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NON-ABELIAN CHIRAL ANOMALIES AND WESS-ZUMINO EFFECTIVE ACTIONS*

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An elementary account is given of the construction of anomalies and effective actions for Goldstone bosons, using the techniques of differential geometry. The emphasis is on simplicity of presentation, comparison of different renormalization schemes and the relationship to bosonization in the case of 2 dimensions.

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CONTENTS

1. *Introduction*

The subject of chiral anomalies [1], abelian and non-abelian [2] as well as their finite counterpart, the Wess-Zumino effective action [3], is far from new. Nevertheless there has been a steady rate of development both in understanding the structural origin of quantum anomalies and in appreciating the regions where these effects may be applied to physical phenomena (for a recent review with many references, see Jackiw's lectures [4]).

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Within the last couple of years this development has been particularly significant. Witten [5, 6] noted a remarkable geometric realization of the Wess-Zumino action and showed how it provided an attractive fermionic interpretation of the old Skyrme-solitons in the Goldstone-Boson field. The ensuing Skyrme-phenomenology of baryons is currently attracting a considerable amount of attention [7, 8].

Polyakov and Wiegmann [9] noted the analogous geometric interpretation of the Wess-Zumino lagrangian in 2 dimensions (see also Ref. [10]). The 2-dimensional model has turned out to be very interesting from the point of view of bosonization [11–14] as well as from other points of view [15].

Meanwhile it was pointed out by Stora [16], Zumino et al. [17], and Chou Kuang-chao et al. [18] (see also Kawai and Tye [19]) how the topological properties of anomalies allowed an essentially complete construction (in any even number of space-time dimension) using the techniques of differential geometry, without having to go through the relatively obscure calculations of 1-loop Feynman diagrams [38, 1, 2]. Also this approach provides a very simple and systematic way of constructing the Wess-Zumino effective action of Goldstone-bosons in a beckground gauge field. The corresponding construction in Ref. [5] was rather intransparent (and contained a few mistakes as has been discussed at length in the literature [18–20]).

The main purpose of these lectures is to give an elementary account of the construction of anomalies and effective actions using the diffrential geometric approach. In particular we discuss the relationship between different renormalization schemes. In fact the scheme mostly employed in Refs. [16–18] is not directly suitable for constructing realistic effective actions for the Goldstone-bosons corresponding to the ordinary pseudoscalar mesons.

There are some very interesting mathematical relations to the Chern-Simons classification of topological invariants. These matters we shall not pursue in any detail (see for example Ref. [33])[1].

In Sect. 2 we set up the problem to be analyzed, that of a set of free Dirac fermions in a non-abelian background gauge field. We distinguish between anomalies and renormalization parts, set up the Wess-Zumino consistency conditions [3] and define the Wess-Zumino effective action.

In Sect. 3 we give a brief treatment of the direct attack on the problem provided by the heat kernel method. We derive some simple results which provide important constraints on the construction in Sect. 5. Also this gives some instructive background insight and allows us to introduce Fujikawa's notion of non-invariance of the quantum measure [21] as the origin of anomalies.

In Sect. 4 we give a short summary of the language of differential geometry.

Sect. 5 contains the central construction of the various objects discussed.

In Sect. 6 we give explicit results in 4 dimensions for purpose of illustration.

In Sect. 7 we discuss the 2-dimensional case, where a complete treatment of the effective action becomes possible, and we indicate the relation to bosonization.

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[1] Very recent further clarification has taken place through works of Alvarez-Gaumé and Ginsparg [42] and Sumitani [43] who stress connections to the index-theorem.

2. *Formulation of the problem*

Let us consider a set of free massless Dirac fermions in a classical non-abelian background in an even number,

$$D = 2n, \tag{1}$$

of space-time dimensions. The action is

$$S = \int d^D x \bar{\psi} \mathbf{D} \psi, \tag{2}$$

where

$$D_\mu \equiv \partial_\mu + V_\mu + \gamma_{D+1} A_\mu, \quad \mathbf{D} = i\gamma_\mu D^\mu, \tag{3}$$

is a covariant derivative. (Although the quantization of the *gauge fields* in general will lead to a renormalizable theory only in $D \leqslant 4$ dimensions, the quantized *Fermi-theory* considered here is renormalizable in all (even) dimensions. It turns out to be instructive to carry out the analysis for general D.) In Minkowski-space

$$\gamma_{D+1} = -i^{n+1} \gamma^0 \gamma^1 \dots \gamma^{D-1} \tag{4}$$

is hermitean. In euclidean space-time it is anti-hermitean. The gauge potentials are matrix-valued fields in the Lie algebra of some group. V_μ and A_μ are the vector- and axial-vector fields[2]. Equivalently we may introduce the left- and right-handed components

$$A_\mathrm{R}^\mu = V^\mu + A^\mu; \quad A_\mathrm{L}^\mu = V^\mu - A^\mu \tag{5}$$

and

$$S = \int d^D x \bar{\psi}_\mathrm{R} \mathbf{D}^\mathrm{R} \psi_\mathrm{R} + \int d^D x \bar{\psi}_\mathrm{L} \mathbf{D}^\mathrm{L} \psi_\mathrm{L}, \tag{6}$$

where

$$D_{\mathrm{R,L}}^\mu = \partial^\mu + A_{\mathrm{R,L}}^\mu \tag{7}$$

act on Weyl-fields ψ_R, ψ_L of definite chirality.

