# WHERE TO LOOK FOR CRITICAL TESTS OF THE PHENOMENOLOGY BEHIND THE SVZ APPROACH TO HEAVY QUARKONIA*

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The point of view is presented that, while the mathematical method used by the ITEP group (SVZ) is very nice, the purely phenomenological model for the gluon condensate, which they use in order to calculate the nonperturbative contributions, is open to doubt. It is suggested that fine splittings in the spectra of heavy quarkonia provide crucial tests for this phenomenology. Present data cast some doubt on the validity of the model, but a deeper analysis and/or data for toponia are necessary, in order to draw final conclusions.

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## 1. *Introduction*

The SVZ sum rules, known also as QCD sum rules, or ITEP sum rules, provide the estimate [1]

$$G = \left(0\left|\frac{\alpha_s}{\pi}\, G^a_{\mu\nu}G^{\mu\nu a}\right|0\right) = 0.012\ \mathrm{GeV}^4. \tag{1}$$

Here the summation over the Lorentz indices $\mu\nu$ and over the colour index $a$ is understood. Tensor $G^a_{\mu\nu}$ is the chromoelectromagnetic field, and the factor $\alpha_s/\pi$ is introduced in order to make $G$ renormalization group invariant. The averaging is over the physical vacuum state, with the counter-terms chosen so that the corresponding average for the perturbative vacuum vanishes. Estimate (1) has been obtained using experimental data for the $\psi$ resonances as input. The experimental error in the input introduces an error of about 20 per cent [2]. Since the numerical value of the matrix element (1) is of great interest, and since many physicists consider that estimate (1) is the best available, it may be useful to analyse teh SVZ argument.

We propose to discuss first the mathematical method of the ITEP group, and then

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the phenomenological model used to calculate the non-perturbative contributions. We shall present the point of view that the mathematics is very nice, but the phenomenology is somewhat controversial. We suggest that studies of fine splittings in the spectra of heavy quarkonia can provide crucial tests for this phenomenology.

## 2. *Mathematical method*

The mathematical method used by the ITEP group is most easily explained on examples from nonrelativistic quantum mechanics. Among the many versions of the method, we choose that from Ref. [3]. The analysis from this and the following Section has been partly presented by the author at the 1983 Smolenice conference [4].

Consider a particle of mass $m$ in a potential field of forces. Suppose for simplicity that all the energy levels are discrete. Define two Euclidean Green functions

$$G(\beta) = \sum_{n=0}^{\infty} |\psi_n(0)|^2 e^{-\beta E_n}, \tag{2}$$

$$G_c(\beta) = \sum_{n=1}^{\infty} |\psi_n(0)|^2 e^{-\beta E_n}. \tag{3}$$

By construction

$$G(\beta) = |\psi_0(0)|^2 e^{-\beta E_0} + G_c(\beta). \tag{4}$$

The SVZ idea is to evaluate $G(\beta)$ and $G_c(\beta)$ in some weak coupling approximation and then to find $|\psi_0(0)|^2$ and $E_0$ from formula (4).

In order to find both $|\psi_0(0)|^2$ and $E_0$ there must be a window (fiducial region)

$$\beta_1 \leqslant \beta \leqslant \beta_2, \tag{5}$$

where the approximations for both $G(\beta)$ and $G_c(\beta)$ are sufficiently good. For $\beta$ too large, the weak coupling approximation breaks down. For $\beta$ too small, $G(\beta)-G_c(\beta)$ becomes the difference of two large almost equal terms, and again the result is unreliable. Consequently, two questions must be answered before applying formula (4):

1. Is there a window with $\beta_1 < \beta_2$?
2. How to find $\beta_1$ and $\beta_2$?

We present the ITEP answer to these questions in the following Section.

We refer to the weak coupling approximation and not to perturbation theory. Indeed, it has been found in practice (cf. e.g. [1] and following Section) that the leading nonperturbative term often is important and should be included.

Note that the method described here can be used to solve rather unusual problems. It is well known that weak coupling methods are good for scattering problems (e.g. Born approximations). Here, however, the knowledge of the weak coupling approximation is seen to yield information about the ground state, which is a bound state. Usually, weak coupling approximations are presented as useless for the study of bound states. The ITEP group has convincingly demonstrated that this in general is not true.

