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QUANTUM MECHANICAL TREATMENT OF AN ISOSPINOR SCALAR IN YANG-MILLS-HIGGS MONOPOLE BACKGROUND

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We examine the quantum mechanics of an isospinor scalar in the background of a one parameter family of solutions of SU(2) Yang-Mills-Higgs system. The Hamiltonian is self-adjoint if one imposes ordinary boundary conditions at the origin. The exact solution is presented for the asymptotic singular monopole when the electric charge is conserved. There exist bound states with an extra degeneration indicating a dynamical symmetry.

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1. Introduction

There is an increasing interest in the dynamics of the fermion-monopole system [1], the second quantized structure of which may be of phenomenological importance [2]. An exact solution of the Jackiw-Rebbi equations [3] has been presented recently [4] for an isospinor fermion in the background of a singular SU(2) Yang-Mills-Higgs (Y-M-H) monopole. One can imagine that the quantum mechanics of a colored scalar also could have some importance. (For example in disappearance of Higgs like scalars from the GUTs via scalar-monopole bound state production.) Besides, this is an interesting problem on its own right which can be solved exactly in the singular monopole case. The classical motion of a colored spinless test particle in the Prasad-Sommerfield (P-S) monopole field [5] has been treated recently and in non-relativistic, large distance limit bounded orbits have been found [6]. Here the quantum analogue of this result will be given. Let us consider as background the monopole solution [7] of the SU(2) Y-M-H model in the P-S limit with gauge field A^a_μ and Higgs field ϕ^a given by:

$$A^a_0 = 0, \quad (a = 1, 2, 3),$$

$$A^a_j = \varepsilon_{ajk} \frac{x^k}{gr} \frac{1-K(r)}{r} \quad (j, k = 1, 2, 3)$$

$$\phi^a = \frac{x^a}{gr}\frac{H(r)}{r}, \quad (\alpha = gF)$$

$$K(r) = \frac{\alpha r}{\sinh(\alpha r + \alpha r_0)}, \quad H(r) = \frac{\alpha r}{\tanh(\alpha r + \alpha r_0)} - 1. \tag{1.1}$$

Here $r_0 \geqslant 0$ is arbitrary, $g > 0$ is the gauge coupling constant and $F > 0$ characterizes the symmetry breaking boundary condition. The $r_0 = \infty$ singular monopole can be used for asymptotical approximation of the $r_0 = 0$ P-S solution. We set up the quantum mechanical equations in the background field (1.1) and find complete solution for the $r_0 = \infty$ asymptotical singular case. The Hamiltonian is self-adjoint for the whole family (1.1) with ordinary boundary condition at the origin. The long range Higgs field gives rise to fermionic monopole-scalar bound states. There is an extra degeneration in the bound state spectrum, similar to that of the non-relativistic Coulomb problem. The electric charge of the test particle is conserved only when $r_0 = \infty$.

2. *Quantum mechanical equations from dimensional reduction*

There are several attempts to derive the symmetry breaking Higgs field as part of a higher dimensional, space-time symmetrical Y-M field [8]. The simplest and almost trivial example is the Y-M-H system in the P-S limit considered as pure Y-M field in five dimensional flat space-time, which has a translational invariance with respect to the fifth direction. Taking this into account the quantum mechanical equation of a Lorentz scalar which belongs to some irreducible representation of SU(2) is given by the usual minimal substitution in the corresponding five dimensional gauge field. Let us denote the coordi nates in the five dimensional space-time $M^5 = M^4 \times S^1_\lambda$ by z^B; $B, C = 0, \ldots, 4$. Here $M^4 = \{x\}$ is the four dimensional Minkowski space and S^1_λ is a circle of radius λ, $z^B = x^\mu$ if $B = \mu = 0, \ldots, 3$; $z^4 = y \in [0, 2\pi\lambda)$, $x^0 = ct$ and $g^{AB} = \mathrm{diag}\,(-1, 1, 1, 1, 1)$. The above mentioned equation for the $\vec{\Psi}$ wave function is

$$\left[g^{BC} D_B D_C - \frac{\kappa^2 c^2}{\hbar^2}\right]\tilde{\Psi}(z) = 0, \tag{2.1}$$

where

$$D_B = \nabla_B + gA_B,\ A_B = A^a_B T_a,\ T^+_a = -T_a,\ [T_a, T_b] = \varepsilon_{abc} T_c.$$

