles that carry low energy. On the basis of this examination, we construct a correction that should be applied to the asymptotic formula described above in order to account approximately for the contribution from low energy particles when Q is not large enough (less than 40 GeV, say).

We will be interested in the contribution to the energy correlation function from events in which one or the other of the observed hadrons has an energy E that is less than some value E_0, so that E is too small to be in the range for which the asymptotic QCD formula was designed. We assume that Q is sufficiently large so that the energy weighted probability that *both* observed hadrons have energy less than E_0 is small enough to be neglected. Thus, we assume that one of the two hadrons has energy larger than E_0.

We will work by comparing the general asymptotic formula with a phenomenological fit to the production of low x hadrons.

The production of low x hadrons at moderate values of Q is fairly well described by the parton model. Let us consider the cross section $d\sigma/dx_A dx_B d\cos\theta$ when x_A is small and x_B is of order 1. We use, following [2], the following distribution for the low energy hadrons

$$dN_A = \frac{d^3p}{E} x_A d_{A/j}(x_A) \frac{1}{\pi\langle P_T^2\rangle} \exp(-P_T^2/\langle P_T^2\rangle). \tag{5.1}$$

Here x_A denotes the energy fraction $2E_A/Q$ and P_T denotes the momentum components transverse to the jet axis, which we take to be along the direction of the high energy hadron B. It is assumed that the hadron distribution function $d_{A/j}(x_A)$ behaves like const/x for small x. The parameter $\langle P_T^2\rangle$ could, in principle, depend on x_A, but we shall assume that it is a constant, e.g. 300 MeV.

We now use this model to construct the cross section $d\sigma/dx_A dx_B d\cos\theta$. We multiply by the cross section $(4\pi\alpha^2/Q^2)e_j^2$ to produce a quark of flavor j, multiply by the decay probability $d_{B/\bar{j}}(x_B)dx_B$ to observe hadron B in the decay of the quark of opposite flavor $\bar{j}$, and sum over flavors j = u, $\bar{\text{u}}$, d, $\bar{\text{d}}$, ... This gives

$$d\sigma = (4\pi\alpha^2/Q^2)\sum e_j^2 d_{B/\bar{j}}(x_B)dx_B$$
$$\frac{d^3p}{E} x_A d_{A/j}(x_A) \frac{1}{\pi\langle P_T^2\rangle} \exp(-P_T^2/\langle P_T^2\rangle).$$

Finally, we replace P_T by $E_A \sin\theta = \frac{1}{2}Qx_A\sin\theta$ and d^3p/E by $\frac{1}{2}\pi Q^2 x_A dx_A d\cos\theta$. This gives the result we need for the cross section in the parton model:

$$d\sigma/dx_A dx_B d\cos\theta = (4\pi\alpha^2/Q^2)\sum e_j^2 d_{A/j}(x_A)d_{B/\bar{j}}(x_B)$$
$$\times (x_A^2 Q^2/2\langle P_T^2\rangle)\exp\{-x_A^2Q^2\sin^2(\theta)/4\langle P_T^2\rangle\}. \tag{5.2}$$

We can use Eq. (5.2) to obtain a parton model estimate for the contribution from low energy particles to the energy correlation function. We multiply by x_A and x_B, integrate over x_A from 0 to $2E_0/Q$ and over x_B from 0 to 1, sum over flavors A and B and multiply by the normalization factor 1/4. This gives the contribution to the energy correlation function from events in which particle A has energy less than E_0. To this we add an equal contribution for the contribution from events in which particle B has low energy. Thus we obtain a parton model estimate for the contribution to the energy correlation function from events in which either one or the other particle has energy less than E_0:

$$\Delta(1/\sigma_T)d\Sigma/d\cos\theta = 2A(0)(E_0/Q)(E_0^2/\langle P_T^2\rangle)f(E_0^2\sin^2\theta/\langle P_T^2\rangle), \tag{5.3}$$

where

$$f(z) = z^{3/2}\int_0^z dy\, y^{1/2}e^{-y}, \tag{5.4}$$

and the constant $A(0)$ is defined as the small x limit of

$$A(x) = \sum_A x d_{A/a}(x; Q^2). \tag{5.5}$$

Here we assume that the parton distributions behave (approximately) like $1/x$ for small x.

