EXACT BRANS-DICKE-BIANCHI TYPE-VI$_h$ SOLUTIONS

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(*Received May 29, 1984; revised version received July 17, 1984*)

We investigate the Brans-Dicke-Bianchi type-VI$_h$ ($n^\alpha_\alpha = 0, h \neq 0, -1/9$) field equations. We present the general vacuum solutions as well as the general stiff matter solution. In addition we derive some special BDT-dust solutions and some special perfect fluid solutions with $1 \leqslant \gamma < 2$.

PACS numbers: 04.50.+h, 98,80.Dr

1. Introduction

In the present paper we consider the Bianchi type-VI$_h$ ($n^\beta_\beta = 0$, $h \neq 0, -1/9$) space-time in the Brans-Dicke theory of gravitation (BDT). The Bianchi type-VI$_h$ metric is given by

$$ds^2 = -dt^2 + R_1^2 dx^2 + \exp(-2q(1+k)x)R_2^2 dy^2 + \exp(-2q(1-k)x)R_3^2 dz^2, \qquad (1)$$

where $k^2 = -1/h$ and $q = \text{const}$. Without loss of generality we may put $q = 1$. (Note that the symbol x has been omitted in the last term in the paper by Collins (1971)). This "diagonal" metric belongs to Class B ($n^\beta_\beta = 0$) models in the Bianchi classification of spatially homogeneous space-times (Ellis and MacCallum, 1969) and includes the axisymmetric type-VI$_0$ models of Class A. The BDT-Bianchi type-VI$_0$ model has been already discussed by us (Lorenz-Petzold, 1984c). In general we have $h \leqslant 0$ (including VI$_{-1}$ = III). We use the notation of our paper (Lorenz-Petzold, 1984a).

2. Field equations

The BDT perfect fluid field equations to be considered are

$$R_{\mu\nu} = [T_{\mu\nu} - ((1+\omega)/(3+2\omega))g_{\mu\nu}T]\phi^{-1} + \omega\phi^{-2}\phi_{,\mu}\phi_{,\nu} + \phi^{-1}\phi_{,\mu;\nu}, \qquad (2)$$

$$\Box\phi = (1/(3+2\omega))T, \qquad (3)$$

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where $R_{\mu\nu}$ denotes the Ricci tensor, $T_{\mu\nu}$ the energy-momentum tensor, T its trace, $g_{\mu\nu}$ the metric tensor, $\Box$ the wave operator, $\phi = \phi(t)$ the BDT-scalar field and ω the BDT-coupling constant. A subscript , or ; after the unknown variable denotes partial or covariant differentiation. The energy momentum-tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{4}$$

where u_μ is the velocity four-vector. An observer comoving with the fluid is assumed to have $u_\mu = \delta_0^\mu$, i.e. we are considering only non-tilted models. The perfect fluid matter is characterized by the equation of state

$$p = (\gamma - 1)\varepsilon, \quad 1 \leqslant \gamma \leqslant 2, \tag{5}$$

where ε and p are, respectively, the density and pressure of the fluid.

From the conservation law $T^\mu_{\nu;\mu} = 0$ we obtain

$$\varepsilon = MR^{-3\gamma}, \quad R^3 = R_1 R_2 R_3, \tag{6}$$

where M = const. and $R_i = R_i(t)$ are the cosmic scale-functions. The corresponding BDT-Bianchi type-VI_h ($h \neq 0, -1/9$) field equations to be solved are

$$\dot{H}_1 + 3HH_1 + H_1(\ln\phi)^{\cdot} - 2(1+k^2)/R_1^2 = \varepsilon[1+\omega(2-\gamma)]/(3+2\omega)\phi, \tag{7a}$$

$$\dot{H}_2 + 3HH_2 + H_2(\ln\phi)^{\cdot} - 2(1+k)/R_1^2 = \varepsilon[1+\omega(2-\gamma)]/(3+2\omega)\phi, \tag{7b}$$

$$\dot{H}_3 + 3HH_3 + H_3(\ln\phi)^{\cdot} - 2(1-k)/R_1^2 = \varepsilon[1+\omega(2-\gamma)]/(3+2\omega)\phi, \tag{7c}$$