### 3. *Example*

Consider a one-dimensional Schrödinger equation for one particle of mass $m$ moving in the field of forces corresponding to the potential

$$V(x) = \begin{cases} V & \text{for} \quad |x| \leqslant 1 \\ \infty & \text{for} \quad |x| > 1 \end{cases}, \tag{6}$$

where $V$ is a constant. It is convenient to choose the units so that $2m = \hbar^2\pi^2$. Then

$$G(\beta) = e^{-\beta V} \sum_{n=0}^{\infty} e^{-\beta(n+0.5)^2}. \tag{7}$$

Expanding $\exp(-\beta V)$ into a power series in $\beta$ and performing the Poisson transformation of the sum over $n$, we find

$$G(\beta) = \sum_{\nu=0}^{\infty} \frac{(-V)^\nu}{\nu!} \beta^\nu \tfrac{1}{2} \sqrt{\frac{\pi}{\beta}} \left[1+2\sum_{n=1}^{\infty} (-1)^n e^{-\pi^2 n^2/\beta}\right]. \tag{8}$$

We now make the usual replacement of the weak coupling by the small $\beta$ (short time) approximation and obtain the estimates

$$G(\beta) = \tfrac{1}{2} \sqrt{\frac{\pi}{\beta}} (1 - V\beta - 2e^{-\pi^2/\beta}), \tag{9a}$$

$$G(\beta) = \sqrt{\frac{\pi}{\beta}} (1 - V\beta + \tfrac{1}{2} V^2\beta^2 - 2e^{-\pi^2/\beta}). \tag{9b}$$

The ITEP authors suggest to estimate $\beta_2$, the upper bound for the fiducial region, from the requirement that neither the first perturbative, nor the first nonperturbative correction should exceed 30 per cent of the leading term. From (9a) this yields

$$\beta < 0.3/|V|; \quad \beta < 5.2. \tag{10}$$

These bounds are expected to ensure that $G(\beta)$ is correct within some 10 per cent, when calculated from (9a). The second perturbative correction is about 4.5 per cent of the main term, thus in this case the bound is very reasonable. The second nonperturbative correction, however, is below 0.11 per cent. Thus, the second bound (10) is much too restrictive.

The recommended approximation for $G_c(\beta)$ is [3]

$$G_c(\beta) = \int_{E_c}^{\infty} \varrho(E) e^{-\beta E} dE, \tag{11}$$

where $E_c$ is a free parameter to be found from identity (4) and the density of states $\varrho(E)$ is found from the "free motion" case

$$\int_0^{\infty} \varrho(E) e^{-\beta E} dE = \tfrac{1}{2} \sqrt{\frac{\pi}{\beta}}. \tag{12}$$

Inverting this relation, one finds $\varrho(E) = E^{-1/2}/2$. Substituting this result into (11)

$$G_c(\beta) = \tfrac{1}{2}\sqrt{\frac{\pi}{\beta}}\,(1-\mathrm{erf}\sqrt{\beta E_c}), \tag{13}$$

where erf($x$) denotes the standard error function (cf. e.g. [5]). This approximation is recommended, when $G_c(\beta)$ does not exceed 30 per cent of $|\psi_0(0)|^2 e^{-\beta E_0}$. Directly from (3) we find numerically

$$\beta > 0.64 \equiv \beta_1. \tag{14}$$

Approximation (13) again is much better than could have been expected from its derivation. It can be shown [4] that it follows from the following two assumptions

1. WKB approximation for the energy levels $E_{n\geqslant 1}$.
2. Conversion of the sum over $n$ in (3) into an integration.

The size of the window defined by formulae (10) and (14) depends on $|V|$. For $|V| > 0.3/0.64$ there is no window at all. When there is a window, we find $E_0$ and $E_c$ ($|\psi_0(0)|$ does not occur in our simple example) from formula (4) with (9) and (13) substituted for $G(\beta)$ and $G_c(\beta)$ at $\beta = \beta_1$ and at $\beta = \beta_2$. The relative erorr $\delta E_0/(E_1 - E_0)$ does not exceed 7.3 per cent for $|V| < 0.35$, when formula (9a) is used. The use of formula (9b) reduces the error to less than 1.5 per cent. Thus the method works very well indeed.

In the example discussed here, the validity of the SVZ method has been checked by comparing the approximation with known exact results. It is controversial (cf. e.g. [3], [6]), whether it is possible to predict a priori the applicability of this method to specific cases. Let us note, however, that even in simple perturbation theory the usual thumb-rule: the method works, when the first nonvanishing correction is small, often fails. For instance consider the effect of a perturbation $\lambda V(x)$ on the ground state of a harmonic oscillator. For $V(x) = $ const, the result is exact, however large is $|\lambda|$, while for $V(x) = -x^4$ the result is completely wrong, however small is $|\lambda|$, provided that $\lambda > 0$.

