ONE-LOOP COUNTERTERMS DERIVATION IN THE FIRST ORDER QUANTUM GRAVITY FORMALISM

By I. L. Buchbinder and I. L. Shapiro

Pedagogical Institute, Tomsk, USSR

(*Received February 6, 1984*)

One-loop counterterms in the first order quantum gravity formalism are found on the basis of the background field method. The structure of one-loop renormalization is investigated.

PACS numbers: 04.60.+n

1. The present paper is devoted to the investigation of one-loop divergences in the first order formalism quantum gravity. The first order formalism, where the fields $g_{\mu\nu}$, $\Gamma^{\lambda}_{\mu\nu}$ are independent, is equivalent on the classical level to the Einstein gravitation theory [1]. However, in the quantum domain it is an alternative formulation of gravity, because of the shell there is no connection between the fields $g_{\mu\nu}$, $\Gamma^{\lambda}_{\mu\nu}$ necessary for the second order formalism. Gravitation theory in the first order formalism contains before renormalization only one interaction vertex in contrast to the second order formalism, where there are an infinity of them. Therefore, it may turn out that the first order formalism is preferable when making quantum calculations.

The main method for finding one-loop counterterms in the quantum gravitation theory is the background field method [2, 3]. In the framework of this method the divergences structure in the pure gravity [4–7] and in the gravity with matter [8–10] was investigated. In this paper the counterterms of first order quantum gravity formalism are found by means of the background field method.

2. According to the background field method the one-loop contribution to the effective action is given.

$$e^{iJ(\Phi)} = \int d\varphi d\bar{c} dc e^{i[S_2(\Phi,\varphi)+S_{GF}(\Phi,\varphi)+S_{GH}(\Phi,\bar{c},c)]}. \tag{1}$$

Here Φ is the set of classical fields, φ is the set of quantum fields, $\bar{c}$, c are the fields of ghost particles, $S_2(\Phi, \varphi)$ is bilinear over the fields φ part of the expression $S(\Phi, \varphi)-S(\Phi)-S'(\Phi)\varphi$. S_{GF} is the gauge fixing action, S_{GH} is the action of the ghost particles. For the calculation of one-loop S-matrix we must substitute the external field Φ which satisfies the classical equation of motion in the expression for $J(\Phi)$.

The gravity action in the first order formalism has the form

$$S(g, \Gamma) = -\frac{1}{\kappa^2}\int d^n x\sqrt{-g}(\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\nu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\tau_{\lambda\tau} - \Gamma^\tau_{\mu\lambda}\Gamma^\lambda_{\tau\nu})g^{\mu\nu}. \tag{2}$$

The decomposition of the fields g, Γ into the classical fields $\Phi \equiv (g, \Gamma)$ and quantum fields $\varphi \equiv (h, \gamma)$ is performed in the following way: $g^{\mu\nu} \to g^{\mu\nu} + \kappa\tilde{h}^{\mu\nu}$, $\Gamma^\lambda_{\mu\nu} \to \Gamma^\lambda_{\mu\nu} + \kappa\gamma^\lambda_{\mu\nu}$. Further it is convenient to pass over to the fields $h^{\mu\nu} = (\delta^{\mu\nu}{}_{,\alpha\beta} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta})\tilde{h}^{\alpha\beta}$ carrying out the local change of variables into the functional integral (1). The expression $S_2(g, \Gamma; h, \gamma)$ is written in the form:

$$S_2(g, \Gamma; h, \gamma) = -\tfrac{1}{2}\int d^n x\sqrt{-g}\,\{h^{\mu\nu}\Lambda_{11\mu\nu,\alpha\beta}h^{\alpha\beta}$$

$$+2h^{\mu\nu}\Lambda_{12\mu\nu,\lambda}{}^{\alpha\beta}\gamma^\lambda_{\alpha\beta} + \gamma^\lambda_{\alpha\beta}\Lambda_{22\lambda}{}^{\alpha\beta}{}_{,\tau}{}^{\mu\nu}\gamma^\tau_{\mu\nu}\},$$

$$\Lambda_{11} = \frac{1}{n-2}g_{\alpha\beta}R_{\mu\nu}(\Gamma) + \frac{1}{n-2}g_{\mu\nu}R_{\alpha\beta}(\Gamma)$$

$$+\tfrac{1}{2}\left(\delta_{\mu\nu,\alpha\beta} - \frac{1}{n-2}g_{\mu\nu}g_{\alpha\beta}\right)g^{\lambda\sigma}R_{\lambda\sigma}(\Gamma),$$