The effective action for the system of quantized Fermi fields in the classical background A_μ^R, A_μ^L is given by

$$e^{iW[A_\mathrm{L}, A_\mathrm{R}]} = \int \mathscr{D}\bar{\psi} \mathscr{D}\psi e^{iS}, \tag{8}$$

where we may regard W as a functional of A_μ^L, A_μ^R or of V_μ, A_μ according to what is most convenient.

The action S is invariant under the $U(N_\mathrm{f}) \times U(N_\mathrm{f})$ transformations

$$\begin{aligned} \psi_\mathrm{L}(x) &\to g_\mathrm{L}^{-1}(x)\psi_\mathrm{L}(x), \qquad \psi_\mathrm{R}(x) \to g_\mathrm{R}^{-1}(x)\psi_\mathrm{R}(x), \\ A_\mathrm{L}^\mu(x) &\to g_\mathrm{L}^{-1}(x) A_\mathrm{L}^\mu(x) g_\mathrm{L}(x) + g_\mathrm{L}^{-1}(x)\partial^\mu g_\mathrm{L}(x) \equiv (A_\mathrm{L}^\mu)^{g_\mathrm{L}}, \\ A_\mathrm{R}^\mu(x) &\to g_\mathrm{R}^{-1}(x) A_\mathrm{R}^\mu(x) g_\mathrm{R}(x) + g_\mathrm{R}^{-1}(x)\partial^\mu g_\mathrm{R}(x) \equiv (A_\mathrm{R}^\mu)^{g_\mathrm{R}}, \end{aligned} \tag{9}$$

where $\psi_{\mathrm{L,R}}$ are regarded as representing a column of N_f (flavour) components and g_L, g_R are elements of $U(N_\mathrm{f})$. In addition we may assume that the Fermi- (quark)- fields possess

[2] $A_\mu = -igT^a A_\mu^a$ where g is a coupling constant, T^a a generator-matrix and A_μ^a is real.

N_c colour degrees of freedom. We take the background flavour gauge fields to be diagonal in colour. For the flavour U(1) field this is unrealistic: the QCD gluon field has non-trivial colour (and is of course flavour diagonal). For simplicity we shall not allow the formalism to explicitly account for that. Thus for some purposes it may be of interest to interpret that mathematical result by interchanging the rôle of colour and flavour.

We shall assume all our fields ($A_{L,R}(x)$, $g_{L,R}(x)$) to have boundary conditions such that in euclidean space-time they are single valued on compactified space. By a stereographic projection this may be mapped on S^D.

Although the action S is invariant under the chiral transformation (9) this is not so of the effective action W:

$$W(A_L^{g_L}, A_R^{g_R}) - W(A_L, A_R) \neq 0 \tag{10}$$

in general. This phenomenon is referred to as the chiral anomaly. As emphasized by Fujikawa [21] it may be blamed on the functional measure $\mathscr{D}\bar{\psi}\mathscr{D}\psi$ not being chirally invariant (see Sect. 3).

For $g_{L,R}$ differing infinitesimally from the unit matrix, Eqs. (9) become

$$\delta A_L^\mu = D_L^\mu v_L \equiv \partial^\mu v_L + [A_L^\mu, v_L]; \qquad \delta A_R^\mu = D_R^\mu v_R \equiv \partial^\mu v_R + [A_R^\mu, v_R],$$

where $v_{L,R}$ are x-dependent infinitesimal scalar matrix valued functions.

When $g_{L,R} \simeq I + v_{L,R}$ we define the anomaly by

$$G(v_L, v_R; A_L, A_R) \equiv \delta_{v_L, v_R} W(A_L, A_R) = W(A_L^{g_L}, A_R^{g_R}) - W(A_L, A_R). \tag{12}$$

From the group property, $(A^{g_1})^{g_2} = A^{(g_1 g_2)}$, follows the Lie algebra property:

$$[\delta_{v_L(1), v_R(1)}, \delta_{v_L(2), v_R(2)}] = \delta_{[v_L(1), v_L(2)], [v_R(1), v_R(2)]} \tag{13}$$

and follow the Wess-Zumino consistency conditions by letting Eq. (13) act on $W(A_L, A_R)$:

$$\begin{aligned} &\delta_{v_L(1), v_R(1)} G(v_L^{(2)}, v_R^{(2)}; A_L, A_R) - \delta_{v_L(2), v_R(2)} G(v_L^{(1)}, v_R^{(1)}; A_L, A_R) \\ &\quad = G([v_L^{(1)}, v_L^{(2)}], [v_R^{(1)}, v_R^{(2)}]; A_L, A_R). \end{aligned} \tag{14}$$

All of these relations may be expressed in terms of V and A if desired.

We can see that the anomalies are related to a non-conservation of currents defined as the response of W to a change in the A's[3]. Thus we define the matrix-valued right-handed current as

$$ij_R^\mu(x) = e^{-iW(A_L, A_R)} \frac{\delta}{\delta A_\mu^R(x)} e^{iW(A_L, A_R)} = i \frac{\delta W}{\delta A_\mu^R(x)} \tag{15}$$

and similarly for the left-handed current or the vector and axial vector currents. Then under a general right-handed change in A^R:

$$\delta_R W(A_L, A_R) = \int d^D x \,\mathrm{tr}\, (j_R^\mu(x) \delta A_\mu^R(x)). \tag{16}$$

[3] These currents differ from covariant ones by local terms. For a very recent discussion see Ref. [45].

