ON SOLUTIONS OF THE DIRAC EQUATION IN THE FIELD OF DIRAC'S ELECTRIC MONOPOLE

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The electromagnetic potential of Dirac's electric monopole is treated as an external field in the Dirac equation. The resulting mathematical problems are described and partially solved. It is shown that when the absolute value of the electric charge is equal to the absolute value of Dirac's magnetic charge, the Dirac equation has no solutions having the Floquet property, very natural in the context.

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1. *Introduction*

In the previous paper [1] we derived an electromagnetic field which has a finite electric charge and simultaneously is a quantized magnetic flux. In this paper we wish to perform the analogue of the Aharonov-Bohm analysis for the magnetic flux in question. As it is well known, there is a special relationship between the so called Aharonov-Bohm effect and the Dirac charge quantization condition: the cross-section for the Aharonov-Bohm effect vanishes exactly when the magnetic flux is a multiple of the Dirac value $2\pi/e$. In our analysis we shall assume from the very beginning that the magnetic flux has the Dirac value; one would expect then, by analogy with the Aharonov-Bohm effect, that such a flux is completely unobservable i.e. it does not scatter point charges. We shall see that a result of this kind can really be proved but on the assumption that a certain differential equation has physically acceptable solutions. The assumption is not obvious; moreover, we shall see that for certain values of charge the differential equation in question does not have physically acceptable solutions.

2. *The equation to be solved*

The potential derived in [1] has the form

$$A_\mu = -\frac{2\pi}{e}\,\Theta\left[\operatorname{tg}\left(eQ\ln\frac{\varrho}{\varrho_0} - \tfrac{1}{2}\varphi\right)\right]\partial_\mu\Theta(x^0-x^3). \tag{1}$$

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Here Θ is the Heaviside step function, e is the elementary charge, Q is the total charge of the field; ϱ and φ are cylindrical coordinates such that

$$x^1 = \varrho \cos\varphi, \quad x^2 = \varrho \sin\varphi.$$

The field changes quickly in time so that to perform the Aharonov-Bohm analysis one has to use the Dirac equation.

Omitting an elementary algebra we shall simply state that the Dirac equation with (1) as an external field is reducible to a single equation for one unknown function ψ:

$$i\frac{\partial\psi}{\partial\tau} + 2\pi\Theta\left[\operatorname{tg}\left(eQ\ln\frac{\varrho}{\varrho_0} - \tfrac{1}{2}\varphi\right)\right]\delta(\tau)\psi$$
$$= -\tfrac{1}{2}\left(\frac{\partial^2\psi}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial\psi}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2\psi}{\partial\varphi^2}\right); \tag{2}$$

τ is a coordinate proportional to $x^0 - x^3$.

It is a simple matter to see that Eq. (2) can have only trivial solution $\psi = 0$: to multiply $\delta(\tau)$ by ψ one has to assume that $\psi(\tau, \varrho, \varphi)$ is continuous for $\tau = 0$; integrating then Eq. (2) over an infinitesimal segment $-\varepsilon \leqslant \tau \leqslant \varepsilon$ one has $i[\psi(\varepsilon, \varrho, \varphi) - \psi(-\varepsilon, \varrho, \varphi)] + 2\pi\Theta\psi(0, \varrho, \varphi) = 0$ which is a contradiction unless $\psi(0, \varrho, \varphi) = 0$; but $\psi(0, \varrho, \varphi) = 0$ and Eq. (2) imply together that $\psi(\tau, \varrho, \varphi) = 0$ for all τ.

