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ON THE INFLUENCE OF THE QCD VACUUM STRUCTURE ON HEAVY QUARKONIUM SPECTRA

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A method of calculating energy levels of heavy quarkonium interacting with vacuum gluon fields is described and tested on a simple, exactly solvable, quantum-mechanical model, which was originally proposed by Zalewski to illustrate some features of the theory of heavy quarkonia. Inferences relevant for calculations of toponium properties are drawn from model results.

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1. *Introduction*

Quarkonia, bound states of heavy quarks and antiquarks, have added a great deal of support to the widespread belief in the correctness of the foundations of quantum chromodynamics (QCD; for a review see e. g. [1]). However, the existing heavy quarkonium families have not become "the hydrogen atom of strong interaction physics", as it was anticipated in early papers dealing with the subject [2]. The reason is that neither the long--distance confining force between the quark and the antiquark, nor their short-range interaction force are directly and unambiguously reflected in properties of the ψ- and Υ-families; they occupy the intermediate distances ($\sim$0.1 to 1 fermi) where the interquark force is poorly known theoretically [3]. The discovery of one more heavy quarkonium family, presumably consisting of top quarks and antiquarks, is therefore eagerly awaited: one could finally probe the short-distance Coulomb-like interaction between quarks and antiquarks, following from the asymptotic freedom of QCD [4].

In QCD the ground-state, the vacuum, is expected to have a rich structure not understandable within the standard weak-coupling perturbation theory (see e. g. [5]). This should be responsible for instance for the permanent quark confinement and low-energy properties

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of hadrons. Without its detailed knowledge, properties of the QCD vacuum are usually parametrized through values of the so called vacuum condensates [6][1].

Very heavy quarkonia, besides being essentially non-relativistic and Coulomb-like systems, are advantageous in another point: non-perturbative effects are expected to show up relatively simply here. The calculation of gluon condensate effects on heavy quarkonia was pioneered by Voloshin [8] and Leutwyler [9] (see also [10]). They show that the effects can be calculated very simply: if the characteristic length of vacuum fluctuations is large compared to the $Q\bar{Q}$ bound state dimensions (i.e. if the quarks are very massive), one assumes in the first approximation that the heavy $Q\bar{Q}$ pair is immersed in the constant (random) vacuum gluon field characterized by the value of the gluon condensate $G^2 \equiv \langle 0| (\alpha_s/\pi) G^a_{\mu\nu} G^{a\mu\nu} |0\rangle$. The whole system must be in a colour-singlet state, hence either the pair and the surrounding medium are both colour singlets, or both octets. The $Q\bar{Q}$ pair lives therefore a part of time in an octet state in which the $Q\bar{Q}$ interaction is repulsive (instead of being attractive as in singlet states), and this changes substantially the heavy quarkonium properties.

If the quark mass is large enough (radius of the bound state very small) the $Q\bar{Q}$ interaction can be considered Coulomb-like in both singlet and octet states, and the interaction of the pair with the vacuum field A^a_μ can be treated using the QCD multipole expansion (including only the electric-dipole-like term). It is easy to estimate e.g. the relative shift of the n, l-th heavy quarkonium level, due to the condensate G^2: the problem is analogous to the quadratic Stark effect in the hydrogen atom. The result must be proportional to:

(*i*) G^2 coming from $E^a_i E^b_j$ through Lorentz and colour invariance of the vacuum;

(*ii*) $r^2 \sim (n^2/m\alpha_s)^2$ from the dipole-like interaction in the second order;

(*iii*) $(m\alpha_s^2/n^2)^{-1}$ originating from the energy denominator in the second -order perturbation formula; and

(*iv*) $(m\alpha_s^2/n^2)^{-1}$, the reciprocal value of the unperturbed energy. One thus gets

$$\frac{\Delta E_{nl}}{|E_{nl}|} \sim \frac{1}{\alpha_s^6 m^4} G^2 n^8$$

in accord with the Leutwyler-Voloshin formula [8, 9]

$$\frac{\Delta E_{nl}}{|E_{nl}|} = \varphi G^2 n^8 a_{nl}, \tag{1}$$

where $\varphi = 4\pi^2/m^4\beta^6$, $\beta = 4\alpha_s/3$, α_s is obviously the QCD fine-structure constant, and a_{nl} is a factor $O(1)$.

However, a more general treatment than that of Leutwyler and Voloshin is necessary at least for two reasons. First, their results are inapplicable to existing heavy quarkonia. The calculation uses the quantum-mechanical perturbation theory, that would be justified if $\Delta E/|E|$ were small. Even an optimistic guess gives $\Delta E/|E| \gg 1$ for charmed and $\gtrsim 1$ for

[1] Actual values of these parameters (especially that of the gluon condensate) extracted using the QCD sum rule approach have been a subject of lively discussions recently; the dust seems not to have completely settled yet (see papers quoted in [7]). I shall briefly return to this point in the concluding Section.

bottom quarks. (Attempts to find bottomonium properties for which the effective perturbation parameter would be smaller have failed, see [11].) Second, even for toponia one shall need some control over corrections to the lowest approximation. It would be interesting to see whether there are some situations in which substantial corrections to the Leutwyler--Voloshin formula could exist.