We now examine the small x behavior of the asymptotic QCD formula as given in Sect. 4. There are two cases to consider: finite angles θ and small angles θ ($\theta \ll 1$). We consider the small angle case first.

Recall that the cross section, differential in x_A and x_B, is given by

$$\frac{1}{\sigma_T}\frac{d\sigma(e^+ + e^- \to ABX)}{dx_A dx_B d\cos\theta} \sim \frac{Q^2}{8\pi}\left(\tfrac{1}{2}\sum_j e_j^2\right)^{-1} W + Y. \tag{5.6}$$

The Y term is important only at finite angles, so we may ignore it in the present discussion. The W term has the form

$$W(k_T, Q, x_A, x_B) = \int d^2b \exp(-ik_T \cdot b)\tilde{W}(b, Q, x_A, x_B), \tag{5.7}$$

where $k_T = Q\sin(\theta/2)$ and

$$\tilde{W}(b; Q; x_A, x_B) = \sum e_j^2 \tilde{W}_{\text{pert},j}(b_*)$$
$$\times \exp\{-\ln(Q^2/Q_0^2) f_1(b) - f_{A/j}(x_A, b) - f_{B/\bar{j}}(x_B, b)\}, \tag{5.8}$$

with $\tilde{W}_{\text{pert}}$ given by Eqs. (4.7) and (4.10) with low order perturbation theory used for the coefficients A, B, and C. The parameter b_*, Eq. (4.13), is approximately equal to b for small b and is always small compared to $1/\Lambda$.

The behavior of the functions f must be determined from a model or from experiment. The function $\exp\{-f_{A/j}(x_A, b)\}$ may be interpreted as the Fourier transform of the transverse momentum distribution of hadron A in parton j; note, however, from Eq. (5.7) that the variable b is Fourier conjugate to $k_T \approx (Q/2)\theta$ instead of the true transverse momentum of the detected low energy hadron, which is $P_T \approx x(Q/2)\theta$. Thus we propose the following model for $f_{A/j}(x_A, b)$:

$$\exp(-f_{A/j}(x_A, b)) = \exp\{-\langle P_T^2\rangle b^2/4x_A^2\}. \tag{5.9}$$

Notice the factor $1/x^2$ in the exponent.

Consider now what happens to $\tilde{W}(b; Q; x_A, x_B)$ when one of the x's, say x_A, becomes small while Q and x_B are held fixed. We see that, when x is small enough, the behavior of $\tilde{W}$ is dominated by the factor $\exp\{-\langle P_T^2\rangle b^2/4x_A^2\}$ in $\tilde{P}_{A/j}(x_A, b)$. Compared to this factor, the function $\tilde{W}_{\text{pert}}$ in Eq. (5.8) is a slowly varying function (except at very small values of b, $b < 1/Q$, which are not important in the small angle case that we are considering). Our numerical studies suggest that the $\tilde{P}$ factor becomes dominant compared to the Sudakov factor in $\tilde{W}_{\text{pert}}$ when $x < 0.1$ at $Q = 30$ GeV. Therefore, when we integrate over b, we may set $\tilde{W}_{\text{pert}}$ equal to its value at small b, where $b \cong b_*$, that is,

$$\tilde{W}_{\text{pert},j}(b_*) \cong d_{A/j}(x_A; Q) d_{B/j}(x_B; Q) + O[\alpha_s(Q)]. \tag{5.10}$$

(We will consider the $O[\alpha_s]$ corrections below, but neglect them for now.) Thus we have

$$\tilde{W}(b; Q; x_A, x_B) \sim d_{A/j}(x_A) d_{B/\bar{j}}(x_B) \exp\{-\langle P_T^2\rangle b^2/4x_A^2\}.$$

Taking the Fourier transform, Eq. (5.7), inserting the result into Eq. (5.6) and multiplying by σ_T gives

$$d\sigma/dx_A dx_B d\cos\theta \approx (4\pi\alpha^2/Q^2) \sum e_j^2 d_{A/j}(x_A) d_{B/\bar{j}}(x_B)$$
$$\times (x_A^2 Q^2/2\langle P_T^2\rangle) \exp\{-x_A^2 k_T^2/\langle P_T^2\rangle\}, \tag{5.11}$$

where $k_T = Q \sin(\theta/2)$. As long as θ is small, so that $\sin(\theta/2) \approx \frac{1}{2}\sin\theta$, this is just the parton model ansatz we used in Eq. (5.2). Integration over the x's then produces the same contribution from low energy paricles to the energy correlation function that we obtained in the parton model, Eq. (5.3).