$$2H_1 = (1+k)H_2 + (1-k)H_3, \tag{7d}$$

$$H_1H_2 + H_1H_3 + H_2H_3 + 3H(\ln\phi)^{\cdot} - (3+k^2)/R_1^2 - (\omega/2)(\ln\phi)^{\cdot 2} = \varepsilon/\phi, \tag{7e}$$

$$(R^3\dot{\phi})^{\cdot} = \varepsilon R^3(4-3\gamma)/(3+2\omega), \tag{7f}$$

where $H_i = \dot{R}_i/R_i$ are the Hubble-parameters, $3H = \Sigma H_i$ and $(\,)^{\cdot} = d/dt$.

By setting $g = R_2R_3\phi$ and $dt = R_1 d\eta$ the field equations can be decoupled to give

$$g'' - 4g = M(2-\gamma)R_1^{2-\gamma}(g/\phi)^{1-\gamma}, \tag{8a}$$

$$y' + y(\ln g)' = 2(1+k) + MR_1^{2-\gamma}\phi^{\gamma-1}g^{-\gamma}[1+\omega(2-\gamma)]/(3+2\omega), \tag{8b}$$

$$\begin{aligned}&\tfrac{1}{2}(1+k)H_2^2 + \tfrac{1}{2}(1-k)H_3^2 + 2H_2H_3 + [\tfrac{1}{2}(3+k)H_2 + \tfrac{1}{2}(3-k)H_3](\ln\phi)' \\ &\quad - (3+k^2) - (\omega/2)(\ln\phi)'^2 = MR_1^{2-\gamma}\phi^{\gamma-1}g^{-\gamma},\end{aligned} \tag{8c}$$

$$(g(\ln\phi)')' = MR_1^{2-\gamma}(g/\phi)^{1-\gamma}(4-3\gamma)/(3+2\omega), \tag{8d}$$

$$R_1^2 = R_2^{(1+k)}R_3^{(1-k)}, \tag{8e}$$

where $y = (\ln R_2)'$ and $(\,)' = d/d\eta$. It follows that Eq. (8a) can be solved simultaneous in the vacuum case ($M = 0$) as well as in the stiff matter case ($\gamma = 2$). Furthermore, Eq.

(8a) is independent of ω! By setting ϕ = const. and $\omega \to \infty$ we thus rediscover the GRT--Bianchi type-VI_h solutions. After solving Eq. (8a) for $g = g(\eta)$ we obtain from Eq. (8b) $R_2 = R_2(\eta)$. Introducing the new variable τ by $g d\tau = \phi d\eta$ we obtain $\phi = \phi(\tau)$ from Eq. (8d) (this procedure is necessary only in the stiff matter case) which may be reexpressed in terms of the variable η via the knowledge of $g = g(\eta)$. It is now an easy matter of calculation to estimate the function $R_3 = g/R_2\phi$ from which we obtain R_1 with the aid of Eq. (8e). All solutions must satisfy the constraint equation (8c).

3. *Exact solutions*

We now present our new solutions. The BDT-Bianchi type-VI_h vacuum solutions are given by

(*i*)

$$\begin{aligned} R_1^2 &= (\sinh 2\eta)^{(1+k^2)}(\tanh \eta)^{mk-n(1-k)/2}, \\ R_2^2 &= (\sinh 2\eta)^{(1+k)}(\tanh \eta)^{m}, \\ R_3^2 &= (\sinh 2\eta)^{(1-k)}(\tanh \eta)^{-(n+m)}, \\ \phi &= \phi_0(\tanh \eta)^{n/2}, \\ 6+2k^2-2m^2&+n((2+\omega)n+2m) = 0, \end{aligned} \tag{9a}$$

(*ii*)

$$\begin{aligned} R_1^2 &= (\cosh 2\eta)^{(1-k^2)} \exp((m/2k-n(1-k)/2) \arcsin(\tanh 2\eta)), \\ R_2^2 &= (\cosh 2\eta)^{(1+k)} \exp(m \arcsin(\tanh 2\eta)), \\ R_3^2 &= (\cosh 2\eta)^{(1-k)} \exp(-(n+m) \arcsin(\tanh 2\eta)), \\ \phi &= \phi_0 \exp((n/2) \arcsin(\tanh 2\eta)), \\ 6+2k^2&+2m^2+n((2+\omega)n+2m) = 0, \end{aligned} \tag{9b}$$