The purpose of the present paper is three-fold: For the sake of completeness I first want to sketch a method of calculating non-perturbative energy-level shifts in heavy quarkonia [12] which, though not fully general, is more general than that of Leutwyler and Voloshin and contains their results as a natural first approximation (Sec. 2). Then (Sec. 3), I illustrate problems of the method on a simple, exactly solvable, quantum-mechanical model proposed by Zalewski [13]. Finally, some conclusions are drawn which could be of relevance for later calculations of heavy quarkonium (toponium) properties (see Sec. 4).

2. *A method of calculating non-perturbative energy-level shifts in heavy quarkonia*

In this Section I shall sketch an approach to the calculation of QCD-vacuum-structure effects on energy levels of heavy quarkonia that is slightly more general than that of Voloshin and Leutwyler (for a more detailed exposition see [12]). Physical assumptions behind the method are roughly identical to Leutwyler's and Voloshin's:

(*i*) the possibility of separating the total Hamiltonian of the singlet system consisting of the quark, the antiquark and the surrounding medium into three parts: the Hamiltonian of the $Q\bar{Q}$ pair (with their Coulomb-like interaction potential due to the exchange of short--wavelength gluons included), that of the gluonic medium, and the interaction Hamiltonian (containing the interaction of the pair with long-wavelength gluonic fluctuations); i.e.

$$H = H_{Q\bar{Q}}+H_G+H_{\rm int}; \tag{2}$$

(*ii*) the $Q\bar{Q}$ bound state radius (of the order of $r_q \sim (m\alpha_s^2)^{-1}$) and period of motion (of the order of $t_q \sim (m\alpha_s^2)^{-1}$) are small compared to characteristic space-time dimensions of non-perturbative vacuum fluctuations. One can then in the first approximation neglect the temporal dependence of the fields, and make use of the QCD multipole expansion [8, 9, 14] of the interaction of quarks with the constant (but random) gluon field; the lowest terms give [8]

$$H_{\rm int} = -Q^aA_0^a(0)-\vec{d}^a\cdot\vec{E}^a(0), \tag{3}$$

where $\vec{E}^a$ is the chromoelectric field, Q^a, $\vec{d}^a$ are colour charge and colour electric dipole moment of the pair, and the centre of mass of the pair is placed in the origin of coordinates.

(The latter assumption has been questioned by Baier and Pinelis [15]. However, recent results on the correlation length of the gluon condensate [16] seem to support the above--described point of view.)

The total Hamiltonian can be separated also in a different way

$$H = H_1+H_8+H', \tag{4}$$

where H_1 (H_8) is the piece of H that does not mix singlet and octet $Q\overline{Q}$-pair and gluonic states, and H' is the part of H that does mix them. Obviously, H_1 gets contributions from $H_{Q\overline{Q}}$ and H_G only, H_8 from $H_{Q\overline{Q}}$, H_G, and $H_{\rm Int}$ too, while H' comes solely from $H_{\rm Int}$.

The quarkonium energy levels in the vacuum field can be found from the "pure quarkonium" Green function, the projection of the full Green function on colour singlet quarkonium states, averaged over the gluonic vacuum $|0_g\rangle$

$$G_Q(E) \equiv \langle 0_g|P_1(H-E)^{-1}P_1|0_g\rangle. \tag{5}$$

(P_1 (P_8) projects on singlet states of the system that consist of the singlet (octet) $Q\overline{Q}$ pair and the singlet (octet) medium). For this a closed equation can be derived (using a single additional assumption, see Eq. (8) below)

$$G_Q(E) = G_Q^{(1)}(E)+G_Q^{(1)}(E)K(E)G_Q(E), \tag{6}$$

where

$$K(E) \equiv \langle 0_g|P_1H'P_8(H_8-E)^{-1}P_8H'P_1|0_g\rangle \tag{7}$$

and $G_Q^{(1)}$ is the Green function for $H_{Q\overline{Q}}^{(1)} = -\Delta/m-4\alpha_s(r)/3r$, the singlet part of $H_{Q\overline{Q}}$.

Instead of giving details of the derivation of Eq. (6) (see [12]) let me explain its content using simple pictures. The notation is fixed in Fig. 1. Fig. 2 shows an identity satisfied by the pure quarkonium Green function $G_Q(E)$. Its meaning is the following: The parts ($Q\overline{Q}$ pair + gluonic medium) either propagate without interacting, or spend some time in singlet states, then jump due to the interaction that mixes singlets and octets to an (overall colour-singlet) state in which the pair is octet and so is the medium, propagate for some time in octet states, and finally jump back to singlet states. Stars at 1's and 8's in Fig. 2 are to remind of the possibility of jumps into virtual excited intermediate states.