If in particular $\delta A^{\mathrm{R}}_\mu$ corresponds to the chiral variation, Eq. (11), we obtain after integrating by parts ($v_{\mathrm{R}} \to 0$ at space-time infinity)

$$\delta_{0,v_{\mathrm{R}}} W(A_{\mathrm{L}}, A_{\mathrm{R}}) = -\int d^D x \operatorname{tr}(v_{\mathrm{R}}(x)(D^{\mathrm{R}}_\mu(x) j^\mu_{\mathrm{R}}(x)), \tag{17}$$

so that

$$G(v_{\mathrm{L}}, v_{\mathrm{R}}; A_{\mathrm{L}}, A_{\mathrm{R}}) = -\int d^D x \operatorname{tr}(v_{\mathrm{R}}(x) D^{\mathrm{R}}_\mu(x) j^\mu_{\mathrm{R}}(x) + v_{\mathrm{L}}(x) D^{\mathrm{L}}_\mu(x) j^\mu_{\mathrm{L}}(x)). \tag{18}$$

Since the fermionic integration, Eq. (8), is Gaussian we get the formal expression for the effective action

$$iW(A_{\mathrm{L}}, A_{\mathrm{R}}) = N_c \operatorname{tr} \log \boldsymbol{D} = \frac{N_c}{2} \operatorname{tr} \log \boldsymbol{D}^2 \tag{19}$$

(N_c being the number of "quark-colours"),

which is not defined until a regularization prescription for the determinant of $\boldsymbol{D}$ has been provided. Such a regularization will give rise to contributions (possibly divergent) depending on the scheme and the regulator. Therefore counterterms must be supplied and the action W is only well defined after renormalization conditions associated with those terms have been imposed. These in turn depend on the physical situation. Such renormalization contributions to W will be D-dimensional integrals over local polynomials in the A's and their derivatives having total dimension D [2, 22, 24, 30]. They give rise to somewhat uninteresting contributions to anomalies: these contributions could be removed by changing to a different renormalization condition. The point about the chiral anomalies is that they contain contributions which *cannot* be described by the variation of a renormalization term in W, they must be present in any renormalization scheme. They may, however, [16–18] be obtained as chiral variations of $(D+1)$-dimensional integrals over local polynomials. This we shall see in Sect. 5.

For $D > 2$ the full determinant of $\boldsymbol{D}$ cannot be obtained in analytic form. It is possible nevertheless to calculate the *chiral variations* of the determinant: the anomalies. This we shall see in Sects. 3 and 5.

Therefore it is also possible to calculate the change in W under a finite chiral transformation, since we may think of the finite transformation as built up of a sequence of infinitesimal ones, and each of those is given by the known anomaly. This was emphasized by Wess and Zumino [3] and we may define their effective action as

$$WZ(g_{\mathrm{L}}, g_{\mathrm{R}}; A_{\mathrm{L}}, A_{\mathrm{R}}) = W(A_{\mathrm{L}}, A_{\mathrm{R}}) - W(A^{g_{\mathrm{L}}}_{\mathrm{L}}, A^{g_{\mathrm{R}}}_{\mathrm{R}}). \tag{20}$$

In general the calculational procedure proposed by Wess and Zumino is very complicated. It was carried out in the 2-dimensional case in Ref. [10]. As pointed out by Witten [5], however, there is a remarkably simple expression for the result in terms of $D+1$-dimensional integrals. The precise relation will be treated in Sect. 5. Notice that 'fields" $g_{\mathrm{L}}(x)$ and $g_{\mathrm{R}}(x)$ can be reached continuously from the identity provided π_D — the D'th homotopy group — is trivial. This is the case for the $U(N_f)$ and $\mathrm{SU}(N_f)$ group in which we shall mostly be interested (when $N_f > D/2$ [14], a counter example is $N_f = 2$ and $D = 4$ for which π_4 (SU(2)) $= Z_2$, cf. Refs. [5, 6] and Sects. 5, 6). Even then, however, there arises a crucial question of global consistency [5, 6, 20]. To this we come back in Sect. 5.

Here we briefly indicate the physical interpretation of the Wess-Zumino effective action [3] (see also Refs. [23–25]).

In the real world global chiral symmetry (quark masses are ignored) is supposed to be spontaneously broken: the vacuum "points" in some chiral direction. A long wave-length Goldstone boson excitation is described by letting the chiral direction of the "vacuum" slowly depend on the space-time point x. This is just what Eq. (20) expresses: the Wess-Zumino lagrangian is an effective description of chiral Goldstone boson fields $(g_L(x), g_R(x))$ interacting with a background gauge field.

If WZ depends in a non-trivial way on all the components of $(g_L(x), g_R(x))$, the chiral symmetry is completely broken. In nature instead diagonal flavour transformations seem to survive the spontaneous symmetry breakdown.

If WZ is independent of diagonal vector-gauge-transformations ($g_L = g_R$) it means that WZ can depend only on the combination

$$U(x) \equiv g_L(x) g_R^{-1}(x). \tag{21}$$

This means that we shall be particularly interested in W's defined in a renormalization scheme where vector gauge invariance is preserved. This scheme we denote the A-(axial) scheme [19]. We take $U(x)$ to be an element of SU(N_f) since we do not want to introduce an axial U(1) Goldstone boson [26, 27].