We are left with a conclusion that is somewhat surprising. The asymptotic QCD formula that we have been using was derived under the assumption that the momentum fractions x to be considered are *not* very small, but are of order 1 as $Q \to \infty$. Presumably, corrections would be needed in order to "sum up" the $\log x$ factors that occur in perturbation theory and are important at small x when Q is very large [9]. However, we are interested here in values of Q that are not particularly large. We have found that, for moderate values of Q in a reasonable model, the nonperturbative factors in the QCD formula take over from the perturbative Sudakov exponential at values of x, $x \approx 0.1$, that are not so small that one must worry about summing the factors of $\log x$ in the perturbative factor. The nonperturbative factors can, if we choose a suitable model for them, have just the right form to reproduce the parton model result at small angles. Thus we conclude that *no* correction is needed in the QCD formula in order to account, at small angles, for the contribution to the energy correlation function from low energy particles.

At this point, we make a small digression. We use the model (5.9) for the functions $f_{A/j}(x, b)$ to obtain the corresponding behavior at small b of the nonperturbative function $f_2(b)$ in Eq. (4.17) for the energy correlation function. We will find that $f_2(b)$ receives a contribution that behaves like b at small b from the $1/x^2$ singularity in $f_{A/j}(x, b)$. We use the definitions of Sect. 4, neglecting the order α_s contribution to the coefficient C, Eq. (4.11). This gives

$$\exp(-\tfrac{1}{2}f_2(b)) = \int_0^1 dx A(x) \exp(-\langle P_T^2\rangle b^2/4x^2),$$

where $A(x)$ is defined in Eq. (5.5). For small b this is approximately

$$\exp(-\tfrac{1}{2}f_2(b)) \approx \int_0^1 dx A(x) - A(0) \int_0^1 dx\{1 - \exp(-\langle P_T^2\rangle b^2/4x^2)\}$$
$$= 1 - \tfrac{1}{2} A(0)\pi^{1/2}\langle P_T^2\rangle^{1/2} b,$$

where $y = \langle P_T^2\rangle^{1/2} b/2$. Thus we identify

$$f_2(b) \sim A(0)\pi^{1/2}\langle P_T^2\rangle^{1/2} b + O(b^2). \tag{5.12}$$

We now return to the main line of our argument. We consider the contributions to the QCD asymptotic formula for the energy correlation function arising from low energy particles when the angle θ is *not* small. First, we consider the function $\tilde{W}(b)$. Since we are interested in values of $k_T = Q \sin(\theta/2)$ that are of order Q, we need only consider values of b that are of order $1/Q$. In this region of b, $b_* \cong b$ and a simple perturbative expansion of $\tilde{W}_{\rm pert}(b_*)$ may be used. However, there is a significant higher twist contribution because of the $1/x^2$ factor in $f_{A/j}(x, b)$. We therefore retain the nonperturbative factors and make the following approximation:

$$\begin{aligned}\tilde{W} \cong & \sum e_j^2[d_{A/j}(x_A; Q)d_{B/\bar{j}}(x_B; Q) + (\alpha_s(Q)/\pi)\tilde{T}^{[1]}_{\rm sing} * d_A * d_B] \\ & \times \exp\{-\ln(Q^2/Q_0^2)f_1(b)\}\exp\{-\langle P_T^2\rangle b^2/4x_A^2 - \langle P_T^2\rangle b^2/4x_B^2\}. \qquad (5.13)\end{aligned}$$

Here $\tilde{T}^{[1]}_{\rm sign}$ is a certain constant times the first order perturbative contribution to the Fourier transform of the part of T, Eq. (4.1), that is at least as singular as $1/\sin^2(\theta/2)$ as $\theta \to 0$. (Recall that the less singular remainder, R, of T is in the Y term in Eq. (5.6), according to Eq. (4.5)). The *'s indicate convolution of $\tilde{T}^{[1]}_{\rm sing}$ with parton decay functions, as in Eq. (4.1).