The last ingredient of Eq. (6) is explained pictorially in Fig. 3. The equation illustrated in Fig. 2 is not closed. A possibility leading to the closed equation is to assume

$$\underset{\sim}{1}^* \cong \underset{\sim}{1}, \text{ or } P_1 \sim P_S|0_g\rangle\langle 0_g|P_S \tag{8}$$

(P_S is the projector on singlet $Q\overline{Q}$-pair states) i. e. to neglect higher singlet gluonic excitations in intermediate states. We then immediately get Eq. (6) (Fig. 3). Of course, some ambiguity is introduced by using the assumption (8); we believe it not to be crucial: the essential point of the situation, the singlet-octet transitions caused by the interaction of the $Q\overline{Q}$ pair with vacuum fluctuations, has still survived. Moreover, Eq. (8) is the simplest possibility, and any more sophisticated assumption is hard to justify taking into account our ignorance of the QCD vacuum. (See also [17].)

The approximate operator equation for $G_Q(E)$ can be used for finding quarkonium energy levels in a straightforward way. Formally, these energies can be found by solving the equation[2]

$$\det\left[(E-\varepsilon_n^{(1)})\delta_{nm}+K_{nm}(E)\right] = 0, \tag{9}$$

[2] For obvious reasons, this equation is reminiscent of the secular equation for finding eigenvalues of a Hamiltonian; the only difference is that K_{nm} themselves depend on E.

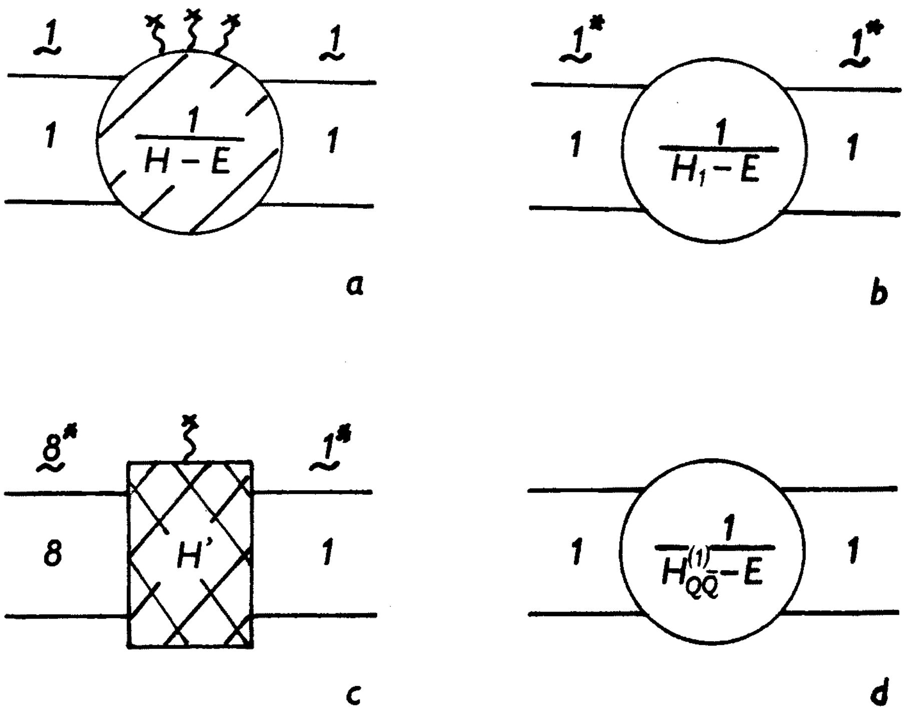

Fig. 1. Elements entering Figs. 2–5: a) pure quarkonium Green function; b) propagation of the quarkonium and the gluonic medium in singlet states (without interacting); c) the interaction of quarkonium with the condensate in which gluonic quantum numbers are transferred; d) propagation of the singlet quarkonium; and similar symbols. ($\underset{\sim}{1}$, $\underset{\sim}{1}^*$, $\underset{\sim}{8}^*$ denote the states of the gluonic background (gluonic vacuum, singlet gluonic excitation, octet gluonic excitation respectively), while 1 and 8 denote the $Q\bar{Q}$-pair colour state (singlet and octet))