In Refs [16–18] a different scheme was considered — mostly for simplicity. Naively one might expect the path-integral, Eq. (8), to factorize into similar pieces depending on right-handed fields and left-handed fields only. As we shall see this is not true in general — it is not true in the A-scheme. It *is* possible, however, to impose this property as a renormalization condition. In that scheme — the (LR)-scheme — we have

$$W_{LR}(A_L, A_R) = \tilde{W}(A_R) - \tilde{W}(A_L) \tag{22}$$

(the origin of the minus sign will be clear in Sect. 5).

In any scheme we can see that the response of the Wess-Zumino action, Eq. (20), to the combined chiral transformation

$$A_L \to A_L^{f_L}, \quad A_R \to A_R^{f_R}, \quad g_L \to f_L^{-1} g_L, \quad g_R \to f_R^{-1} g_R \quad \text{or} \quad U \to f_L^{-1} U f_R \tag{23}$$

in the A-scheme, is the same as the response of the underlying quark-theory. Indeed the last term in Eq. (20) is trivially invariant under the transformations, Eq. (23), since

$$A_L^{g_L} \to (A_L^{f_L})^{f_L^{-1} g_L} = A_L^{g_L} \text{ etc.}$$

Hence the response of $WZ(g_L, g_R; A_L, A_R)$ is the same as the response of $W(A_L, A_R)$.

An important difference between the effective action and the underlying Fermi-theory is that the non-invariance under chiral transformations is hidden in the quantum measure in the Fermi-theory, but made explicit in the effective theory. Thus, in a quantization of the Goldstone-boson modes, a chirally *invariant* measure should be employed so as to not create additional anomalies.

Finally we should emphasize, that if the background gauge fields are to be considered as parts of a bigger theory in which these are quantized (path-integrated over) with some gauge-invariant Yang-Mills interaction, then this full theory can be consistent only if the anomalies considered here are cancelled, for example by additional fermions, say leptons. It follows that any effective theory obtained by integrating out the quarks in favour of meson degrees of freedom, must respond to chiral transformations the same way the quark theory did (see for instance Jackiw [4] for a detailed discussion).

3. *The heat kernel method* [22, 28–30]

Let us attempt a direct calculation of the fermionic determinant in euclidean space--time:

$$e^{-W} = (\det \boldsymbol{D})^{N_c} = \exp\left\{\frac{N_c}{2}\,\mathrm{tr}\log \boldsymbol{D}^2\right\}.$$

Using the "proper-time" expression for the logarithm of an eigenvalue this gives[4]

$$W[V, A] = \frac{N_c}{2}\int_{\varepsilon}^{\infty}\frac{ds}{s}\,\mathrm{tr}\,(e^{-s\boldsymbol{D}^2}), \tag{24}$$

where $\varepsilon \to 0^+$ is the proper time cut off. As emphasized in the preceding section, we consider gauge fields that are single valued on compactified euclidean space S^D. Then there are no zero modes of the Dirac operator and Eq. (24) may be used. For gauge fields with more complicated (instanton-like) topology, zero modes are present. However, then the partition function becomes zero due to the integration rule for Grassmann numbers ($\int \mathscr{D}\psi_0 = 0$, ψ_0 zero mode of $\boldsymbol{D}$). Nevertheless it is possible to analyze the divergence of the axial current also in that case. This in fact leads to an alternative elegant way of normalizing the anomalies (see for example Refs. [17, 21, 32, 42]). If we write (replacing γ_{D+1} by $i\gamma_{D+1}$ in the euclidean case, cf. Eq. (4))

$$D_\mu = \partial_\mu + \mathscr{A}_\mu, \qquad \mathscr{A}_\mu = V_\mu + i\gamma_{D+1}A_\mu \tag{25}$$

we get for a general change in $\mathscr{A}_\mu$ (using the operator identity,

$$\delta \boldsymbol{D}^2 = \delta\underline{\mathscr{A}}\boldsymbol{D} + \boldsymbol{D}\delta\underline{\mathscr{A}}, \qquad \text{and} \qquad [\boldsymbol{D}, e^{-s\boldsymbol{D}^2}] = 0$$

(where $\boldsymbol{D} = i\gamma_\mu D^\mu$ and $\underline{\mathscr{A}} = i\gamma_\mu\mathscr{A}^\mu$)

as well as the cyclic property of the trace),

$$W[\mathscr{A}] = -N_c\int_{\varepsilon}^{\infty} ds\,\mathrm{tr}\,(e^{-s\boldsymbol{D}^2}\boldsymbol{D}\delta\underline{\mathscr{A}}).$$

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[4] $\log\lambda = -\int_{\varepsilon}^{\infty} ds/s\,e^{-\lambda s} + \text{const.}$, for $\lambda > 0$.

For an arbitrary chiral variation we may write (cf. Eq. (11))

$$\delta\mathscr{A}_\mu = [D_\mu, \alpha] + i[D_\mu, \beta]\gamma_{D+1} \tag{26}$$

where $\alpha(x)$ and $\beta(x)$ are matrix-valued functions, but Dirac and Lorentz-scalars, and where the operator D_μ is given in Eq. (25). For $\beta = 0$ this gives first (using again $[\boldsymbol{D}, e^{-s\boldsymbol{D}^2}] = 0$ and the cyclic property of the trace)

$$\delta_V W = -\int_\varepsilon^\infty ds \operatorname{tr}(e^{-s\boldsymbol{D}^2}\boldsymbol{D}(\boldsymbol{D}\alpha - \alpha\boldsymbol{D})) = 0. \tag{27}$$

Thus the regularization scheme, Eq. (24), automatically gives us a vector gauge invariant definition of the determinant: we are in the A-scheme.