Consider first the contribution at zeroth order in α_s, which we obtain by neglecting the $\tilde{T}^{[1]}_{\rm sing}$ term. We cannot neglect the third factor, even though it tends to 1 as b becomes small. This is because we are integrating over x's and there are $1/x^2$ singularities in the exponent, which make this factor more important at small b after integrating over the x's than it was at fixed values of the x's. After performing the x integrals, as in Eq. (5.12), we obtain a factor that behaves like $1 + \text{const} \times [\vec{b}^2]^{1/2}$ at small b. With b of order $1/Q$ at finite angles, the order b term is power suppressed compared to the order α_s term in the first factor — it is "higher twist". Nevertheless, numerical estimates indicate that this power suppressed term is roughly as large as the perturbative term in the interesting energy range $Q \approx 20$ GeV. Thus we retain the contribution from the zero order perturbative term times the third factor in Eq. (5.13).

We shall *assume* that $f_1(b)$ behaves like b^2 at small b, so that the contribution from the second factor when $b \approx 1/Q$ is negligible even at $Q \approx 20$ GeV. Of course, this assumption should be verified by measuring $f_1(b)$ experimentally.

It may be helpful to discuss the contribution from the third factor as a function of angle θ, after Fourier transforming. Because of the nonanalytic behavior at small b, the Fourier transform of this factor has a long tail extending from small angles out to the finite angle region in which we are interested. (The Fourier transform behaves like $[\sin(\theta/2)]^{-3}$.) The contribution from this tail is suppressed compared to the perturbative terms by a power of Q — it is "higher twist" — but it is numerically important at moderate values of Q.

In summary, then, the contribution to $\tilde{W}$ at zeroth order in α_s may be approximated by the quark decay functions d times the third factor in Eq. (5.13).

Now we turn to the contribution to $\tilde{W}$, that is proportional to $\alpha_s(Q)$. Since this contribution is already small, we shall not worry about the higher twist corrections to it: we replace the second and third factors in Eq. (5.13) by 1.

We can now assemble our result for the energy correlation function as given by the QCD formula. There are first of all perturbative contributions proportional to α_s from W and from Y in Eq. (5.6). These terms give the contribution, according to first order QCD perturbation theory, to the energy correlation function from three jet events, in which there is a hard gluon emission from one of the quarks. Such a contribution is, of course, expected, although it is not included in the parton model (5.2) for the cross section.

Next, there is the contribution from the higher twist term, proportional to α_s^0, that we have isolated in $\tilde{W}$:

$$\Delta\tilde{W} = \sum e_j^2 d_{A/j}(x_A; Q) d_{B/\bar{j}}(x_B; Q)$$
$$\times \exp\{-\langle P_T^2\rangle b^2/4x_A^2 - \langle P_T^2\rangle b^2/4x_B^2\}. \tag{5.14}$$

Consider this contribution when one of the x's, say x_A, is small and the other x is finite. Then performing the Fourier transformation gives the contribution to the cross section:

$$\Delta d\sigma/dx_A dx_B d\cos\theta = (4\pi\alpha^2/Q^2)\sum e_j^2 d_{A/j}(x_A) d_{B/\bar{j}}(x_B)$$
$$\times (x_A^2 Q^2/2\langle P_T^2\rangle)\exp\{-x_A^2 Q^2 \sin^2(\theta/2)/\langle P_T^2\rangle\}. \tag{5.15}$$

This order α_s^0 term is to be compared to the two jet parton model expression (5.2). We see that they are almost identical. The only difference is that, because of the small angle approximations made in deriving the QCD formula, the higher twist contribution from the QCD formula has 2 sin ($\theta/2$) replacing sin (θ) in the exponent. This makes no difference in the small angle case that we investigated earlier, but it does make a difference at finite angles.