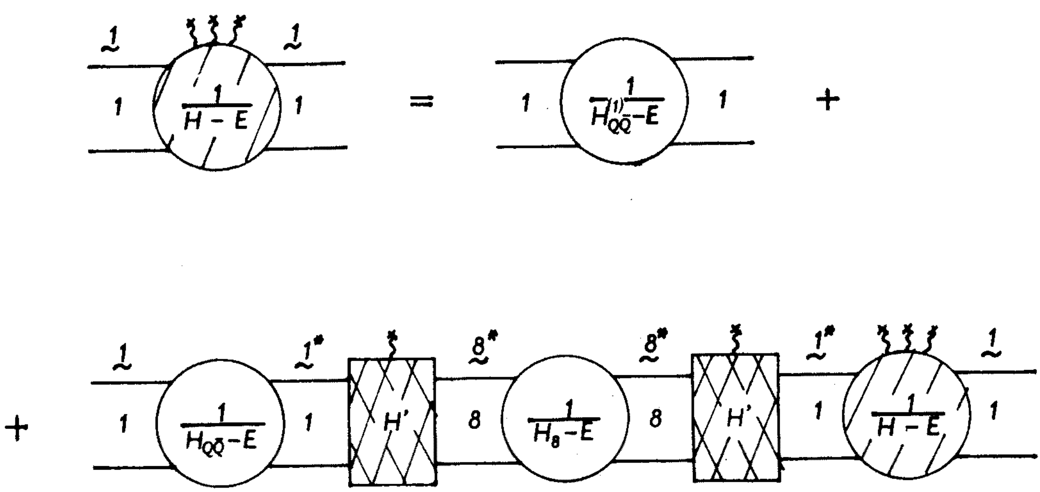

Fig. 2. Pictorial representation of an identity for the pure quarkonium Green function

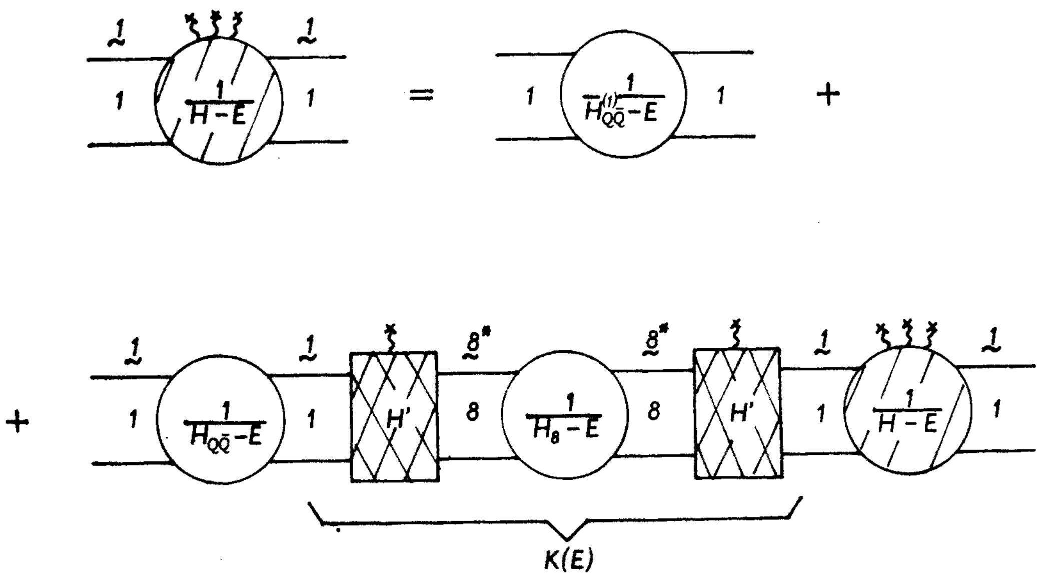

Fig. 3. Taking $\underset{\sim}{1}^* \simeq \underset{\sim}{1}$ at intermediate stages leads to the closed equation for $G_Q(E)$

where n represents the set of all quantum numbers characterizing the unperturbed (Coulomb-like) singlet state with energy $\varepsilon_n^{(1)}$, K_{nm} is $\langle n|K(E)|m\rangle$, the matrix element of K (Eq. (7)) between two such states.

The Leutwyler-Voloshin formula, Eq. (1), is a result of the natural and straightforward first approximation to Eq. (9). One just has

(*i*) to linearize Eq. (9) by neglecting all non-diagonal elements of $K(E)$ and putting $K_{nn}(E) \simeq K_{nn}(\varepsilon_n^{(1)})$; the energy shifts are then simply

$$\Delta E_n = -\langle n|\, \langle 0_g|P_1 H' P_8 (H_8 - \varepsilon_n^{(1)})^{-1} P_8 H' P_1 |0_g\rangle\, |n\rangle; \tag{10}$$

and

(*ii*) to neglect all except the quarkonium part of H_8: $H_8 \simeq H_{Q\bar{Q}}^{(8)} = -\Delta/m + \alpha_s/6r$. This, after using colour and rotation invariance of the gluonic vacuum $[\langle 0_g|E_i^a E_j^b|0_g\rangle \sim -G^2\delta^{ab}\delta_{ij}]$ and after straightforward calculating of the relevant matrix element, leads to Eq. (1). The essence of the Leutwyler and Voloshin approximation is illustrated in Fig. 4.