Then for $\beta \neq 0$ we find

$$\delta_A W = -iN_c \int_\varepsilon^\infty ds \operatorname{tr}(e^{-s\boldsymbol{D}^2}\boldsymbol{D}(\boldsymbol{D}\beta\gamma_{D+1} - \beta\boldsymbol{D}\gamma_{D+1})) = -2iN_c \int_\varepsilon^\infty ds \operatorname{tr}(e^{-s\boldsymbol{D}^2}\boldsymbol{D}^2\beta\gamma_{D+1})$$

$$= +2iN_c \int_\varepsilon^\infty ds \frac{d}{ds} \operatorname{tr}(e^{-s\boldsymbol{D}^2}\beta\gamma_{D+1}) = -2iN_c \operatorname{tr}(e^{-\varepsilon\boldsymbol{D}^2}\beta\gamma_{D+1}). \tag{28}$$

This expression is interesting in connection with Fujikawa's way of looking at the anomaly [27, 31].

Under the axial gauge transformation which we consider, the fermionic quantum action is unchanged provided we change variables

$$\psi \to e^{i\beta\gamma_{D+1}}\psi, \quad \bar\psi \to \bar\psi e^{i\beta\gamma_{D+1}}.$$

But under that change of variables the quantum measure changes by a Jacobian (the rule for changing Grassmann-variables is "upside-down" compared to the rule for ordinary variable change)

$$\mathscr{D}\bar\psi\mathscr{D}\psi \to \mathscr{D}\bar\psi\mathscr{D}\psi(\det e^{2i\beta\gamma_{D+1}})^{-N_c}.$$

This precisely corresponds to Eq. (28) in the formal limit $\varepsilon \to 0$. Eq. (28) represents the change in the measure in the proper-time regularization.

Eq. (28) involves the heat kernel

$$h(x, y; s) \equiv e^{-s\boldsymbol{D}^2}(x, y)$$

satisfying the "heat equation"

$$\partial_s h = -\boldsymbol{D}^2 h, \quad h(x, y; 0) = \delta^D(x-y). \tag{29}$$

When $\mathscr{A}_\mu \equiv 0$, the solution is a Gaussian

$$h_0(x, y; s) = (4\pi)^{-n}s^{-n}\exp(-(x-y)^2/4s), \quad D = 2n. \tag{30}$$

In general h may be found by the ansatz

$$h = h_0 \times \sum_l a_l(x, y)s^l$$

and the heat equation allows the coefficients a_l to be determined by recursion from the equation

$$((x-y)_\mu D_\mu + l)a_l(x, y) = -D^2 a_{l-1}(x, y). \tag{31}$$

Calculations quickly become rather involved. However, the simple case of no axial field, $A_\mu \equiv 0$ is nearly trivial due to the γ-matrix algebra

$$\operatorname{tr}(\gamma_{D+1}\gamma_{\mu_1}\cdots\gamma_{\mu_{2l}}) = \begin{cases} 0 & \text{for} \quad l < n = D/2 \\ (-)^{n+1}\, i^{n+1}\, 2^n\, \varepsilon^{\mu_1\cdots\mu_D} & \text{for} \quad l = n \end{cases} \tag{32}$$

(cf. Eq. (4); Eq. (32) holds in euclidean space-time). Then the first coefficient contributing becomes a_n and the terms with $l > n$ vanish in the limit $\varepsilon \to 0$ (Eq. (30)). Using the normalization condition $a_0(x, x) = 1$, Eq. (29), we get from Eq. (31)

$$a_n(x, x) = \frac{1}{n!}\left(\tfrac{1}{4}[\gamma_\mu, \gamma_\nu]F_{\mu\nu}\right)^n + \ldots,$$

where the dots denote terms not contributing in the trace, Eq. (28), and where we used

$$-D^2 = \tfrac{1}{2}\{\gamma_\mu, \gamma_\nu\}\,\tfrac{1}{2}\{D_\mu, D_\nu\} + \tfrac{1}{2}[\gamma_\mu, \gamma_\nu]\,\tfrac{1}{2}[D_\mu, D_\nu].$$

Putting all this together gives [17, 38]

$$\delta_A W = N_c(-)^n i^{n+1}\frac{2}{(4\pi)^n n!}\,\varepsilon^{\mu_1\cdots\mu_D}\operatorname{tr}(\beta F_{\mu_1\mu_2}\cdots F_{\mu_{D-1}\mu_D}) \tag{33}$$

when the axial field $A_\mu \equiv 0$.