It is easy to correct the QCD formula for this difference. The contribution to the QCD formula for the energy correlation function from the higher twist term (5.15) integrated over the region in which either one or the other of the detected particles has energy less than a cutoff E_0 is given by Eq. (5.3) with sin (θ) replaced by 2 sin ($\theta/2$). We have merely to subtract this expression from the QCD formula, then add back the correct expression (5.3) as given by the parton model. Therefore we add to the asymptotic QCD formula for the energy correlation function the net higher twist correction, which is important only at finite angles θ,

$$\Delta(1/\sigma_T)d\Sigma/d\cos\theta = 2A(0)(E_0/Q)(E_0^2/\langle P_T^2\rangle)$$
$$\times\{f(E_0^2\sin^2\theta/\langle P_T^2\rangle) - f(E_0^2 4\sin^2(\theta/2)/\langle P_T^2\rangle)\}, \tag{5.16}$$

where

$$f(z) = z^{-3/2}\int_0^z dy\, y^{1/2} e^{-y}, \tag{5.17}$$

and the constant $A(0)$ is defined as the small x limit of

$$A(x) = \sum_A x d_{A/a}(x; Q^2). \tag{5.18}$$

Experimentally [10], $A(0) \approx 4$. The results should not be sensitive to the choice of the cutoff E_0; a reasonable choice might be $E_0 \approx 600$ MeV.

6. *Summary of results for $d\sigma(M, N)/d\cos\theta$*

In this section, we summarize the results discussed in Section 4 for the two particle inclusive cross section. We state the results for the (M, N) moment of the cross section instead of for the cross section as a function of x_A and x_B since the results take their simplest form when presented in this manner. We will summarize the results for the energy-energy correlation function, including the parton model corrections, in the following section.

In this section on the (M, N) moment, we state the values of the perturbative coefficients that have been calculated previously and of three coefficient functions that we have calculated at first order in α_s in order to complete the formula at this order.

The (M, N) moment of the inclusive cross section for $e^+e^- \to A+B+X$ is defined by

$$\frac{d\sigma_{AB}(M, N)}{d\cos\theta} = \int_0^1 dx_A x_A^M \int_0^1 dx_B x_B^N \frac{d\sigma(e^+e^- \to ABX)}{dx_A dx_B d\cos\theta} . \tag{6.1}$$

In this section we can, if we like, broaden the meaning of the notation to allow A and B to denote not just particle types but types of idealized detectors. That is, A can denote a detector that detects only protons, A = *proton*, or A can denote a detector that detects all particles, A = *all*, or A can denote a detector that detects all charged particles, A = *all charged*, et cetera. One goes from the case of A = *particle flavor* to the other cases by performing the appropriate sum over particle flavors and then redefining the nonperturbative functions that appear in the formula.

Our result for the cross section is

$$\begin{aligned}
&\frac{1}{\sigma_T}\frac{d\sigma_{AB}(M, N)}{d\cos\theta} \\
&= \frac{Q^2}{8\pi}\left(\tfrac{1}{2}\sum_j e_j^2\right)^{-1} \int d^2b\, e^{-ik_T\cdot b} \\
&\times \exp\left\{-\int_{C_1^2/b_*^2}^{C_2^2Q^2} \frac{d\bar\mu^2}{\bar\mu^2}\left[\ln\left(\frac{C_2^2Q^2}{\bar\mu^2}\right)A(g(\bar\mu); C_1)+B(g(\bar\mu); C_1, C_2)\right]\right\} \\
&\times \sum_j e_j^2 \sum_a C_{a/j}(M, b_*; g(\mu), \mu; C_1/C_2)d_{A/a}(M; \mu) \\
&\times \sum_b C_{b/\bar{j}}(N, b_*; g(\mu), \mu; C_1/C_2)d_{B/b}(N; \mu) \\
&\times \exp\{-\ln(Q^2/Q_0^2)f_1(b)-f_{A/j}(M, b)-f_{B/\bar{j}}(N, b)\}+Y(M, N; \theta).
\end{aligned} \tag{6.2}$$

We will come to the wide angle term Y in a moment. The sums over j run over quark flavors and antiflavors, j = u, ū, d, d̄, ... The sums over a and b run over all parton types: quark flavors and antiflavors and gluon, g. The parameters C_1 and C_2 should be of order 1, and may be adjusted to improve the convergence of perturbation theory. The transverse momentum k_T is defined by $k_T = Q\sin(\theta/2)$. The adjusted transverse position b_* may be taken as $b_* = b/(1+b^2/b_{max}^2)^{1/2}$, with b_{max} of order 1 GeV^{-1}.