In trying to further exploit Eq. (9) one shall have to encounter two problems:

(A) to reasonably truncate the (formally) infinite determinant in Eq. (9);

(B) to make some reasonable guess or model of the octet Hamiltonian H_8, which enters $K(E)$ and is not fully known:

$$H_8 = H_{Q\bar{Q}}^{(8)} + H_G^{(8)} + H_{\text{int}}^{(8)}. \tag{11}$$

One will have to find a compromise between two extreme possibilities indicated in Fig. 5: to take the minimum of information on H_8, but take into account the coupling to other singlet states (Fig. 5a), or to forget completely about the coupling and try to consider

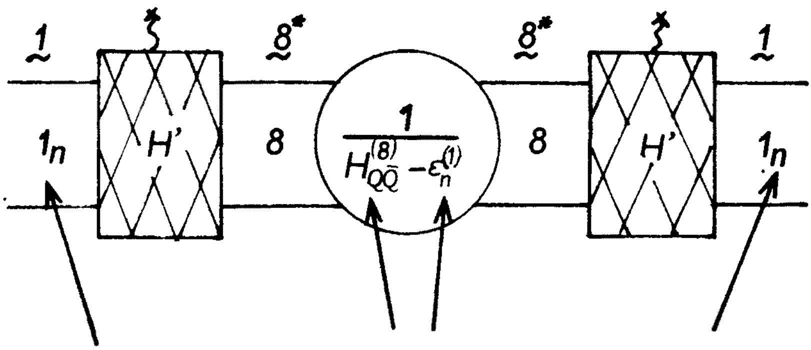

Fig. 4. The content of the Leutwyler-Voloshin approximation. Only diagonal elements of $K(E)$ are nonzero, E is replaced by $\varepsilon_n^{(1)}$, $H_8 \simeq H^{(8)}_{Q\bar{Q}}$ (see arrows)

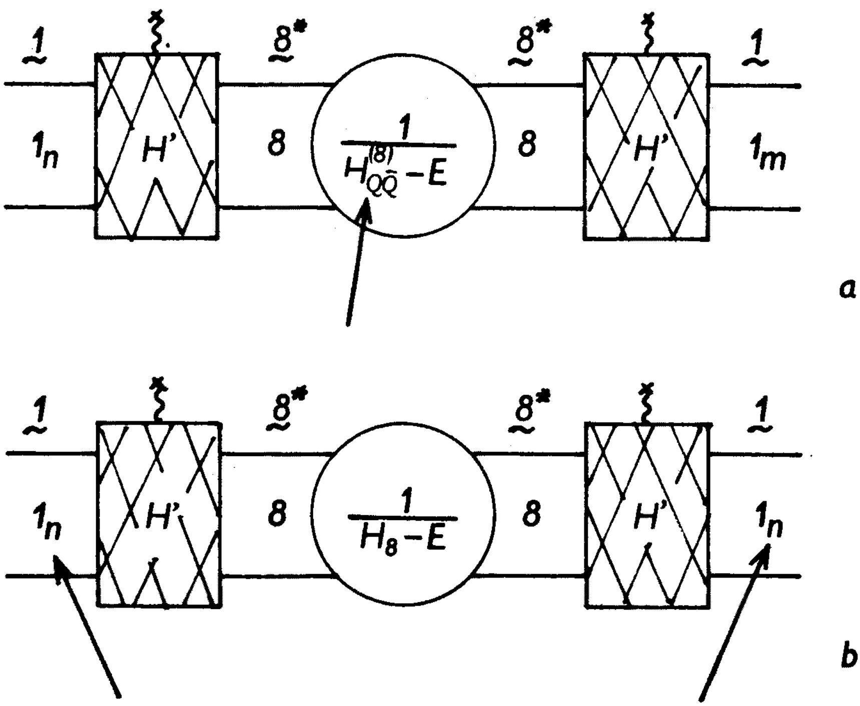

Fig. 5. Two extreme possibilities in solving Eq. (9): a) minimum information on H_8, non-diagonal elements of K included, b) more information on H_8, coupling to other singlet states neglected

in more detail the behaviour of the system during the time when the $Q\bar{Q}$ pair and the gluon medium are in octet states (Fig. 5b).

It is not a priori self-evident which of the problems (A), (B) is more important for an improved calculation of toponium properties. In the next section I shall therefore study an exactly solvable model in which different approximations (including the two above--mentioned ones) can be used and results can be compared with the exact solution.

3. *The two-channel harmonic oscillator model*

Zalewski [13] has recently used an illustrative (one-dimensional) generalized harmonic oscillator model to pin-point some controversial moments in the theory of heavy quarkonia. I shall not try to resolve the controversy; instead I shall make use of the model to shed some light on problems in calculating heavy quarkonium energy levels.