(*iii*)

$$\begin{aligned} R_1^2 &= \exp(2(1+k^2)\eta-(mk-n(1-k)/2) \exp(-2\eta)), \\ R_2^2 &= \exp(2(1+k)\eta-m \exp(-2\eta)), \\ R_3^2 &= \exp(2(1-k)\eta+(n+m) \exp(-2\eta)), \\ \phi &= \phi_0 \exp((-n/2) \exp(-2\eta)), \\ 2m^2&+n((2+\omega)n+2m) = 0, \end{aligned} \tag{9c}$$

where m, n = const.

By setting $n = 0$, i.e. $\phi = \text{const.}$, our solution (9a) reduces to the GRT-Bianchi type-VI_h vacuum solution first given by Ellis and MacCallum (1969) (see also MacCallum (1971) and Kramer et al. (1980)). If $k = 0$, $n \neq 0$, we obtain from (9a)–(9c) the BDT--Bianchi type-V vacuum solutions discussed by us recently (Lorenz-Petzold, 1984a, 1984b). The GRT-limit $k = n = 0$ of (9a) is nothing but the Joseph (1966) solution. However, our solution (9b) has no GRT-limit. This is in contrast to the case (9c), from which we obtain the GRT-limit ($n = m = 0$) first given by Collins (1971, 1977), Ruban (1977b) (see also Ruban et al. (1981)) and Belinskii et al. (1982) in a somewhat different form. This special case was noted earlier by Lifshitz and Khalatnikov (1963a, 1963b), without reference to homogeneous models (see also Evans (1974, 1978), Siklos (1978, 1980, 1981a, 1981b) and the recent paper by Wainwright (1983)).

The case $k = m = n = 0$ gives the "open" FRW ($k = -1$) vacuum solutions (Lorenz--Petzold, 1984d), the particular case $k^2 = 1$ corresponds to the Bianchi type-III model. By replacing the hyperbolic functions by trigonometric one obtains the BDT-Kantowski--Sachs (Kantowski, 1966; Kantowski and Sachs, 1966) vacuum solutions.

We now turn to the stiff matter case ($\gamma = 2$). The corresponding solutions are given by

(*i*)

$$R_1^2 = (\sinh 2\eta)^{(1+k^2)}(\tanh \eta)^{kn/m}\phi^{-1},$$

$$R_2^2 = (\sinh 2\eta)^{(1+k)}(\tanh \eta)^{n/m}\phi^{-1},$$

$$R_3^2 = (\sinh 2\eta)^{(1-k)}(\tanh \eta)^{-n/m}\phi^{-1},$$

where

$$\phi = \phi_1(\ln(\tanh \eta))^{-2}, \quad \Delta = 0,$$

$$\phi = \phi_0(\cos(\ln(\tanh \eta)^{\sqrt{\Delta}/2m})^{-2} \quad \Delta > 0,$$

$$\phi = \phi_0(\cosh(\ln(\tanh \eta)^{-\sqrt{-\Delta}/2m})^{-2}, \quad \Delta < 0,$$

$$m^2(3+k^2)-n^2+\Delta(3+2\omega) = 0, \tag{10a}$$

(*ii*)

$$R_1^2 = (\cosh 2\eta)^{(1+k^2)} \exp[(nk/m) \arctan(\sinh 2\eta)]\phi^{-1},$$

$$R_2^2 = (\cosh 2\eta)^{(1+k)} \exp[(n/m) \arctan(\sinh 2\eta)]\phi^{-1},$$

$$R_3^2 = (\cosh 2\eta)^{(1-k)} \exp[-(n/m) \arctan(\sinh 2\eta)]\phi^{-1},$$

where

$$\phi = \phi_0[\cos((\sqrt{\Delta}/2m) \arctan(\sinh 2\eta))]^{-2}, \quad \Delta > 0,$$