### 4. *Phenomenology*

In order to use the SVZ method it is necessary to know a reliable weak coupling approximation for the Euclidean Green function. The calculation of perturbative contributions in the framework of QCD is a well established algorithm. It may cost much work, but if done properly it is not controversial. The calculation of nonperturbative corrections, on the other hand, is still a kind of art.

Experts agree that in the limit $m_q \to \infty$ the $q\bar{q}$ system becomes coulombic. The question is: what are the leading corrections? The ITEP group proposed that the leading corrections are due to the interaction of the quarks with the gluon condensate filling the physical vacuum. Practical calculations are performed for the model (SVZ model), where the $q\bar{q}$ system moves in a uniform, stochastically varying chromoelectric field. Thus one calculates the Stark shifts of the energy levels and averages the result over the external field. The leading order corrections to Green functions [1] and to the energy levels [7, 8] are proportional to the matrix element (1). We interpret the SVZ model as pure phenomenology and we will discuss what we think are the crucial experimental tests for it. The ITEP authors, however, present arguments in favour of their model. In the following Section we report these arguments and explain, why we do not find them conclusive.

As a test of the SVZ model, we propose to discuss fine splittings in the spectra of heavy quarkonia. According to Voloshin [7] and Leutwyler [8] for very large quark masses this splitting is proportional to $n^6$, where $n$ is the principal quantum number of the coulombic system. The c and b quarks are too light to expect quantitative agreement between the Voloshin-Leutwyler formulae and data for the $\psi$ and $\Upsilon$ families. The qualitative result that fine splittings rapidly increase with $n$ should have, however, a much wider validity range. On the other hand, simple potential models (cf. e.g. [9]) ascribe fine splittings to a perturbing potential linear in $r$. This leads to fine splittings, which do not depend on $n$.

The experimental data are [10]

$$\Delta_1(c\bar{c}) = (161 \pm 0.6)\ \text{MeV};$$

$$\Delta_1(b\bar{b}) = (122 \pm 3)\ \text{MeV}; \quad \Delta_2(b\bar{b}) = (94 \pm 3)\ \text{MeV}. \tag{15}$$

Here $\Delta_n = E_n(^3S) - E_n(P)$, where $n$ denotes the principal quantum number of the corresponding coulomb system and $P$ stands for the centre of gravity of the $^JP$ states. Data (15) seem to favour the potential picture. This observation should be perhaps correlated with Voloshin's result [11]. Using the SVZ model he found $\Delta_1(b\bar{b}) = (190 \pm 30)$ MeV. Since there was no difficulty with the $n = 0$ states (where of course there is no fine splitting), this overestimate of $\Delta_1$ may reflect a systematic overestimate of the effect on fine splittings of increasing $n$. This according to potential models should be a characteristic feature of the SVZ model.

It should be stressed again at this point that b-quarks are too light to make the data (15) conclusive. Data for $\Delta_2(c\bar{c})$ would show the trend with increasing quark mass, but the data on $\Delta_n(t\bar{t})$ seem necessary to reach a firm conclusion.

We conclude that present data cast some doubt on the SVZ model used to evaluate the nonperturbative contributions to the SVZ sum rules. In order to make a stronger statement, however, it would be necessary to make a more quantitative analysis of the fine splittings predicted by the SVZ model for moderate quark masses. Data for $t\bar{t}$ systems would make this task much easier.

## 5. *The SVZ arguments*

The SVZ procedure for charmonia involves the following steps.

1. From experiment one finds the sum

$$\sum_i \Gamma_i/(M_i)^{n+1}; \quad n = 1, 2, \ldots, \tag{16}$$

where the sum extends over all the $c\bar{c}$ resonances coupled to the photon, $\Gamma_i$ is (approximately) the electronic width of the resonance and $M_i$ its mass. Sum (16) corresponds to $|\psi_0(0)|^2 e^{-\beta E_0}$ from Section 2. The parameter $n$ corresponds to $\beta$.

2. From perturbation theory one finds the moments

$$M_n = \int_0^\infty \frac{R_i(s)ds}{s^{n+1}} \qquad n = 1, 2, ..., \tag{17}$$

where $s$ is the centre of mass energy squared and $R_i$ is the ratio of the cross-section $e^+e^- \to c\bar{c}$ to the cross-section $e^+e^- \to \mu^+\mu^-$ calculated in the lowest Born approximation. This corresponds to $G(\beta)$. Also from perturbation theory one finds the contributions to $M_n$ from the continuum. This corresponds to $G_c(\beta)$.

3. One tests an equation analogous to (4) and finds that it is good for low $n$, but fails for larger $n$. The range of $n$, where the equation is approximately satisfied, can be significantly increased by including into the calculated $M_n$ nonperturbative terms proportional to $G$. This is one argument in favour of the SVZ model. Another argument is that according to the principles of the operator product expansion (OPE) method, the leading nonperturbative corrections should be those, which contain the matrix elements of lowest dimension. The dimension of $G$ is $m^4$, while other matrix elements, which one could try, have dimensions $m^a$ with $a > 4$.