The model Hamiltonian

$$H = -\tfrac{1}{2}\frac{d^2}{dx^2}+\tfrac{1}{2}x^2-\tfrac{1}{2}\omega^2x^2\sigma_3-\lambda\omega^2x^2\sigma_1 \tag{12}$$

(σ_1, σ_3 are standard Pauli matrices) bears much resemblance to the situation encountered in the previous section. It can be separated into the "singlet", the "octet" and the interaction pieces

$$H = H_1+H_8+H', \tag{13}$$

where

$$H_1 = \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix}; \quad H_8 = \begin{pmatrix} 0 & 0 \\ 0 & h_8 \end{pmatrix}$$

$$h_{1,8} = -\tfrac{1}{2}\frac{d^2}{dx^2}+\tfrac{1}{2}\omega_{1,8}^2x^2,$$

$$\omega_{1,8} = \sqrt{1\mp\omega^2}, \tag{14}$$

and

$$H' = \begin{pmatrix} 0 & -V \\ -V & 0 \end{pmatrix}, \quad V(x) = \lambda\omega^2x^2. \tag{15}$$

H' represents the "interaction with the condensate" that mixes "singlet" (upper) and "octet" (lower) components. Without the interaction ($\lambda = 0$) the model Hamiltonian has two sets of eigenfunctions:

(*i*) "pure singlet" ones:

$$\phi_n^{(1)}(x) = \begin{pmatrix} \psi_n^{(\omega_1)}(x) \\ 0 \end{pmatrix} \text{ with energies } \varepsilon_n^{(1)} = (n+\tfrac{1}{2})\omega_1; \tag{16a}$$

and

(*ii*) "pure octet" ones:

$$\phi_N^{(8)}(x) = \begin{pmatrix} 0 \\ \psi_N^{(\omega_8)}(x) \end{pmatrix} \text{ with energies } \varepsilon_N^{(8)} = (N+\tfrac{1}{2})\omega_8. \tag{16b}$$

Here $\psi_n^{(\omega)}(x)$ are normalized wave-functions of the one-dimensional linear harmonic oscillator with frequency ω, see e. g. [18].

After turning on the interaction in the original "pure singlet" states an "octet" admixture evolves

$$\phi_n^{(1)}(x) = \begin{pmatrix} \psi_n^{(\Omega_1)}(x)\cos\theta \\ \psi_n^{(\Omega_1)}(x)\sin\theta \end{pmatrix} \tag{17a}$$

and their energies change to

$$E_n^{(1)} = (n+\tfrac{1}{2})\Omega_1, \tag{17b}$$

while "pure octets" become

$$\phi_N^{(8)}(x) = \begin{pmatrix} -\psi_N^{(\Omega_8)}(x)\sin\theta \\ \psi_N^{(\Omega_8)}(x)\cos\theta \end{pmatrix} \quad \text{with } E_N^{(8)} = (N+\tfrac{1}{2})\Omega_8, \tag{17c}$$

where

$$\Omega_{1,8} = (1\mp\omega^2\sqrt{1+\mathrm{tg}^2 2\theta})^{1/2} \tag{18}$$

and

$$\theta = \tfrac{1}{2}\,\mathrm{arctg}\, 2\lambda. \tag{19}$$

$\phi_n^{(1)}(x)$, $\phi_N^{(8)}(x)$ are exact eigenfunctions of the model Hamiltonian, $E_n^{(1)}$, $E_N^{(8)}$ are the corresponding exact energy eigenvalues.

One can try to solve the model along the lines of the method described in Sec. 2. The analogon of the pure quarkonium Green function (see Eq. (5)) is the "singlet" Green function

$$G(E) \equiv P_1(H-E)^{-1}P_1; \quad P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{20}$$

for which an *exact* equation holds (cf. Eq. (6))

$$G(E) = G_1(E)P_1 + G_1(E)K(E)G(E), \tag{21}$$

where

$$G_1(E) \equiv (h_1-E)^{-1},$$

$$K(E) \equiv P_1V(h_8-E)^{-1}VP_1. \tag{22}$$

"Singlet" energies can be found from Eq. (21) by solving

$$\det\left[(E-\varepsilon_n^{(1)})\delta_{nm}+K_{nm}(E)\right] = 0, \tag{23}$$

in full analogy with Eq. (9)[3]. Here

$$K_{nm}(E) = \int dxdy\,\psi_n^{(\omega_1)}(x)V(x)G_8(x,y;E)V(y)\psi_m^{(\omega_1)}(y), \tag{24}$$

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[3] It is necessary to stress, however, that in contrast to Eqs. (6) and (9), Eqs. (21) and (23) are exact. No additional assumption like (8) is needed in the oscillator model.

and

$$G_8(x, y; E) \equiv \langle x| (h_8 - E)^{-1} |y\rangle = \sum_{N=0}^{\infty} \frac{\psi_N^{(\omega_8)}(x)\psi_N^{(\omega_8)}(y)}{\varepsilon_N^{(8)} - E}. \tag{25}$$

Again two problems arise (cf. (A), (B) at the end of Sec. 2)

(A) that of truncating the infinite determinant;

(B) that of approximating the "octet" Green function.