$$\Lambda_{12} = -\delta^{\alpha\beta}{}_{\mu\nu}\nabla_\lambda + \delta^{\alpha\beta}{}_{\lambda(\mu}\nabla_{\nu)} + 2A^{(\alpha}_{\lambda(\mu}\delta^{\beta)}_{\nu)} - A^{(\alpha}_{\mu\nu}\delta^{\beta)}_\lambda - A^\sigma_{\lambda\sigma}\delta^{\alpha\beta}_{\mu\nu},$$

$$\Lambda_{22} = \delta^{\mu\nu,\alpha}_{\lambda}\delta^\beta_\tau + \delta^{\mu\nu,\beta}_{\lambda}\delta^\alpha_\tau - g^{\mu\nu}\delta^{\alpha\beta}_{\lambda\tau} - g^{\alpha\beta}\delta^{\mu\nu}{}_{\lambda\tau},$$

$$A^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\mu\nu}(g). \tag{3}$$

Here $\Gamma^\alpha{}_{\mu\nu}(g)$ is the Christoffel symbol which is the basis for constructing the covariant derivatives ∇_α, $R_{\mu\nu}(\Gamma)$ is the Ricci tensor built only from $\Gamma^\alpha_{\beta\gamma}$. Note that $A^\alpha_{\mu\nu}$ is the tensor. Let us choose the gauge condition so that the terms S_{GF}, S_{GH} may have the form

$$S_{\mathrm{GF}} = -\tfrac{1}{2}\int d^n x\sqrt{-g}\,h^{\alpha\beta}g_{\nu\beta}\nabla_\alpha\nabla_\mu h^{\mu\nu},$$

$$S_{\mathrm{GH}} = \int d^n x\sqrt{-g}\,\bar{c}^\mu(g_{\mu\nu}\Box + R_{\mu\nu})c^\nu. \tag{4}$$

As the action $S_2(g, \Gamma; h, \gamma)$ is bilinear over the fields and the coefficient Λ_{22} (3) is local and does not contain derivatives, in the functional integral (1) we may perform integration over the fields γ. After the integration we shall get

$$e^{iJ(g,\Gamma)} = \int dh d\bar{c} dc\, e^{i\{\bar{S}_2(g,\Gamma;h) + S_{\mathrm{GH}}(g,\bar{c},c)\}}. \tag{5}$$

Here

$$\bar{S}_2(g, \Gamma; h) = -\tfrac{1}{2}\int d^n x\sqrt{-g}\,h^{\mu\nu}\left\{\tfrac{1}{2}\left(\delta_{\mu\nu,\alpha\beta} - \frac{1}{n-2}g_{\mu\nu}g_{\alpha\beta}\right)\cdot\Box\right.$$

$$+2\left[A^\lambda_{\alpha\mu}g_{\lambda\nu}\delta^\sigma_\beta + A^\lambda_{\alpha\lambda}g_{\nu\beta}\delta^\sigma_\mu + \left(\delta_{\alpha\beta,\nu\lambda} - \frac{1}{n-2}g_{\alpha\beta}g_{\nu\lambda}\right)A^\lambda_{\mu\tau}g^{\tau\sigma}\right]\nabla_\sigma$$

$$+\frac{2}{n-2}\,g_{\alpha\beta}R_{\mu\nu}(\Gamma)+\tfrac{1}{2}\,g^{\lambda\sigma}R_{\lambda\sigma}(\Gamma)\delta_{\mu\nu,\alpha\beta}-\frac{1}{2(n-2)}\,g_{\mu\nu}g_{\alpha\beta}g^{\lambda\sigma}R_{\lambda\sigma}(\Gamma)$$

$$-g_{\nu\beta}R_{\mu\alpha}+R_{\mu\alpha\nu\beta}+\left(\delta_{\alpha\beta,\nu\lambda}-\frac{1}{n-2}\,g_{\alpha\beta}g_{\nu\lambda}\right)\nabla^{\tau}A^{\lambda}_{\mu\tau}-g_{\nu\beta}\nabla_{\lambda}A^{\lambda}_{\mu\alpha}$$