When $A_\mu \neq 0$ the calculation becomes far more involved [30]. We shall derive (the relevant part of) the result in Sect. 5. But it is instructive to discuss it already here. One finds in 4 dimensions [30]

$$\delta_A W = -\frac{N_c i}{16\pi^2}\operatorname{tr}(\beta(G^{(1)}+G^{(2)})),$$

$$\begin{aligned} G^{(1)} &= 4\varepsilon_{\mu\nu\varrho\sigma}\big[\tfrac{1}{4}V_{\mu\nu}V_{\varrho\sigma} + \tfrac{1}{12}A_{\mu\nu}A_{\varrho\sigma} - \tfrac{2}{3}(A_\mu A_\nu V_{\varrho\sigma} + A_\mu V_{\nu\varrho}A_\sigma \\ &\qquad + V_{\mu\nu}A_\varrho A_\sigma) + \tfrac{8}{3}A_\mu A_\nu A_\varrho A_\sigma\big], \\ G^{(2)} &= 2\big[\tfrac{4}{3}\{D^V_\mu A_\nu + D^V_\nu A_\mu, A_\mu A_\nu\} - \tfrac{2}{3}\{D^V_\mu A_\mu, A^2\} + \tfrac{4}{3}[A_\mu, D^V_\lambda V_{\mu\lambda}] \\ &\qquad - \tfrac{1}{3}[A_{\mu\lambda}, V_{\mu\lambda}] + \tfrac{2}{3}D^V_\varrho D^V_\varrho(D^V_\mu A_\mu) + 4A_\lambda(D^V_\mu A_\mu)A_\lambda\big]. \end{aligned} \tag{34}$$

Here { , } denotes the anticommutator and

$$\begin{aligned} V_{\mu\nu} &\equiv \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] + [A_\mu, A_\nu], \\ A_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [V_\mu, A_\nu] - [V_\nu, A_\mu], \\ D^V_\mu f &\equiv \partial_\mu f + [V_\mu, f]. \end{aligned} \tag{35}$$

The contribution from $G^{(1)}$ was given by Bardeen [2] on the basis of Feynman diagram calculations. The contribution from $G^{(2)}$ is a renormalization part since it may be derived from a local counter term in 4 dimensions [2, 30]:

$$\Delta S = 2 \int d^4x \, \mathrm{tr} \, \{\tfrac{2}{3} (D_\mu^V A_\nu)^2 - (D_\mu^V A_\mu)^2 - \tfrac{2}{3} [A_\mu, A_\nu]^2 + \tfrac{2}{3} A_\mu A_\nu A_\mu A_\nu + \tfrac{1}{2} V_{\mu\nu}^2\} \tag{36}$$

(which itself is vector gauge invariant).

In contrast $G^{(1)}$ or Eq. (33) *cannot* be obtained in this way (see Sect. 5).

Notice also that the $G^{(1)}$ term contains an even number of A-fields whereas $G^{(2)}$ contains an odd number of A-fields. This property follows from the parity invariance of $W(A_L, A_R)$:

$$\vec{x} \to -\vec{x}, \quad t \to t$$

$$A_L^\mu \leftrightarrow A_R^\mu \quad (\text{i.e. } V^\mu \to V^\mu, \quad A^\mu \to -A^\mu). \tag{37}$$

Since β is a pseudoscalar, it follows that the genuine anomaly parts must involve the ε-symbol and be even under $A_L \leftrightarrow A_R$. Correspondingly the Wess-Zumino action is *odd* under $A_L \leftrightarrow A_R$ and $U \leftrightarrow U^{-1}$ (also cf. Ref. [5]). *Thus by studying anomalies from a general point of view we can only hope to learn about those parts of the action which have this property.*

In a realistic effective action for Goldstone bosons (with or without background fields), there will in general be terms even under $U \leftrightarrow U^{-1}$ ($A_L \leftrightarrow A_R$). These represent perhaps the most important physical phenomena. Examples are:

$$-\frac{F_\pi^2}{16} \int d^4x \, \mathrm{tr} \, (\partial_\mu U \partial_\mu U^{-1}), \qquad U = \exp\left[\frac{2i}{F_\pi} \pi(x)\right], \tag{38}$$

($\pi(x)$ is the matrix-valued Goldstone boson field and $F_\pi = 190$ MeV is the PCAC-constant). This is the chiral action describing low energy current algebra.

$$\int d^4x \, \mathrm{tr} \, ([U^{-1}\partial_\mu U, U^{-1}\partial_\nu U]^2), \tag{38'}$$

is the Skyrme-term stabilizing the classical Skyrme-soliton solutions of Eq. (38). Similarly the Wess-Zumino-action derived from Eq. (36) belongs to this category. These important contributions cannot be derived from simple considerations of anomalies. They describe processes involving an even number of Goldstone bosons.

In contrast the anomaly parts give information about processes involving an odd number, such as the famous examples $\pi^0 \to 2\gamma$, $K^+K^- \to \pi^+\pi^0\pi^-$, $\gamma\pi^+\pi^-\pi^0$ vertices etc.

4. *Notation of differential geometry*

Here we give the briefest possible description of the language of forms which we shall employ (see also Refs. [33, 17]).

Antisymmetric matrix-valued tensor fields of rank p are made into matrix-valued p-forms by contracting with p antisymmetrized dx's:

$$\omega_p = \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}.$$

Antisymmetrized direct products are usually denoted by wedge-products, $\omega_p \wedge \omega_q$ etc. However, a very convenient way of avoiding the book keeping of combinatorics and antisymmetrization consists in treating the dx's formally as Grassmann-variables [17] until at the very end of the calculation. This we shall do.