In order to see that the improved fit does not necessarily show that the correction term is correctly introduced, let us consider the function $f = (1-x)^{-1}$. As seen from Fig. 1, the truncated power series (perturbative) approximations rapidly deteriorate with increasing $x$ and little is gained by small increases of the degree of the polynomial. On the other hand the "nonperturbative term" $70.6 \exp(-2.475/x)$ improves the fit very significantly. The reason for this success, however, is not the analytic structure of $f$, but the two free parameters in the "nonperturbative" term. Thus, also in the SVZ case the improvement may well be due to the new free parameter $G$ and not to the realism of their model.

The OPE argument is applicable for $\beta \to 0$. This, however, is outside the fiducial region and has no obvious relevance for the situation considered by the ITEP group.

There has been much work showing that the extension of the OPE argument to the region of interest fails. Thus, Nikolaev and Radyushkin [12] estimated the terms of dimension $m^8$ and found that for the $\psi$ family their contribution is comparable to that of the term proportional to $G$. The ITEP group pointed out [3] that the result [12] cannot be rigorously derived, this, however, is a long way from proving that the $m^8$ terms are negligible as assumed by SVZ. The importance of these terms has been confirmed by Shuryak [13] in a paper supporting SVZ, but advising against the use of OPE. The $m^8$ corrections are reduced by a factor of about 100, when going from the $c\bar{c}$ to the $b\bar{b}$ family. There, however, the SVZ method gives for $G$ results contradicting those for the $\psi$ family and unplausible [14].

By construction, the SVZ model may work, if the gluon condensate has only low frequency oscillations. It must fail, when the oscillation frequencies are high. The important question, of course, is where is the border line between "low" and "high" frequencies. According to Gromes [15], a frequency 100 MeV is enough to invalidate the SVZ picture. Since it is difficult to believe that all the fluctuations of the gluon condensate have frequencies much smaller than $m_\pi$, it seems that a theoretical derivation of the SVZ model would

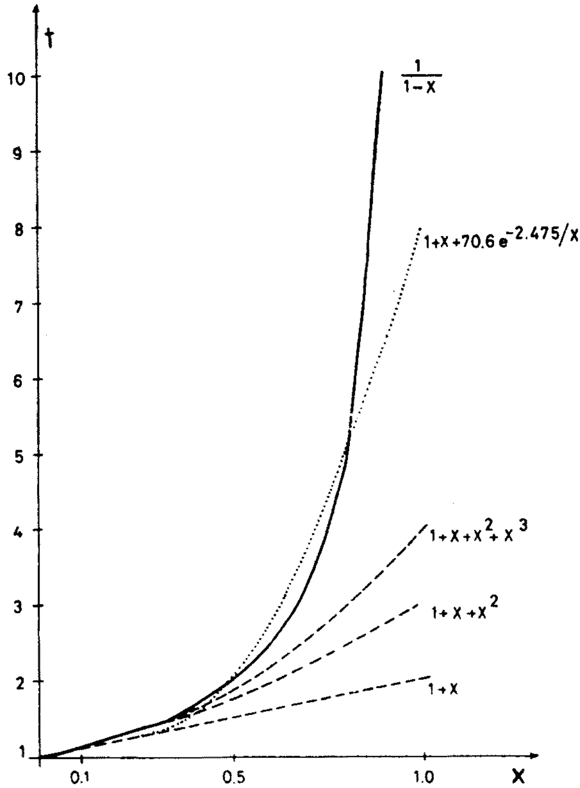


Fig. 1. Function $(1-x)^{-1}$ and its perturbative and nonperturbative approximations

be difficult. Results similar to those of Gromes have been derived using a different method by Baiers and Pinelis [16].

For all these reasons we prefer to discard the presently available theoretical arguments in favour of the SVZ model and to consider this model as a purely phenomenological guess, which needs support from experiment to become credible.

## 6. *Conclusion*

The SVZ method of estimating the matrix element (1) is based on a very nice mathematical idea and on very controversial phenomenology. A study of fine splittings in the spectra of heavy quarkonia seems particularly suitable to settle the point, whether or not the SVZ model is realistic. Present data on fine splittings in the $c\bar{c}$ and $b\bar{b}$ spectra give no support for the model, but a good quantitative analysis of the problem would be of great interest and should be able to eliminate: either the model, or one of its most striking difficulties. For an attempt in this direction cf. [17].

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