$$-g_{\lambda\beta}\nabla_{\nu}A^{\lambda}_{\mu\alpha}+g_{\nu\beta}\nabla_{\mu}A^{\lambda}_{\alpha\lambda}-\tfrac{1}{2}\left(\delta_{\mu\nu,\alpha\beta}-\frac{1}{n-2}\,g_{\mu\nu}g_{\alpha\beta}\right)\nabla^{\lambda}A^{\tau}_{\lambda\tau}$$

$$+(2A^{\varrho}_{\alpha\tau}\delta^{\sigma}_{\beta}-A^{\varrho}_{\alpha\beta}\delta^{\sigma}_{\tau}-A^{\lambda}_{\tau\lambda}\delta^{\varrho\sigma}_{\alpha\beta})\Lambda^{-1\tau}_{2\,2\varrho\sigma,}{}^{\gamma}_{\delta\omega}(2A^{\delta}_{\mu\gamma}\delta^{\omega}_{\nu}-A^{\delta}_{\mu\nu}\delta^{\omega}_{\gamma}-A^{\kappa}_{\gamma\kappa}\partial^{\delta\omega,}_{\mu\nu})\Big\}\,h^{\alpha\beta}. \qquad (6)$$

The structure of expression (6) is suitable for using the general Fradkin-Vilkovisky algorithm [11, 10] finding divergences $J(g,\Gamma)$. We shall give only the final result

$$J_{\rm div}(g,\Gamma)=-\frac{L^2}{32\pi^2}\int d^4x\sqrt{-g}\,\{6R-8R_{\mu\nu}(\Gamma)g^{\mu\nu}+A^{\tau}_{\mu\nu}(4\delta^{\alpha\beta,}{}_{\tau\lambda}g^{\mu\nu}$$

$$-\delta^{\beta}_{\tau}\delta^{\mu\nu,\alpha}{}_{\lambda}-g_{\lambda\tau}g^{\mu\nu}g_{\alpha\beta}+5g_{\lambda\tau}\delta^{\mu\nu,\alpha\beta}+12\delta^{\nu}_{\tau}\delta^{\alpha\beta,\mu}{}_{\lambda}A^{\lambda}_{\alpha\beta}\}$$

$$-\frac{\ln L^2/\mu^2}{32\pi^2}\int d^4x\sqrt{-g}\,\{-\tfrac{53}{5}\,R_{\mu\nu}R^{\mu\nu}+\tfrac{127}{60}\,R^2-5g^{\mu\nu}g^{\alpha\beta}R_{\mu\nu}(\Gamma)R_{\alpha\beta}(\Gamma)$$

$$+\tfrac{14}{3}\,Rg^{\mu\nu}R_{\mu\nu}(\Gamma)+8R^{\mu\nu}\nabla_{\lambda}A^{\lambda}_{\mu\nu}-\tfrac{13}{3}\,R\nabla^{\lambda}A^{\alpha}_{\lambda\alpha}+(\nabla_{\varrho}A^{\tau}_{\mu\nu})$$

$$\times(-g^{\mu\nu}\delta^{\varrho}_{\tau}\delta^{\sigma}_{\lambda}g^{\alpha\beta}-\tfrac{5}{6}\,\delta^{\mu\nu,\alpha\beta}\delta^{\varrho}_{\tau}\delta^{\sigma}_{\lambda}-\tfrac{7}{3}\,\delta^{\mu\nu,\alpha}{}_{\tau}g^{\varrho\sigma}\delta^{\beta}_{\lambda}+\tfrac{2}{3}\,g^{\alpha\beta}g^{\varrho\sigma}\delta^{\mu\nu}_{\tau\lambda}$$

$$-\tfrac{5}{6}\,g^{\varrho\sigma}\delta^{\nu}_{\lambda}\delta^{\alpha\beta,\mu}{}_{\tau}-\tfrac{1}{6}\,g_{\lambda\tau}g^{\varrho\sigma}\delta^{\mu\nu,\alpha\beta}+\tfrac{4}{3}\,\delta^{\varrho}_{\tau}\delta^{\mu\nu,\gamma\sigma}\delta^{\alpha\beta}{}_{\lambda\gamma}$$

$$+\tfrac{2}{3}\,\delta^{\mu\nu,\sigma}{}_{\tau}\delta^{\alpha\beta,\varrho}{}_{\lambda}+\tfrac{1}{6}\,\delta^{\mu\nu,\varrho}{}_{\lambda}\delta^{\alpha\beta,\sigma}{}_{\tau}+\tfrac{1}{3}\,\delta^{\sigma}_{\lambda}\delta^{\alpha\beta,\gamma}{}_{\tau}\delta^{\mu\nu,\varrho}{}_{\gamma}-\tfrac{1}{3}\,\delta^{\varrho}_{\tau}g^{\alpha\beta}\delta^{\mu\nu,\sigma}{}_{\lambda}$$