We remark that if one had taken the classical Wong's equation [9], i.e. the geodesic equation of the underlying metrized SU(2) fibre bundle [10], as starting point, and had tried to work with the corresponding scalar Klein-Gordon equation of the bundle then an infinite tower of interacting particle fields belonging to irreps of SU(2) would have been obtained [11]. If A_B is translational invariant in the fifth direction then we can reinterpret it as

$$A_\mu(z) = A_\mu(x), \quad A_4(z) = \phi(x), \quad z = (x, y) \tag{2.2}$$

Y-M and Higgs fields over M^4. Using the Fourier expansion of $\tilde{\Psi}$ on S^1_λ

$$\tilde{\Psi}(x, y) = \sum_{N=-\infty}^{\infty} \Psi_N(x) \exp\left(iN \frac{y}{\lambda}\right) \tag{2.3}$$

the following equation is obtained for the N-th component:

$$\left\{g^{\mu\nu}(i\hbar\nabla_\mu + gA^a_\mu Q_a)(i\hbar\nabla_\nu + gA^b_\nu Q_b) + \left(N\frac{\hbar}{\lambda} - g\phi^a Q_a\right)^2 + \kappa^2 c^2\right\}\Psi_N = 0. \tag{2.4}$$

Here $Q_a = i\hbar T_a$ are Hermitian generators of the Lie algebra of SU(2). In the case of the monopole field (1.1) let us introduce the following notations:

$$V_N = g\eta^a Q_a\left[g\eta^b Q_b - 2\left(N\frac{\hbar}{\lambda} - g\phi^b_\infty Q_b\right)\right],$$

$$\hat{m}^2_N = \kappa^2 + \frac{1}{c^2}\left(N\frac{\hbar}{\lambda} - g\phi^a_\infty Q_a\right)^2, \tag{2.5}$$

where

$$\phi^a_\infty = \phi^a_\infty(\hat{\vec{r}}) = \frac{\alpha x^a}{gr}, \quad \phi^a = \phi^a_\infty + \eta^a, \quad \hat{\vec{r}} = \frac{\vec{r}}{r}. \tag{2.6}$$

These definitions will be convenient since the potential V_N approaches to zero at the infinity. With the above formulae we are looking for stationary states

$$\Psi_N(x) = \psi_N(\vec{r}) \exp\left(-i\frac{Ex^0}{c\hbar}\right) \tag{2.7}$$

of the equation

$$\{g^{\mu\nu}(i\hbar\nabla_\mu + gA^a_\mu Q_a)(i\hbar\nabla_\nu + gA^b_\nu Q_b) + \hat{m}^2_N c^2 + V_N\}\Psi_N = 0. \tag{2.8}$$

This gives rise to

$$\hat{H}_N \psi_N(\vec{r}) = \left[\left(\frac{E}{c}\right)^2 - \kappa^2 c^2\right]\psi_N(\vec{r}), \tag{2.9}$$

where the formal Hamiltonian with (1.1) is

$$\hat{H}_N = -\hbar^2 \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}[\hat{J}^2 - \hat{W}^2]$$

$$+ \frac{K^2}{r^2}[\hat{Q}^2 - \hat{W}^2] + \left[N\frac{\hbar}{\lambda} - \frac{H}{r}\hat{W}\right]^2 - \frac{2K}{r^2}[\hat{Q}^2 - \hat{W}^2 + (\vec{\mathscr{L}}, \vec{Q})]. \tag{2.10}$$

Here $\hat{W} = (\hat{\vec{r}}, \vec{Q}) = \frac{1}{r} x^a Q_a$, $\vec{\mathscr{L}}$ is the usual orbital angular momentum $\mathscr{L}_j = \frac{\hbar}{i} \varepsilon_{jkm} x^k \partial_m$ and

$$\vec{J} = \vec{\mathscr{L}} + \vec{Q} \tag{2.11}$$

is the combined angular momentum of the monopole-test particle system. $\vec{J}$ is conserved as a result of the SO(3) symmetry of the monopole [8]. When the particle carries a half integer SU(2) charge we get just the famous "spin from isospin" phenomenon [12]. To solve (2.9) the standard quantum mechanical methods will be applied. For simplicity in the following we are going to deal with an isospinor scalar. This means that now

$$Q_a = \tfrac{1}{2} \hbar \sigma_a \tag{2.12}$$

and with the Pauli matrices σ_a the identity

$$\hat{W}^2 = \tfrac{1}{4} \hbar^2 \tag{2.13}$$

is obtained.