This difficulty was investigated by Dr Wojciech Karaś in his Ph. D. Thesis [2], from which the above analysis is taken. Dr Karaś finds that one can define generalized solutions of first order equations like Eq. (2), if one approximates the δ-function by a less singular function, solves the equation and takes the limit in the solution.[1]

In accordance with Dr Karaś' prescription we replace $\delta(\tau)$ in Eq. (2) by

$$\delta_\varepsilon(\tau) = \begin{cases} 0 & \text{for} \quad \tau < 0, \\ 1/\varepsilon & \text{for} \quad 0 \leqslant \tau \leqslant \varepsilon, \\ 0 & \text{for} \quad \tau > \varepsilon. \end{cases}$$

Eq. (2) becomes the two-dimensional Schrödinger equation in which the static potential

$$V = -\frac{2\pi}{\varepsilon}\Theta\left[\operatorname{tg}\left(eQ\ln\frac{\varrho}{\varrho_0} - \tfrac{1}{2}\varphi\right)\right] \tag{3}$$

is turned on for $\tau = 0$ and turned off for $\tau = \varepsilon$ and it is obvious, in principle, how to handle the problem.

There remain technical difficulties, however, and we shall presently solve the first one. The potential (3) is discontinuous and the magnitude of discontinuity is very large. It is well known [3] that for discontinuous potentials there are qualitative differences

[1] It should be noted that this prescription, apparently obvious, does not work for higher order equations for which a similar contradiction occurs. Thus, there is no simple way to define generalized solutions for the equation $\psi''(\tau) + \lambda\delta'(\tau)\psi(\tau) = 0$, $\lambda = \text{const}$.

between the classical and the quantum mechanical motion. The difference relevant for the solution of our problem consists in the following: in the quantum mechanics a large discontinuity of potential acts as an impenetrable barrier both in the repulsive case and in the attractive case while in the classical mechanics only a large repulsive discontinuity is impenetrable. Therefore the problem of motion in the potential (3) can be simplified as follows.

A particle moves freely in two separate domains, the domain in which

$$\Theta\left[\operatorname{tg}\left(eQ\ln\frac{\varrho}{\varrho_0}-\tfrac{1}{2}\varphi\right)\right]=0$$

and the domain in which

$$\Theta\left[\operatorname{tg}\left(eQ\ln\frac{\varrho}{\varrho_0}-\tfrac{1}{2}\varphi\right)\right]=1.$$

At the boundary of each domain we impose the Dirichlet boundary condition $\psi = 0$ corresponding to complete impenetrability. Denote by $\psi_n^{(0)}$ a complete set of stationary states in the first domain and by $\psi_n^{(1)}$ a complete set of stationary states in the second domain, $E_n^{(0)}$ and $E_n^{(1)}$ being the corresponding energies. (The notation does not imply, of course, that the energy spectrum is discrete). The time evolution of the wave function for $0 \leqslant \tau \leqslant \varepsilon$ is

$$\psi(\tau)=\sum_n c_n^{(0)}\psi_n^{(0)}e^{-iE_n^{(0)}\tau}+\sum_n c_n^{(1)}\psi_n^{(1)}e^{-i\left(E_n^{(1)}-\frac{2\pi}{\varepsilon}\right)\tau},$$

$c_n^{(0)}$ and $c_n^{(1)}$ being constants. It is seen that for $\tau = \varepsilon$ the wave function reproduces exactly the value it would have for the vanishing potential, which means that the force which acted for $0 \leqslant \tau \leqslant \varepsilon$ left no observable changes.

This is exactly what one would expect by analogy with the Aharonov-Bohm effect. However, our argument was based on the assumption that there are physically acceptable stationary states: this assumption is investigated in the next section.

3. *The problem of stationary states*

We have to investigate the following problem: a nonrelativistic particle moves freely in the part of Euclidean plane bounded by two logarithmic spirals

$$\varrho=\varrho_0\exp\frac{\varphi}{2eQ}\quad\text{and}\quad\varrho=\varrho_0\exp\frac{\varphi+\pi}{2eQ};$$

at the boundary it is elastically reflected It is useful to investigate first the classical motion.