$$+\tfrac{2}{3}\,\delta^{\mu\nu,\sigma}{}_{\lambda}\delta^{\alpha\beta,\varrho}{}_{\tau}-\tfrac{1}{3}\,\delta^{\sigma}_{\tau}\delta^{\alpha\beta,\gamma}{}_{\lambda}\delta^{\mu\nu,\varrho}{}_{\gamma}-\tfrac{1}{3}\,\delta^{\sigma}_{\tau}\delta^{\mu\nu,\gamma}{}_{\lambda}\delta^{\alpha\beta,\varrho}{}_{\gamma})\nabla_{\sigma}A^{\lambda}_{\alpha\beta}$$

$$+R_{\mu\nu}(\Gamma)\,(F_{1\lambda,}^{\mu\nu,\alpha\beta,}{}_{\tau}\nabla^{\lambda}A^{\tau}_{\alpha\beta}+F_{2}^{\mu\nu,\alpha\beta,\varrho\sigma}{}_{\lambda\ \tau}A^{\lambda}_{\alpha\beta}A^{\tau}_{\varrho\sigma})$$

$$+F_{3}^{\mu\nu,\gamma,\alpha\beta,\varrho\sigma}{}_{\omega\ \lambda\ \tau}(\nabla_{\gamma}A^{\omega}_{\mu\nu})A^{\lambda}_{\alpha\beta}A^{\tau}_{\varrho\sigma}+F_{4\gamma}^{\mu\nu,\alpha\beta,\varrho\sigma,\delta\xi}{}_{\lambda\ \tau\ \kappa}A^{\gamma}_{\mu\nu}A^{\lambda}_{\alpha\beta}A^{\tau}_{\varrho\sigma}A^{\kappa}_{\delta\xi}$$

$$+F_{5}^{\mu\nu,\alpha\beta,\varrho\sigma}{}_{\lambda\ \tau}R_{\mu\nu}A^{\lambda}_{\alpha\beta}A^{\tau}_{\varrho\sigma}\}. \qquad (7)$$

Here L^2 is the regularization parameter [11, 10]. The transition to generally used dimensional regularization is carried out in the following way: $L^2\to 0$, $\ln L^2/\mu^2\to\dfrac{2}{4-n}$; F_1, F_2, F_3, F_4, F_5 are expressions depending only upon $g_{\mu\nu}$, the explicit form being not written. The action of one-loop counterterms differs from $J_{\rm div}(g,\Gamma)$ in the sign. Note that expression (7) does not contain all the invariants which may be constructed from the fields g, Γ taking into account dimensions and general covariants.

3. One-loop divergences of S-matrix are obtained from the expression $J_{\rm div}(g,\Gamma)$ (7) when the fields g, Γ satisfy classical equations of motion $R_{\mu\nu}(\Gamma)=0$, $A^{\alpha}_{\beta\gamma}=0$. Then it is evident that the one-loop S-matrix is finite, similar to the second order formalism [4].

The expression for the counterterms (7) shows that though before the renormalization the action was polynomial over the fields $\hat{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$, $\Gamma^{\alpha}_{\beta\gamma}$ and had only one interaction vertex, after the renormalization it becomes essentially non-linear.

Let us rewrite the expression from the counterterms (7) in the form

$$S_c = \frac{L^2}{16\pi^2}\kappa^2 S(g,\Gamma) + \frac{L^2}{32\pi^2}\int d^4x\sqrt{-g}\,A^{\tau}_{\mu\nu}K^{\mu\nu}_{\tau}(g,\Gamma)$$
$$-\frac{\ln L^2/\mu^2}{32\pi^2}\int d^4x\sqrt{-g}\,\{R_{\mu\nu}(\Gamma)M^{\mu\nu}(g,\Gamma)+A^{\tau}_{\mu\nu}N^{\mu\nu}_{\tau}(g,\Gamma)\}. \tag{8}$$

The explicit form of K, M, N can be easily found from the comparison of equalities (8) and (7). After some transformations one may show that the action $S(g,\Gamma)+S_c(g,\Gamma)$ is obtained from the classical action $S(g,\Gamma)$ (2) with the help of the renormalization of the gravitational field.