3. *Exact solution for the asymptotical singular monopole*

Now $K(r) = 0$ which is equivalent to the fact that $\hat{W} = \frac{1}{2} \hbar \hat{U} - \hat{U} = (\hat{\vec{r}}, \vec{\sigma})$— commutes with the Hamiltonian (2.10). It is easy to see in an Abelian gauge [13] that $\hat{W}$ is the charge operator. So the electric charge is conserved only in the $r_0 = \infty$ case. For $N = 0$ one can choose as a complete system of commuting variables

$$\hat{H}_N, \hat{J}^2, \hat{J}_3, \hat{\mathscr{L}}^2, \tag{3.1}$$

where $\hat{\mathscr{L}}^2$ characterizes the parity too or another one

$$\hat{H}_N, \hat{J}^2, \hat{J}_3, \hat{U}, \tag{3.2}$$

but for $N \neq 0$ only the second works. It is natural to use spherical coordinates (r, ϑ, φ). Denote $\chi_{J,J_3,l}(\vartheta, \varphi)$ and $\chi_{J,J_3,U}(\vartheta, \varphi)$ the elements of the complete orthonormal systems of $L^2(S^2, C^2)$ corresponding to the angular parts of (3.1) and (3.2) respectively. Let C^2 be the isospinor space with standard basis vectors τ_+, τ_-. Then $l = J \pm \frac{1}{2}$ and we can use for (3.1)

$$\chi_{J,J_3,l=J-\frac{1}{2}} = \left(\frac{J+J_3}{2J}\right)^{\frac{1}{2}} Y_{J-\frac{1}{2}}^{J_3-\frac{1}{2}} \tau_+ + \left(\frac{J-J_3}{2J}\right)^{\frac{1}{2}} Y_{J-\frac{1}{2}}^{J_3+\frac{1}{2}} \tau_-, \tag{3.3}$$

$$\chi_{J,J_3,l=J+\frac{1}{2}} = \left(\frac{J+1-J_3}{2J+2}\right)^{\frac{1}{2}} Y_{J+\frac{1}{2}}^{J-\frac{1}{2}} \tau_+ - \left(\frac{J+1+J_3}{2J+2}\right)^{\frac{1}{2}} Y_{J+\frac{1}{2}}^{J_3+\frac{1}{2}} \tau_-.$$

Here $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$; $J_3 = -J, (-J+1), \ldots, J$ and for the spherical harmonics $Y_l^m(\vartheta, \varphi)$ the phase convention is

$$Y_l^{-m}(\vartheta, \varphi) = (-1)^m Y_l^{m*}(\vartheta, \varphi). \tag{3.4}$$

The reader can readily verify that with the $\hat{U}$ pseudoscalar operator

$$\chi_{J,J_3,U} = \frac{1}{\sqrt{2}}\{\chi_{J,J_3,l=J-\frac{1}{2}}+U\chi_{J,J_3,l=J+\frac{1}{2}}\}; (U = \pm 1) \tag{3.5}$$

is applicable for (3.2). We can use the following relations:

$$\hat{U}\chi_{J,J_3,l=J\pm\frac{1}{2}} = \chi_{J,J_3,l=J\mp\frac{1}{2}}, \tag{3.6}$$

$$(\vec{\mathscr{L}}, \vec{\sigma})\chi_{J,J_3,U} = -\chi_{J,J_3,U}+(J+\tfrac{1}{2})\chi_{J,J_3,-U}.$$

We remark that in the case of $r_0 \neq \infty$ (3.1) is a good observable system for $N = 0$ and

$$\hat{H}_N, \hat{J}^2, \hat{J}_3 \tag{3.7}$$

can be chosen at $N \neq 0$, which leads to two coupled radial equations. As in the present case $r_0 = \infty$ we look for eigenstates of the following form

$$\psi_{N,J,J_3,U}(\vec{r}) = \frac{R_{J,u}(r)}{r}\chi_{J,J_3,U}(\vartheta, \varphi), \tag{3.8}$$

(1.1), (2.10) with $r_0 = \infty$ give rise to the radial equation

$$\frac{d^2R_{J,U}}{dr^2} +2\left[S-\left(\frac{J(J+1)}{2r^2}+\frac{q_{N,U}}{r}\right)\right]R_{J,U} = 0, \tag{3.9}$$

where

$$q_{N,U} = \tfrac{1}{2}\left(\frac{N}{\lambda}U-\tfrac{1}{2}\alpha\right),$$

$$S = \frac{1}{2\hbar^2}\left[\frac{E^2}{c^2} -\kappa^2c^2-\left(N\frac{\hbar}{\lambda}-\tfrac{1}{2}\hbar U\alpha\right)^2\right] = \frac{1}{2\hbar^2}\left[\frac{E^2}{c^2}-m_{N,U}^2c^2\right]. \tag{3.10}$$