The logarithmic spiral meets all its radii at the same angle [4]; using this basic property one can easily show that apart from the obvious energy integral there is another integral of motion

$$I=Et-eQJ-\tfrac{1}{2}m\varrho\dot{\varrho}.$$

Here E is the kinetic energy, t is time, J is the angular momentum and m is the mass. The same integral exists also in the more general case when the potential is constant on both sides of the logarithmic spiral and has a finite discontinuity across the spiral so that the particle's trajectory can be reflected or refracted. The integral I is not a Noether integral so that its existence is rather unexpected. One feels that the existence of the second integral of motion should play a role in the quantum case. The integral I is not directly applicable because it does not commute with the energy: in the quantum mechanics

$$E = -\frac{1}{2m}\left(\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\varphi^2}\right),$$

$$I = Et+ieQ\frac{\partial}{\partial\varphi}+\frac{i}{2}\left(\varrho\frac{\partial}{\partial\varrho}+1\right)$$

and $[E, I] = iE$. Let us take, however, the identity [5]

$$e^A Be^{-A} = B+\frac{1}{1!}[A, B]+\frac{1}{2!}[A,[A, B]]+\dots$$

and put $B = E$, $A = cI$, where c is a constant. Then

$$e^{cI}E = Ee^{cI}e^{-ic}$$

and, putting $c = 2\pi n$, n being an integer, we find that the operator $\exp(2\pi nI)$ does commute with the energy.

To apply the operator $\exp(2\pi nI)$ to stationary states we note that if

$$[A, B] = cB, \quad c = \text{const},$$

then

$$\exp(A+B) = \exp A\cdot\exp\frac{1}{c}(1-e^{-c})B. \tag{4}$$

I suppose this identity is known although I cannot indicate a reference; when c is so small that its square can be neglected, the identity becomes the Baker-Hausdorff identity. Take

$$A = 2\pi n\left[ieQ\frac{\partial}{\partial\varphi}+\frac{i}{2}\left(\varrho\frac{\partial}{\partial\varrho}+1\right)\right],$$

$$B = 2\pi nEt$$

so that

$$2\pi nI = A+B, \quad [A, B] = -2\pi niB.$$

Applying the identity (4) we have

$$\exp 2\pi n I = \exp 2\pi n\left[ieQ\frac{\partial}{\partial\varphi}+\frac{i}{2}\left(\varrho\frac{\partial}{\partial\varrho}+1\right)\right].$$

Guided by the classical analogy we pose the eigenvalue problem

$$-\frac{1}{2m}\left(\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\varphi^2}\right)\psi = E\psi, \tag{5}$$

$$\exp 2\pi ni\left(eQ\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\varrho\frac{\partial}{\partial\varrho}\right)\psi = \lambda_n\psi, \tag{6}$$

where E and λ_n are eigenvalues.

It will be convenient to change coordinates in such a way that the operator in the exponent is transformed into a simple differentiation operator. To this end we put

$$\varrho = \varrho_0 e^{2eQ\xi+\zeta},$$

$$\varphi = -\xi+2eQ\zeta,$$

where ξ and ζ are the new coordinates and ϱ_0 is a constant introduced for dimensional reasons; it may be put equal to ϱ_0 in the equation of the logarithmic spiral. Then

$$\frac{\partial\psi}{\partial\zeta} = \varrho\frac{\partial\psi}{\partial\varrho}+2eQ\frac{\partial\psi}{\partial\varphi}$$

and Eq. (6) becomes

$$\psi(\xi,\zeta+\pi ni) = \lambda_n\psi(\xi,\zeta),$$

which means that the stationary state is a function which is, up to a factor, periodic in ζ with the imaginary period $i\pi$. Solutions of this kind are known in the theory of ordinary differential equations as the Floquet solutions [6]. Guided by analogy with the Floquet theory we put

$$\psi(\xi,\zeta) = \sum_{n=0}^{\infty}\psi_n(\xi)e^{(2n+\nu)\zeta}, \tag{7}$$

where ν is the characteristic exponent to be determined. The Helmholtz equation (5) in the new coordinates is

$$e^{-2(2eQ\xi+\zeta)}\left(\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\zeta^2}\right)+[1+(2eQ)^2]\varrho_0 2mE\psi = 0;$$

the boundary condition is

$$\psi = 0 \quad \text{for} \quad \xi = 0 \quad \text{and} \quad \xi = \frac{\pi}{1+(2eQ)^2}.$$