Scalars are denoted 0-forms. The vector-gauge-field becomes the 1-form

$$A = A_\mu dx^\mu$$

and the 1-form:

$$d = dx^\mu \partial_\mu$$

denotes the exterior derivative for which

$$d^2 = 0$$

since $\partial_\mu \partial_\nu$ is symmetric and $dx^\mu dx^\nu$ is antisymmetric. Acting on a product of forms, d acts by the rules of anti-derivation:

$$d(AB) = (dA)B + (-)^p A(dB)$$

if A is a p-form. Also this rule follows immediately when the dx's are treated as Grassmann numbers.

The identity (U is a matrix-valued field)

$$dU^{-1} = -U^{-1} dU U^{-1}$$

will be used throughout.
From the cyclic property of the trace of matrices follows

$$\operatorname{tr}(AB) = \pm \operatorname{tr}(BA),$$

when B and A are forms, and where the sign is minus, if both forms are odd ("fermionic"). Similarly the commutator of two forms is defined by

$$[A, B] = AB \pm BA$$

with + if both forms are odd. Then the trace of a commutator always vanishes and for the 1-form, $A = A_\mu dx^\mu$

$$[A, A] = 2A^2.$$

Also then, d and $[A, \cdot]$ act the same way on polynomials. The field strength 2-form is given by

$$F = dA + A^2 = dx^\mu dx^\nu \partial_\mu A_\nu + dx^\mu dx^\nu A_\mu A_\nu$$

$$= \tfrac{1}{2} dx^\mu dx^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) = \tfrac{1}{2} dx^\mu dx^\nu F_{\mu\nu}. \tag{39}$$

Defining the covariant derivative 1-form by

$$Df = df + [A, f]$$

we find the Bianchi-identity

$$DF = 0 \tag{40}$$

since

$$dF = dA^2 = dA \cdot A - AdA = -[A, dA] = -[A, dA + A^2] = -[A, F].$$

A p-form in D space time dimensions vanishes identically if $p > D$.

Let Σ_{p+1} be a $p+1$ dimensional differential orientable manifold with boundary Σ_p, a p-dimensional orientable manifold. And let ω_p be a p-form. Then Stoke's theorem says

$$\int_{\Sigma_{p+1}} d\omega_p = \int_{\Sigma_p} \omega_p. \tag{41}$$

Here the substitution, for example $dx_{\mu_1} \ldots dx_{\mu_p} \to \varepsilon_{\mu_1 \ldots \mu_p} d^pV$ (d^pV the volume element of Σ_p) should be made. If the $p+1$ form ω_{p+1} is a total derivative of a p-form:

$$\omega_{p+1} = d\omega_p \quad (\omega_{p+1} \text{ is exact})$$

then

$$d\omega_{p+1} = d^2\omega_p = 0 \quad (\omega_{p+1} \text{ is closed}). \tag{42}$$

The converse is also true locally (Poincaré's lemma): if $d\omega_{p+1} = 0$, then locally there exists a p-form such that Eq. (42) holds.

5. *Differential geometric construction of anomalies and Wess-Zumino actions*

Finally we come to the central section[5]. According to the discussion of the end of Sect. 3 we are looking for anomalies in D-dimensions of the form

$$G(v_L, v_R; A_L, A_R) = c_n \int_{S^D} \omega_D^1(v_L, v_R; A_L, A_R), \tag{43}$$

where ω_D^1 is a D-form linear in the chiral variations v_L and v_R, and a polynomial of dimension D in A's and their derivatives; c_n is a normalization constant. As before, we assume boundary conditions on the fields at infinity such that they are well defined on compactified space. Thus in euclidean space E^D we may use coordinates on the D-dimensional unit sphere S^D obtained by a stereographic mapping.

These anomalies must satisfy the Wess-Zumino consistency conditions, Eq. (14). This is ensured provided they can be derived from a generating functional

$$G(v_L, v_R; A_L, A_R) = \delta_{v_L, v_R} W^0(A_L, A_R), \tag{44}$$

where W^0 may differ from the true effective action by terms that are completely chirally invariant. The consistency conditions do not in any way constrain the form of W^0. However,

[5] Our treatment simplifies and generalizes somewhat Refs. [16, 19]. Also cf. Ref. [33] (Sects. 6.1 and 8.3 in particular).

as we have argued, ω_D^1 should be a local Lorentz-invariant polynomial of dimension D and

$$W^0(A_L, A_R) = -W^0(A_R, A_L). \tag{45}$$

These requirements very strongly limit the possible forms of the G's. (Nevertheless, the uniqueness aspect of the following construction seems to need further clarification; for discussion cf. Refs. [39–41]). In fact we shall see that the requirements are fulfilled by constructing W^0 as a certain $D+1$-dimensional integral

$$W^0(A_L, A_R) = c_n \int_{B_{D+1}} \omega_{D+1}^0(A_L, A_R), \tag{46}$$

where ω_{D+1}^0 is a $D+1$ form in $D+1$ dimensions satisfying

$$\omega_{D+1}^0(A_L, A_R) = -\omega_{D+1}^0(A_R, A_L). \tag{47}$$

B_{D+1} may be thought of as the $D+1$ dimensional unit ball having S^D as its boundary. This construction implies that we extend the fields $A_L(x)$ and $A_R(x)$ from S^D to B_{D+1} in some "arbitrary" way. We shall see that the whole construction finally becomes consistent.