The functions $d_{A/a}(M;\mu)$ are moments of parton decay functions at renormalization scale μ, defined as in [4] using $\overline{MS}$ renormalization. They must be measured experimentally, for instance in $e^+ + e^- \to A + X$; their μ dependence follows from the Altarelli-Parisi equation. The non-perturbative functions $f_1(b)$ and $f_{A/j}(M, b)$ must also be measured experimentally, with the constraint that these functions vanish as $b \to 0$. The function f_1 is the most important of these functions because it gives the Q dependence of the non-perturbative factors in the cross section and is also numerically important at very high energies when $\ln(Q^2)$ is large. Notice that this function is independent of hadron type and of moment number.

The functions A, B, and C have perturbative expansions of the form

$$F(g(\mu),\mu) = \sum_N F^{(N)}(\mu)\left[\frac{\alpha_s(\mu)}{\pi}\right]^N. \tag{6.3}$$

The expansions for A and B start at order α_s. The order α coefficients are [9, 7, 11]

$$A^{(1)} = \tfrac{4}{3}, \tag{6.4}$$

$$B^{(1)} = \tfrac{8}{3}\ln\left[\frac{C_1}{C_2}\tfrac{1}{2}e^{\gamma-3/4}\right], \tag{6.5}$$

where γ is Euler's constant, $\gamma = 0.577\ldots$. An expression has also been given by Kodaira and Trentadue [11] for the second order contribution to A:

$$A^{(2)} = \tfrac{67}{9} - \frac{\pi^2}{3} - \tfrac{10}{27}N_f + \tfrac{8}{3}\left(\frac{33-2N_f}{12}\right)\ln(C_1\tfrac{1}{2}e^\gamma). \tag{6.6}$$

The expansions for C start at order α_s^0:

$$C_{a/j}^{(0)}(N, b_*; \mu; C_1/C_2) = \delta_{aj}. \tag{6.7}$$

The first order contributions to C can easily be calculated by combining the results of [5] with the calculation of [12]:

$$C_{k/j}^{(1)}(N, b_*; \mu; C_1/C_2) = \delta_{kj}\tfrac{4}{3}\int_0^1 dZ Z^N\left\{-\left[\frac{1+Z^2}{1-Z}\right]_+\ln\left(\frac{\mu b_*}{2Z}e^\gamma\right) + \tfrac{1}{2}(1-Z)\right.$$

$$\left. + \delta(1-Z)\left[\frac{\pi^2}{4} - \tfrac{29}{16} - \ln^2\left(\frac{C_1}{2C_2}e^{\gamma-3/4}\right)\right]\right\}$$

$$= \delta_{kj}\tfrac{4}{3}\left\{\left(\sum_{n=1}^{N}\frac{1}{n} + \sum_{n=3}^{N+2}\frac{1}{n}\right)\ln(\tfrac{1}{2}\mu b_* e^\gamma)\right.$$

$$+\left(\sum_{n=1}^{N}\frac{1}{n^2}+\sum_{n=1}^{N+2}\frac{1}{n^2}-\frac{\pi^2}{3}\right)+\frac{1}{2(N+1)(N+2)}$$

$$+\left[\frac{\pi^2}{4}-\tfrac{29}{16}-\ln^2\left(\frac{C_1}{2C_2}e^{\gamma-3/4}\right)\right]\Bigg\}, \tag{6.8}$$