Making a translation in the ζ-direction (such a translation does not change the boundary) we can reduce the Helmholtz equation to the form

$$e^{-2(2eQ\xi+\zeta)}\left(\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\zeta^2}\right)+[1+(2eQ)^2]\psi = 0. \tag{8}$$

Note that in Eq. (8) all dimensional parameters have disappeared; this reflects the fact that the logarithmic spiral contains no dimensional parameters with which the de Broglie wave length could be compared.

Putting the series (7) into the Helmholtz equation (8) we obtain

$$\psi_0''(\xi)+v^2\psi_0(\xi) = 0, \tag{9}$$

$$\psi_n''(\xi)+(2n+v)^2\psi_n(\xi)+[1+(2eQ)^2]e^{4eQ\xi}\psi_{n-1}(\xi) = 0. \tag{10}$$

These equations and the boundary conditions

$$\psi_n(\xi) = 0 \quad \text{for} \quad \xi = 0 \quad \text{and} \quad \xi = \frac{\pi}{1+(2eQ)^2}, \tag{11}$$
$$n = 0, 1, 2, \ldots,$$

should determine the sequence $\psi_0(\xi)$, $\psi_1(\xi)$, $\psi_2(\xi)$, ... In particular, the characteristic exponent is determined as

$$v = m[1+(2eQ)^2], \qquad m = 1, 2, \ldots .$$

We choose $v > 0$ to make the solution regular at the origin; for each natural number m we obtain a formal solution. Thus the integral $\exp(2\pi nI)$ is seen to play the role of the angular momentum, namely it splits the energy degenerate problem into states which are not degenerate anymore.

The convergence of the formal series determined by Eqs (7), (9), (10) and (11) seems to be a difficult question; we hope to be able to comment upon it in a separate paper. In this paper we wish to indicate only that for certain values of charge the series in question is not even formally constructible.

The first term is

$$\psi_0(\xi) = \sin(v\xi)$$

and, of course, can be always determined. The second term is (I put $\beta = 2eQ$ to save space)

$$\psi_1(\xi) = \frac{1}{4[m(v+2)+1]}\left\{e^{2\beta\xi}[\beta m\cos v\xi-(m+1)\sin v\xi]\right.$$

$$\left.-\beta m\cos(v+2)\xi+\beta m\,\frac{\cos\dfrac{2\pi}{1+\beta^2}-\exp\dfrac{2\pi\beta}{1+\beta^2}}{\sin\dfrac{2\pi}{1+\beta^2}}\sin(v+2)\xi\right\}.$$

We see that we have to assume that $\beta^2 \neq 1$; for $\beta^2 = 1$ the formal Floquet solution does not exist.

The value $\beta^2 = 1$ is physically remarkable; for $\beta^2 \neq 1$

$$e^2 Q^2 \neq \tfrac{1}{4}.$$

Since, from the Dirac relation which obviously holds also in our case,

$$e^2 g^2 = \tfrac{1}{4},$$

we conclude that

$$Q^2 \neq g^2.$$

Thus the present theory, which is simply the Aharonov-Bohm theory for the magnetic flux in question, excludes the equality of electric and magnetic charges and thus excludes the exact symmetry between electricity and magnetism which was originally the motivation to introduce magnetic charges.

Investigating higher terms one sees that all rational values of β^2 have to be excluded[2]; for irrational β^2 the formal series (7) *is* constructible. The question of its convergence is, of course, open and seems to be a difficult one.

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[2] Except $\beta^2 = 0$, of course; the case $\beta^2 = 0$ is solvable by means of the Bessel functions.