To investigate their importance and to find the more serious of them, I tried the following approximations:

(I) to take the lowest term in $G_8(x, y; E)$, and

(Ia) linearize Eq. (23) (this mimics what Leutwyler and Voloshin did for heavy quarkonia, see Fig. 4);

(Ib) calculate with it $K_{nm}(E)$, and then solve Eq. (23) for finite determinant size N (see Fig. 5a)[4];

(II) to neglect all non-diagonal elements of $K(E)$ and take into account more ($M \geqslant 2$) terms in the "octet" Green function G_8 (see Fig. 5b); one then solves

$$(E - \varepsilon_n^{(1)}) + K_{nn}(E) = 0; \tag{26}$$

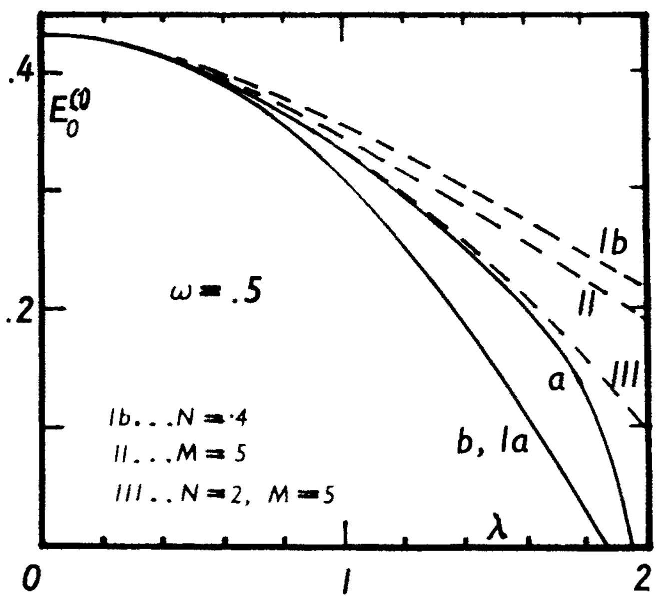

Fig. 6. The ground-state "singlet" energy in the Zalewski model for $\omega = 0.5$. Full lines represent the exact ground state energy (a) and its expansion around $\lambda = 0$ (up to terms $\sim\lambda^2$, (b)). Broken lines correspond to different approximations and are denoted Ia, Ib, II and III in accord with Sec. 3 (in fact, (b) coincides with Ia on the plot)

[4] Matrix elements of K necessary for this analysis are listed in the Appendix.

TABLE I

(a) Approximation Ib Dependence on the determinant size N		(b) Approximation II Dependence on the number of terms M in G_8	
(both for $\omega = 0.5$, $\lambda = 1.0$)			
N	$E_0^{(1)}$	M	$E_0^{(1)}$
1	0.3556	1	0.3556
2	0.3524	2	0.3422
3	0.3523	5	0.3418
6	0.3523	30	0.3418
exact	0.3320	exact	0.3320

(III) to retain $M \geqslant 2$ terms in $G_8(x, y; E)$ and solve Eq. (23) for finite determinant size N (in fact I took $N = 2$ only)[4].

Results of the calculation of the ground state energy in the model of Zalewski are summarized in Fig. 6 for an illustrative value of $\omega = 0.5$. The following is clearly visible:

1. The "Leutwyler-Voloshin" approximation (Ia) leads to rather poor results except for small values of λ, in fact, it coincides with what one gets from expanding the exact ground state energy around $\lambda = 0$.

2. Taking higher size of the determinant in Eq. (23) does not help to improve the agreement with the exact solution if one neglects all higher terms in the "octet" Green function G_8 (approximation (Ib)). This is further illustrated in Table Ia.

3. However, taking account of a few higher terms in the Green function G_8 (see Eq. (25)) brings a considerable improvement even if non-diagonal elements of $K(E)$ are neglected. Five terms are quite enough; then the value of $E_0^{(1)}$ stops improving, see Table Ib.

4. A few terms in the "octet" Green function together with coupling to the nearest level provides almost the exact ground state energy even for high values of λ.

To conclude: to get satisfactory agreement between the ground state energy and the value obtained from Eq. (23) one inevitably needs some information on the "octet" sector of the model. Coupling to higher "singlet" states plays a relatively minor role in the calculation. Conclusions following from this result will be discussed in Sec. 4.