$$C^{(1)}_{g/j}(N, b_*; \mu; C_1/C_2) = \tfrac{4}{3}\int_0^1 dZZ^N\left\{-\frac{Z^2-2Z+2}{Z}\ln\left(\frac{\mu b_*}{2Z}e^{\gamma}\right)+\tfrac{1}{2}Z\right\}$$

$$=\tfrac{4}{3}\left\{-\left(\frac{1}{N+2}-\frac{2}{N+1}+\frac{2}{N}\right)\ln\left(\tfrac{1}{2}\mu b_* e^{\gamma}\right)\right.$$

$$\left.-\left(\frac{1}{(N+2)^2}-\frac{2}{(N+1)^2}+\frac{2}{N^2}\right)+\frac{1}{2(N+2)}\right\}. \tag{6.9}$$

The value of the $\overline{\mathrm{MS}}$ renormalization scale μ in C and d in Eq. (6.2) may be chosen at will (C times d is a renormalization group invariant). In order to keep high order terms in C from being large, we choose $\mu = C_1/b_*$.

We now turn to the large angle term Y in Eq. (6.2). It has a factorized form

$$Y(M, N; \theta) = \sum_{a,b} R_{ab}(M, N, \theta, Q; g(\mu), \mu)\, d_{A/a}(M; \mu) d_{B/b}(N; \mu). \tag{6.10}$$

Here again μ is arbitrary. We choose $\mu = C_2 Q$. The remainder function R has a perturbative expansion beginning at order α_s. The first order coefficient functions $R^{(1)}$ can be calculated simply by taking the well known expressions [13] for the cross section for e^+e^- annihilation into two partons, taking moments of these functions, and subtracting the terms that behave like $1/\sin^2(\theta/2)$ times possible logarithms as $\theta \to 0$. The result is most simply expressed in terms of the functions

$$F(Z; M, N, I) = \frac{1}{[1-Z]^{M+N+1}}\sum_{m=0}^{M}\frac{M!}{m!(M-m)!}(-1)^m\sum_{n=0}^{N}\frac{N!}{n!(N-n)!}(-1)^n$$

$$\times\left\{\theta(m \neq n+I)\frac{1}{n+I-m}[Z^m - Z^{n+I}]+\theta(m = n+I)Z^m\ln\left(\frac{1}{Z}\right)\right\}. \tag{6.11}$$

One finds

$R^{(1)}_{jk}(M, N, \theta, Q; \mu)$

$$= e_j^2\left(\tfrac{1}{2}\sum_{i} e_i^2\right)^{-1}\delta_{kj}\,\tfrac{4}{3}\,\frac{1}{4\sin^2(\theta/2)}\{F(\sin^2(\theta/2); M, N+2, 0)$$

$$+F(\sin^2(\theta/2); M+2, N, 0))\}$$

$$-e_j^2\left(\tfrac{1}{2}\sum_i e_i^2\right)^{-1}\delta_{kj}\,\tfrac{4}{3}\,\frac{1}{4\sin^2(\theta/2)}\left\{2\ln\left(\frac{1}{\sin^2(\theta/2)}\right)\right.$$

$$\left.-\sum_{m=1}^{M}\frac{1}{m}-\sum_{m=1}^{M+2}\frac{1}{m}-\sum_{n=1}^{N}\frac{1}{n}-\sum_{n=1}^{N+2}\frac{1}{n}\right\}. \tag{6.12}$$

$$R_{jg}^{(1)}(M, N, \theta, Q; \mu) = R_{gj}^{(1)}(N, M, \theta, Q; \mu)$$

$$= e_j^2\left(\tfrac{1}{2}\sum_i e_i^2\right)^{-1}\tfrac{4}{3}\,\frac{1}{4\sin^2(\theta/2)\cos^2(\theta/2)}$$

$$\times\ \{F(\sin^2(\theta/2); M+2, N-1, 1)+F(\sin^2(\theta/2); M, N-1, 1)$$

$$-2\cos^2(\theta/2)F(\sin^2(\theta/2); M+1, N, 1)$$

$$+\cos^4(\theta/2)F(\sin^2(\theta/2); M+2, N+1, 1)\}$$

$$-e_j^2\left(\tfrac{1}{2}\sum_i e_i^2\right)^{-1}\tfrac{4}{3}\,\frac{1}{4\sin^2(\theta/2)}\left\{\frac{2}{N}-\frac{2}{N+1}+\frac{1}{N+2}\right\}. \tag{6.13}$$

$$R_{gg}^{(1)} = 0.$$

7. *Summary of results for energy-energy correlation function*

In this section we bring together the results discussed in the previous sections in the case of the energy-energy correlation function defined in Eq. (3.2).