$$\phi = \phi_0[\cosh(-\sqrt{-\Delta}/2m) \arctan(\sinh 2\eta))]^{-2}, \quad \Delta < 0,$$

$$m^2(3+k^2)+n^2-\Delta(3+2\omega) = 0, \tag{10b}$$

(*iii*)

$$R_1^2 = \exp[2(1+k^2)\eta - (nk/m)\exp(-2\eta)]\phi^{-1},$$

$$R_2^2 = \exp[2(1+k)\eta - (n/m)\exp(-2\eta)]\phi^{-1},$$

$$R_3^2 = \exp[2(1-k)\eta + (n/m)\exp(-2\eta)]\phi^{-1},$$

where

$$\phi = \phi_2 \exp(-4\eta), \quad \Delta = 0,$$

$$\phi = \phi_0[\cos((\sqrt{\Delta}/2m)\exp(-2\eta))]^{-2}, \quad \Delta > 0,$$

$$\phi = \phi_0[\cosh((\sqrt{-\Delta}/2m)\exp(-2\eta))]^{-2}, \quad \Delta < 0,$$

$$n^2 - \Delta(3+2\omega)s = 0, \quad s = 1 \ (\Delta > 0), \quad s = -1 \ (\Delta < 0). \tag{10c}$$

In addition we have the relations

$$\Delta = -((3+2\omega)b^2 + 4Mc)/(3+2\omega), \quad \phi_0 = -(3+2\omega)\Delta/4M,$$
$$\phi_1 = -4(3+2\omega)/M, \quad \phi_2 = -(3+2\omega)M, \tag{10d}$$

where b, c are constants arising from the solution of Eq. (8d).

The GRT-limit ϕ = const. of (10a) was first given by Ruban (1978) (see also Collins (1971) and Wainwright et al. (1979) and Kramer et al. (1980)). The GRT-Bianchi type-V ($k = 0$) stiff matter solution was found by Ruban (1977a, 1977b) (note that these papers are not quoted in Ref. Kramer et al. (1980)) and later by Maartens and Nel (1978) (in the locally rotationally symmetric (LRS)-case with tilt; note however that their solution is valid only if $b = 0$ (see also Kramer et al. (1980), (12.16)). Our solution (10b) has no analogy in the GRT. However, the GRT-limit of (10c) is well defined and seems to be new! By taking $k = 0$ our solutions (10a)–(10c) reduce to the BDT-Bianchi type-V stiff matter solutions obtained by us recently (Lorenz-Petzold, 1984a). The $k^2 = 1$ solutions are of Bianchi type-III and can be extended to the BDT-Kantowski-Sachs space-times.

It follows from Eq. (8a) that it is impossible (at least with the aid of the method described in this paper) to obtain the general radiation ($\gamma = 4/3$) BDT-Bianchi type-VI$_h$ ($h \neq -1$) solution (see also Collins (1971)). This is in contrast to the Bianchi type-V model where such a general solution is allowed (Lorenz-Petzold, 1984a, 1984e). The corresponding GRT-Bianchi type-V radiation solution was first given by Ruban (1977a, 1977b). The $k^2 = 1$ Bianchi type-III (Kantowski-Sachs) radiation solution is due to Kantowski (1966) (see also Collins (1971), MacCallum (1971)).

We now consider the dust case ($\gamma = 1$). By setting

$$y = g'' - 4g \tag{11}$$

we obtain from equations (7a) and (8a)

$$(\ln y)'' + (\ln y)'(\ln g)' = ((1+\omega)/(3+2\omega))(y/g) + 2(1+k^2). \tag{12}$$

After solving this complicated fourth-order differential equation the most general Bianchi type-VI_h dust solution would arise, both in the GRT ($\omega \to \infty$) as well as in the BDT. The only known general GRT-dust solutions of Eq. (11) and (12) are the Bianchi type-III solutions ($k^2 = 1$):

$$y = MR_1 = 2aM(\cosh 2\eta - 1),$$

$$g = (Ma/4)\,[2\sinh 2\eta - 2(\cosh 2\eta - 1) + \sinh 2\eta], \tag{13}$$

where a = const. (see Vajk and Eltgroth (1970) and Lorenz (1983)) and the Bianchi type-V ($k = 0$) solution first given by Heckmann and Schücking (1958, 1962) (see also Ellis and MacCallum (1969)) in terms of elliptic functions. There is also the special GRT-Bianchi type-III dust solution found by Ftaclas and Cohen (1978) (see also Lorenz (1983)):

$$y = 8b \exp 2\eta, \qquad g = b \exp 2\eta[\ln (c \exp 2\eta)], \tag{14}$$

where b, c = const.