This is just like the radial equation of the non-relativistic Coulomb problem but J runs through the half integers now. At least one of the "effective coupling constants" $q_{N,U}$ ($U = \pm 1$) is surely negative and produces bound states. It is clear from the radial equation (3.9) that the Hamiltonian is self-adjoint with the usual $R_{J,U}(0) = 0$ boundary condition at the origin as it happened for fermionic test particle too [4]. This is true for the whole family (1.1) as well. The general solution of (3.9) is easily obtainable for arbitrary N but let us investigate the $N = 0$ case only. For $N \neq 0$ very large $\frac{N^2}{\lambda^2}$ mass terms appear (2.5) which underlines the exceptional role of the $N = 0$ symmetrical wave function case. The allowable ($R_{J,U}(\infty) = 0$) negative values of S give a discrete spectrum with bound states. These can be characterized by a half integer "principal quantum number" $n \in \{\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots\}$. For fixed $n \geqslant \frac{3}{2}$ J runs from $\frac{1}{2}$ to $(n-1)$ and $U = \pm 1$. S and so the energy depend only on n

$$S_n = -\left(\frac{\alpha}{4}\right)^2\frac{1}{2n^2}. \tag{3.11}$$

The extra degeneration in the discrete spectrum (3.11) is very similar to that of the non-relativistic Coulomb problem. So we can point out a dynamical symmetry of the quantum mechanical equation (2.1) in the $r_0 = \infty$ singular monopole background [14]. The wave function is given in terms of the confluent hypergeometrical function

$$R_{n,J}(r) = c_{n,J}\varrho^{J+1}e^{-\frac{\varrho}{2}}F(J+1-n, 2J+2, \varrho), \tag{3.12}$$

where $\varrho = 2(-2S_n)^{\frac{1}{2}} \cdot r$ and $c_{n,J}$ is a normalization constant. Each S_n is doubly degenerate with respect to U therefore we can use these radial functions with the basis $\chi_{J,J_3,l}$ as well. The multiplicity of S_n equal to

$$M_n = 2\sum_{J=\frac{1}{2}}^{n-1}(2J+1) = 2(n^2-\tfrac{1}{4}). \tag{3.13}$$

This is the multiplicity of the $\pm E_n$ energy eigenvalues

$$E_n^2 = \kappa^2c^4 + \left(\frac{\alpha\hbar c}{2}\right)^2\left(1-\frac{1}{(2n)^2}\right) = m_0^2c^4 - \left(\frac{\alpha\hbar c}{2}\right)^2\frac{1}{(2n)^2} \tag{3.14}$$

at the same time. There is a gap in the energy spectrum even in the case of $\kappa = 0$. For $S > 0$ a continuous spectrum is present. With the aid of the $k = (2S)^{\frac{1}{2}}$ wave number the scattering solutions apart from normalization can be written as

$$R_{J,k}(r) = c_{k,J}^{\pm}(2kr)^Je^{\mp ikr}F\left(\pm\frac{i}{4}\frac{\alpha}{k}+J+1, 2J+2, \pm ikr\right) \tag{3.15}$$

corresponding to in- and outgoing spherical waves.

4. *Concluding remarks*

For arbitrary r_0 and $N = 0$ using the angular basis (3.3) we get the radial equation

$$\frac{d^2R_{J,l}}{dr^2} + 2[S - V_{J,l}^{\text{eff}}(r)]R_{J,l} = 0;$$

$$V_{J,l}^{\text{eff}} = \frac{J(J+1)}{2r^2} + \tfrac{1}{8}\left[\left(\frac{H}{r}\right)^2 - \frac{1}{r^2} - \alpha^2\right] - \frac{K}{2r^2}\left[J(J+1) - l(l+1) + \tfrac{1}{4} - \frac{K}{2}\right]. \tag{4.1}$$

Neglecting the exponentially descending terms the large distance asymptotic form of $V_{J,l}^{\text{eff}}$ is $\left[\frac{J(J+1)}{2r^2} - \frac{\alpha}{4r}\right]$, this appeared exactly in (3.9). Close to the origin in the most interesting case of the P-S monopole one can use

$$V_{J,l}^{\text{eff}}(r) = \frac{l(l+1)}{2r^2} + \frac{\alpha^2}{12}[J(J+1) - l(l+1) - \tfrac{9}{4}] + O(r). \tag{4.2}$$

The self-adjointness of the Sturm-Liouville problem of (4.1) is ensured for arbitrary partial wave by (4.2) but it allows only a numerical study. It would be interesting to see how the dynamical symmetry does work if $r_0 = \infty$ and what kind of splitting appears in the spectrum of (2.10) when $r_0 \neq \infty$. As it have been remarked $\hat{W}$ is not conserved for $r_0 \neq \infty$. The charge exchange processes between the test particle and the monopole (dyon) can be traced properly within the framework of the second quantized theory. We shall investigate the general case of (1.1) and give a detailed investigation of the scattering phenomena with incorporation of dyons elsewhere.

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