We first of all divide the correlation function into two parts, a QCD part and a parton model correction that is designed to account for the contribution of low energy final state particles. The parton model correction is not important when Q is large or θ is small.

$$\frac{1}{\sigma_T}\frac{d\Sigma}{d\cos\theta} = \left[\frac{1}{\sigma_T}\frac{d\Sigma}{d\cos\theta}\right]_{QCD}$$

$$+2\frac{E_0}{Q}A(0)\frac{E_0^2}{\langle P_T^2\rangle}\left\{f\left(\frac{E_0^2}{\langle P_T^2\rangle}\sin^2\theta\right)-f\left(\frac{E_0^2}{\langle P_T^2\rangle}4\sin^2(\theta/2)\right)\right\} \tag{7.1}$$

$$f(z) = z^{-3/2}\int_0^z dy\, y^{1/2}e^{-y}. \tag{7.2}$$

The QCD contribution is further divided into a small angle (i.e. small P_T) part and a finite angle correction Y:

$$\left[\frac{1}{\sigma_T}\frac{d\Sigma}{d\cos\theta}\right]_{QCD} = \frac{Q^2}{16\pi}\int d^2b\, e^{-ik_T\cdot b}\tilde{W}(b, Q)+Y(b, Q). \tag{7.3}$$

Here $k_T = Q \sin(\theta/2)$. The finite angle correction may be computed from Eqs. (6.12) and (6.13) and is [2, 7]:

$$Y = \tfrac{4}{3}\frac{\alpha_s(C_2Q)}{\pi}\left\{\frac{1}{4x}\left[\left(\frac{3}{y^5}-\frac{4}{y^4}\right)\ln x+\frac{3}{y^4}-\frac{5}{2y^3}-\frac{1}{y^2}\right]\right.$$

$$\left.-\frac{1-2x}{8x}\left[\left(\frac{12}{y^5}-\frac{16}{y^4}+\frac{4}{y^3}\right)\ln x+\frac{12}{y^4}-\frac{10}{y^3}\right]+\frac{1}{4x}\left[\ln x+\tfrac{3}{2}\right]\right\}, \tag{7.4}$$

where $x = \sin^2(\theta/2)$ and $y = 1-x = \cos^2(\theta/2)$. We write

$$\tilde{W}(b, Q) = \tilde{W}(b_*, Q)_{\text{pert}} \exp(-\ln(Q^2/Q_0^2)f_1(b)-f_2(b)), \tag{7.5}$$

where

$$b_* = b+([1-b^2/b_{\max}^2]^{1/2} \tag{7.6}$$

and the functions $f_1(b)$ and $f_2(b)$ are to be fitted to experiment with the constraint that $f_1(0) = f_2(0) = 0$.

The QCD perturbative part is given by

$$W(b_*, Q)_{\text{pert}} = \exp\left\{-\int_{C_1^2/b_*^2}^{C_2^2Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\ln\left(\frac{C_2^2Q^2}{\mu^2}\right)A(g(\bar{\mu}))+B(g(\bar{\mu}))\right]\right\}$$

$$\times[C(g(C_1/b_*))]^2, \tag{7.7}$$

where A and B are as given in Sec. 6 and C is the appropriate sum of first moments of the coefficient C given in Sec. 5:

$$C(g) = 1+\sum_{N=1}^{\infty} C^{(1)}\left[\frac{\alpha_s}{\pi}\right]^N, \tag{7.8}$$

$$C^{(1)} = -\tfrac{13}{12}-\frac{\pi^2}{9}-\tfrac{4}{3}\ln^2\left(\frac{C_1}{C_2}\tfrac{1}{2}e^{\gamma-3/4}\right). \tag{7.9}$$

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