$$g^{\mu\nu} = g^{\mu\nu}_{R} + \frac{\kappa^2_R \ln L^2/\mu^2}{32\pi^2}M^{\mu\nu}(g_R,\Gamma_R)$$

$$\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{R\mu\nu} - \frac{\kappa^2_R L^2}{32\pi^2}\Lambda^{-1\alpha\ \beta}_{22\mu\nu,\lambda\sigma}K^{\lambda\sigma}_{\beta}(g_R,\Gamma_R) + \frac{\kappa^2_R\ln L^2/\mu^2}{32\pi^2}\Lambda^{-1\alpha\beta}_{22\mu\nu,\lambda\sigma}N^{\lambda\sigma}_{\beta}(g_R,\Gamma_R) \tag{9}$$

and the renormalization of the gravity constant

$$\kappa^2 = \kappa^2_R\Big/1+\frac{L^2\kappa^2_R}{16\pi^2}\,. \tag{10}$$

Equality (9) shows that the renormalization of gravitational wave functions is a non--linear reparametrisation. This agrees with Voronov and Tyutin's general results [12] that renormalization in quantum gravity contains (generally speaking, non-linear) reparametrisation of the wave function. From expression (10) follows a confirmation of the Fradkin-Vilkovisky hypothesis about the possible absence of the zero-charge problem in the gravity theory [18].

Note that a different variant of the renormalization is possible when the gravitational constant is not renormalized and the whole renormalization reduces only to the reparametrisation of the wave functions. In this variant one should exclude equality (10) and add the terms $-\frac{L^2}{16\pi^2}\kappa^2 g^{\mu\nu}_R$ to the expression $g^{\mu\nu}$ (9).

It is interesting to consider the structure of the counterterms (7) in the sector of the fields g (the classical equation of motion for the fields Γ is performed: $A^{\alpha}_{\beta\gamma}=0$) and in the sector of the fields g (the equation of motion for the fields g is performed: $R_{\mu\nu}(\Gamma)=0$). In the first case we shall obtain the counterterms

$$S_c = -\frac{L^2}{16\pi^2}\int d^4x\sqrt{-g}\,R - \frac{\ln L^2/\mu^2}{32\pi^2}\int d^4x\sqrt{-g}\,\{\tfrac{53}{3}R_{\mu\nu}R^{\mu\nu}+\tfrac{107}{60}R^2\}. \tag{11}$$

The difference of coefficients at $R_{\mu\nu}R^{\mu\nu}$ and R^2 from those of paper [4] is accounted for by a different parametrisation of the quantum field. The counterterms in the sector of the field Γ are not polynomial and depend upon the fields g. From this it follows that for the renormalization of the Green functions of operators depending only upon the fields Γ (for example, the curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$) it is necessary to include the terms depending upon the fields g in the renormalized action.

The authors are grateful to G. A. Vilkovisky, I. V. Tyutin, E. S. Fradkin for numerous discussions.

REFERENCES

[1] Ch. W. Misner, K. S. Torn, J. A. Wheeler, *Gravitation*, W. H. Freeman and Company 1973.
[2] B. S. De Witt, *Phys. Rev.* **162**, 1195 (1967).
[3] R. Kallosh, *Nucl. Phys.* **B78**, 293 (1974).
[4] G. t'Hoopt, M. Veltman, *Ann. Inst. H. Poincare* **20**, 69 (1974).
[5] E. S. Fradkin, G. A. Vilkovisky, *Phys. Lett.* **73B**, 209 (1978).
[6] R. E. Kallosh, O. V. Tarasov, I. V. Tyutin, *Nucl. Phys.* **B137**, 145 (1978).
[7] S. M. Christensen, M. J. Duff, *Nucl. Phys.* **B170**, 480 (1980).
[8] S. Deser, P. van Nieuwenhuizen, *Phys. Rev.* **10D**, 401, 411 (1974).
[9] E. Sezgin, P. van Nieuwenhuizen, *Phys. Rev.* **22D**, 301 (1980).
[10] A. O. Barvinsky, G. A. Vilkovisky, *Nucl. Phys.* **B191**, 237 (1981).
[11] E. S. Fradkin, G. A. Vilkovisky, Inst. Theor. Phys. Bern. Preprint, 1976.
[12] B. L. Voronov, I. V. Tyutin, *Sov. J. Nucl. Phys.* **33**, 1710 (1981) (in Russian).