4. *Discussion*

In Sec. 2 I described a method of calculating QCD vacuum condensate effects on heavy quarkonium energy levels. It contains the results of Voloshin and Leutwyler, obtained using different approaches, as a first approximation. The starting point is similar as theirs: characteristic space-time dimensions of vacuum fluctuations are assumed large compared to characteristic dimensions of the $Q\overline{Q}$ bound state. A natural procedure leads to an equation for finding quarkonium energy levels (Eq. (9)). The price to pay is only an additional simplifying assumption that neglects the influence of singlet gluonic excitations on quarkonium states. As discussed in Sec. 2, I believe this assumption not to be crucial, since it does not spoil the physical picture of the interaction of the $Q\bar{Q}$ pair with the vacuum condensate.

A few remarks concerning the method are necessary:

— Since the starting point is similar to Leutwyler's and Voloshin's, I also completely neglect the finite correlation length of the vacuum condensates [16, 19]. This is a serious flaw of the method which has to be improved in a fully realistic calculation.

— A better understanding or a model of the QCD vacuum and its excitations is urgently needed for finding corrections to the results of Voloshin and Leutwyler. One lacks more detailed information on the dynamics of quarkonium when the $Q\bar{Q}$ pair and the surrounding gluon medium are both in octet states (problem (B) of Sec. 2).

The model of Zalewski is ideal for illustrating the above point. While for small values of λ (the parameter controlling the strength of interaction between "singlet" and "octet" states in the model) all approximations we tried in Sec. 3 (including that analogous to Leutwyler's and Voloshin's) give values of the ground state energy of the model in accordance with the exact solution, for higher values of λ the situation is different. The "Leutwyler--Voloshin" approximation becomes poor for higher λ's, and to get a reasonable agreement between approximate and exact values of the ground state energy one necessarily needs some information on the "octet" states (on the Green function corresponding to the "octet" Hamiltonian H_8). Taking account of the coupling of the ground state to higher "singlet" states helps to improve the agreement, but plays a subdominant role.

Of course, there is no direct way of relating the value of λ in the model of Zalewski to the gluon condensate G^2, and the similarity of the model with essential features of heavy quarkonia may well turn to be illusive. The model results can thus be considered only a warning that if the value of G^2 were high enough (see Footnote 1) then one could expect sizeable corrections to the simple Leutwyler-Voloshin formula (Eq. (1)) even for toponia.

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APPENDIX

Here I list some necessary matrix elements of $K(E)$ (Eq. 24)). The notation is fixed as follows:

$$K_{nm}^{(M)}(E) = \sum_{N=0}^{M-1} \frac{1}{\varepsilon_N^{(8)} - E} \int dxdy \psi_n^{(\omega_1)}(x) V(x) \psi_N^{(\omega_8)}(x) \psi_N^{(\omega_8)}(y) V(y) \psi_m^{(\omega_1)}(y). \tag{A1}$$

The following formulae were used in solving Eq. (23):

(*i*) in the approximations (Ia), (Ib)

$$K^{(1)}_{2k,2l}(E) = \frac{1}{\frac{1}{2}\omega_8 - E} \frac{\lambda^2}{4\omega^2} \sqrt{\omega_1\omega_8}\,(\omega_8-\omega_1)$$

$$\times(-\tfrac{1}{2})^{k+l} \sqrt{\binom{2k}{k}\binom{2l}{l}} \left(\frac{\omega_8-\omega_1}{\omega_8+\omega_1}\right)^{k+l} [\omega_8-(4k+1)\omega_1]\,[\omega_8-(4l+1)\omega_1]; \qquad \text{(A2)}$$

(*ii*) in the approximations (II) and (III)

$$K^{(M)}_{2k,2l}(E) = \tfrac{1}{2}\lambda^2 \sqrt{\omega_1\omega_8}\,(\omega_8-\omega_1)$$

$$\times \sum_{N=0}^{M-1} \frac{1}{(4N+1)\omega_8 - 2E} \binom{2N}{N} \left[\frac{1}{2}\left(\frac{\omega_8-\omega_1}{\omega_1+\omega_8}\right)\right]^{2N} \left(-\frac{1}{2\sqrt{2}}\right)^{k+l}$$

$$\times\left[(4N+1)\frac{\omega_8}{\omega} - \frac{\omega_1}{\omega}\right]^{2-k-l} \left[(4N+1)\frac{\omega_8^3}{\omega^3} - (32N^2+16N+7)\frac{\omega_1\omega_8^2}{\omega^3}\right.$$

$$\left. + 11(4N+1)\frac{\omega_1^2\omega_8}{\omega^3} - 5\frac{\omega_1^3}{\omega^3}\right]^{k+l} \qquad \text{(A3)}$$

for $k,\ l \in \{0, 1\}$.

Only "even-even" matrix elements are listed above; the odd-even and even-odd ones vanish (because of parity), the odd-odd elements were not necessary for finding the ground state energy in the model, the problem I concentrated on.

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