To proceed in the more complex BDT-Bianchi type-VI_h ($k^2 \neq 1$) case we assume the special relation y/g = const. Equation (12) can be rewritten as

$$y'' - 2(1+k^2)y = y[(\ln y)'^2 - (\ln y)'(\ln g)' + ((1+\omega)/(3+2\omega))\,(y/g)], \tag{15}$$

which in the present case may be reduced to

$$y'' - 2by = 0, \qquad b = ((k^2(3+2\omega)+1)/(2+\omega)). \tag{16}$$

We obtain the following solutions

$k^2 \neq 1$:

$$R_i = r_i \exp(n_i\eta), \qquad \phi = \phi_0 \exp(m\eta),$$

$$n_1^2 = (2/(2+\omega))\,[k^2(3+3\omega)+1],$$

$$n_2 = (2/n_1(2+\omega))\,[k^2(1+\omega)+k(2+\omega)+1],$$

$$n_3 = (2/n_1(2+\omega))\,[k^2(1+\omega)-k(2+\omega)+1],$$

$$m = (2/n_1(2+\omega))\,(k^2-1). \tag{17}$$

If $k = 0$ we obtain the BDT-Bianchi type-V (FRW) solution discussed by us recently (Lorenz-Petzold, 1984a).

We finally consider some special power type solutions in case of $1 \leqslant \gamma < 2$. By setting

$$R_i = a_i t^{p_i}, \qquad \phi = \phi_0 t^r, \qquad a_i, p_i, \phi_0, \qquad r = \text{const}. \tag{18}$$

it follows from (7a)–(7f) (after somewhat lengthy calculations)

$$p_1 = 1,$$

$$p_2 = (1/2k\gamma)\,[(k-1)\,(2-r-\gamma)+2\gamma],$$

$$p_3 = (1/2k\gamma)\,[(k+1)\,(2-r-\gamma)-2\gamma],$$

$$a_1^2 = 4q^2k^2\gamma^2[2-\gamma+r(\gamma-1)]^{-1}\,[3\gamma-2+r]^{-1},$$

$$(M/\phi_0(a_1a_2a_3)^\gamma) = (1/2k^2\gamma^2)\,(3+2\omega)\,[1+\omega(2-\gamma)]^{-1}$$

$$\times[2-\gamma+r(\gamma-1)]\,[k^2(2-\gamma-r)-(3\gamma-2+r)],$$

$$r^2(\gamma-1)\,\{2k^2\gamma[1+\omega(2-\gamma)]+(1+k^2)\,(4-3\gamma)\}$$

$$+r\{(2-\gamma)\,[2k^2\gamma(1+\omega(2-\gamma))+(1+k^2)\,(4-3\gamma)]$$

$$+(\gamma-1)\,(4-3\gamma)\,[3\gamma-2+k^2(\gamma-2)]\}+(2-\gamma)\,(4-3\gamma)\,[3\gamma-2+k^2(\gamma-2)] = 0. \qquad (19)$$

If $r = 0$ we obtain the special GRT-Bianchi type-VI_h perfect fluid solutions first given by Collins (1971). In particular, we have the following simple expression in the dust case ($\gamma = 1$):

$$r = ((k^2-1)/[k^2(3+2\omega)+1]). \qquad (20)$$

The dust solution (20) turns out to be identical with our solution (17). The GRT-Bianchi type-VI space-time has been recently reviewed by Rosquist (1984). For the sake of completeness we mention also the works of Batakis (1981), Carmeli and Charach (1980), Carmeli et al. (1981) and McIntosh (1978) of Bianchi type-VI_h cosmologies.

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