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NON-LINEAR FOULING TRANSFORMATIONS FOR HARMONIC OSCILLATOR

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The examples of nonlinear fouling transformations for two-dimensional harmonic oscillator are given.

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Recently, several papers [1–7] have been devoted to the problem of equivalent Lagrangians of dynamical systems. It is the well-known fact that the classical equations of motion do not uniquely determine their Lagrangians. We know the Lagrangians such that the corresponding Euler-Lagrange equations are not identical, but result in the same set of solutions. These Lagrangians are called solution-equivalent (s-equivalent) [1].

Our aim is to show the relationship between solution-equivalent Lagrangians for two-dimensional harmonic oscillator and the noncanonical transformations in the phase space which preserve the Hamilton canonical equations of motion.

For two s-equivalent Lagrangians L and $\bar{L}$, we have the following relationship between Euler-Lagrange equations [8, 9]:

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}_i} - \frac{\partial \bar{L}}{\partial q_i} = \sum_{k=1}^{n} \Lambda_{ik}(q, \dot{q}, t)\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}\right), \quad i = 1, 2, \dots n. \tag{1}$$

Hojman and Harleston [1] calculated immediately from the Euler-Lagrange equations the form of matrix Λ:

$$\Lambda_{ik} = \sum_{j=1}^{n}\left(\frac{\partial^2 \bar{L}}{\partial \dot{q}_i \partial \dot{q}_j}\right)\left(\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_k}\right)^{-1}, \tag{2}$$

and proved that traces of all integer powers of Λ are constants of the motion.

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Let us denote by H and $\overline{H}$ the Hamiltonians obtained via Legendre transformation from the s-equivalent Lagrangians L and $\overline{L}$. Currie and Saletan [8] for $n = 1$ and Santilli [7] for $n > 1$ showed that the transformation connecting $H(q, p)$ and $\overline{H}(\overline{q}, \overline{p})$ must be the fouling transformation [10]:

$$\overline{q}_i = q_i, \tag{3a}$$

$$\overline{p}_i = \overline{p}_i(q, p). \tag{3b}$$

If we calculate canonical equations for the H and $\overline{H}$ and eliminate p_i and $\overline{p}_i$ from them, we obtain the second-order differential equations for q_i and $\overline{q}_i$, respectively. These equations will have the same solutions. Therefore we can obtain for $\overline{H}$:

$$\frac{\partial \overline{H}}{\partial q_i} = \sum_{j=1}^{n} \left[\frac{\partial \overline{p}_i}{\partial p_j} \frac{\partial H}{\partial q_j} + \left(\frac{\partial \overline{p}_j}{\partial q_i} - \frac{\partial \overline{p}_i}{\partial q_j} \right) \frac{\partial H}{\partial p_j} \right], \tag{4}$$

$$\frac{\partial \overline{H}}{\partial p_i} = \sum_{j=1}^{n} \frac{\partial \overline{p}_j}{\partial p_i} \frac{\partial H}{\partial p_j}. \tag{4}$$

Gelman and Saletan [10] demonstrated the existence of linear fouling transformations for the two-dimensional oscillator. In this paper we present more general transformations.

We have the standard Lagrangian L

$$L = \tfrac{1}{2} \sum_{i=1}^{2} (\dot{q}_i^2 - q_i^2), \tag{6}$$

and two s-equivalent Lagrangians given by Hojman and Harleston [1], namely

$$\overline{L}^{(1)} = \tfrac{1}{24} (\dot{q}_1^4 + \dot{q}_2^4) + \tfrac{1}{4} \dot{q}_1^2 \dot{q}_2^2 + \tfrac{1}{4} (q_1^2 + q_2^2)(\dot{q}_1^2 + \dot{q}_2^2)$$
$$+ q_1 q_1 \dot{q}_1 \dot{q}_2 - \tfrac{3}{4} q_1^2 q_2^2 - \tfrac{1}{8} (q_1^4 + q_2^4), \tag{7}$$

$$\overline{L}^{(2)} = \tfrac{1}{6} \dot{q}_1 \dot{q}_2 (\dot{q}_1^2 + \dot{q}_2^2) + \tfrac{1}{2} \dot{q}_1 \dot{q}_2 (q_1^2 + q_2^2) + \tfrac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2) q_1 q_2 - \tfrac{1}{2} q_1 q_2 (q_1^2 + q_2^2). \tag{8}$$

The expressions for $\overline{H}^{(1)}$ and $\overline{H}^{(2)}$ are, respectively

$$\overline{H}^{(1)} = \tfrac{1}{8} (p_1^4 + p_2^4 + q_1^4 + q_2^4) + \tfrac{3}{4} (q_1^2 q_2^2 + p_1^2 p_2^2) + \tfrac{1}{4} (q_1^2 + q_2^2)(p_1^2 + p_2^2) + q_1 q_2 p_1 p_2, \tag{9}$$

$$\overline{H}^{(2)} = \tfrac{1}{2} (q_1 q_2 + p_1 p_2)(p_1^2 + p_2^2 + q_1^2 + q_2^2). \tag{10}$$

By inserting (9) into (4) and (5), we obtain the system of equations for $\overline{p}_1^{(1)}$, $\overline{p}_2^{(1)}$. By solving them, we get the explicit expression for the fouling transformation:

$$\overline{p}_1^{(1)} = \tfrac{1}{6} p_1^3 + \tfrac{1}{2} p_1 p_2^2 + \tfrac{1}{2} (q_1^2 + q_2^2) p_1 + q_1 q_2 p_2, \tag{11}$$

$$\overline{p}_2^{(1)} = \tfrac{1}{6} p_2^3 + \tfrac{1}{2} p_1^2 p_2 + \tfrac{1}{2} (q_1^2 + q_2^2) p_2 + q_1 q_2 p_1. \tag{12}$$

By the same way for $\bar{H}^{(2)}$, we obtain

$$\bar{p}_1^{(2)} = \frac{1}{6}p_2^3+\frac{1}{2}(p_1^2+q_1^2+q_2^2)p_2+p_1q_1q_2, \tag{13}$$

$$\bar{p}_2^{(2)} = \frac{1}{6}p_1^3+\frac{1}{2}(p_2^2+q_1^2+q_2^2)p_1+p_2q_1q_2. \tag{14}$$

The formulas (11), (12) and (13), (14) agree with those calculated from the Lagrangians $\bar{L}^{(1)}$, $\bar{L}^{(2)}$:

$$\bar{p}_i^{(a)} = \frac{\partial \bar{L}^{(a)}}{\partial \dot{q}_i}, \quad a = 1, 2. \tag{15}$$

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