First we must demand that

$$\delta_{v_L,v_R}\omega_{D+1}^0(A_L, A_R) = d\omega_D^1(v_L, v_R; A_L, A_R) \tag{48}$$

in order that the anomaly becomes

$$\begin{aligned}\delta_{v_L,v_R} W^0(A_L, A_R) &= c_n \int_{B_{D+1}} d\omega_D^1(v_L, v_R; A_L, A_R)\\ &= c_n \int_{S^D} \omega_D^1(v_L, v_R; A_L, A_R) = G(v_L, v_R; A_L, A_R),\end{aligned} \tag{49}$$

and thus lives in D-dimensions the way it should.

Remarkably these requirements lead to a nearly unique construction of ω_{D+1}^0 up to renormalization terms. In fact, consider extending the definition of A_L and A_R one more dimension — into $D+2$ dimensional space. Then define in $(D+2)$ dimensions

$$\Omega_{D+2}(A_L, A_R) = d\omega_{D+1}^0(A_L, A_R). \tag{50}$$

This is a $(D+2)$-form, odd under $A_L \leftrightarrow A_R$ and exact. Further, Ω_{D+2} is completely chirally invariant. In fact

$$\delta_{v_L,v_R}\Omega_{D+2} = d\delta_{v_L,v_R}\omega_{D+1}^0 = d^2\omega_D^1 = 0.$$

An obvious solution which we shall soon analyze is [17, 19][6]

$$\begin{aligned}\Omega_{D+2}(A_L, A_R) &= \mathrm{tr}\,(F_R^{n+1}) - \mathrm{tr}\,(F_L^{n+1})\\ F_L = dA_L + A_L^2; &\quad F_R = dA_R + A_R^2.\end{aligned} \tag{51}$$

[6] Ω_{D+2} is related to the Chern character and ω_{D+1}^0 to the Chern-Simons secondary form (cf. Ref. [33] Sects. 6.1 and 8.3 and [16, 17]).

This is not the most general solution. In fact any polynomial of dimension $D+2$ built from tr (F^p) and tr (F^q) factors, odd in $L \leftrightarrow R$ will do [33]. As we shall soon see, however, the form Eq. (51) is required by comparison with our heat kernel calculation, Eq. (33), for $A_L = A_R$.

In Refs. [16, 17] the similarity of Ω_{D+2} to the global, abelian, pure vector anomaly Eq. (33) was emphasized but the connection remained mysterious. Very recently Alvarez--Gaumé and Ginsparg [42] (see also Sumitani [43]) have clarified the connection: A_L (and A_R) have to be reinterpreted in $D+2$ dimensions as *vector* fields coupling to Dirac fermions of both chiralities in $D+2$ dimensions. The corresponding global, abelian anomaly may be analyzed using the Atiyah-Singer index-theorem, and a relation to the local, pure left Weyl fermion, non-abelian anomaly in D dimensions is established.

Now we must solve Eq. (50) for ω^0_{D+1}. Clearly a solution is only determined up to an additive term of the form $d\varrho_D(A_L, A_R)$ so that

$$\int\limits_{B_{D+1}} \omega^0_{D+1} \to \int\limits_{B_{D+1}} \omega^0_{D+1} + \int\limits_{S^D} \varrho_D,$$

showing that ϱ_D has the form of a renormalization contribution.

A particular solution ω^0_{D+1} may be constructed as follows: for any space time point x, let γ be a curve in field space connecting A_L and A_R

$$\gamma: t \in [t_a, t_b] \to A(t) \text{ s.t. } A(t_a) = A_L, \quad A(t_b) = A_R. \tag{52}$$

Then

$$(F(t) = dA(t) + A(t)^2, \quad \dot{A}(t) = \frac{d}{dt} A(t), \text{ etc.}).$$

$$\Omega_{D+2} = \text{tr}\,(F_R^{n+1} - F_L^{n+1}) = \int\limits_{t_a}^{t_b} dt \frac{d}{dt} \text{tr}\,(F(t)^{n+1})$$

$$= (n+1) \int\limits_{t_a}^{t_b} dt \,\text{tr}\,(\dot{F}(t)F(t)^n) = (n+1) \int\limits_{t_a}^{t_b} dt \,\text{tr}\,[(d\dot{A}(t) + \dot{A}(t)A(t) + A(t)\dot{A}(t))F(t)^n]$$

$$= (n+1) \int\limits_{t_a}^{t_b} dt \,\text{tr}\,[d\dot{A}(t)F^n(t) + \dot{A}(t)\,[A(t), F^n(t)]] = (n+1) \int\limits_{t_a}^{t_b} dt \,\text{tr}\,[d\dot{A}(t)F^n(t) - \dot{A}(t)dF^n(t)],$$

where we used the cyclic property of the trace and the Bianchi identity Eq. (40) for $F(t)$ and $F^n(t)$ (both d and $[A, \cdot]$ act the same way on products). But then we have (cf. Sect. 4)

$$\Omega_{D+2}(A_L, A_R) = (n+1) \int\limits_{t_a}^{t_b} dtd \,\text{tr}\,[\dot{A}(t)F^n(t)]$$

giving the particular solution

$$\omega^0_{D+1}(A_L, A_R; \gamma) \equiv (n+1) \int\limits_{t_a}^{t_b} dt \,\text{tr}\,[\dot{A}(t)F^n(t)], \tag{53}$$

where we have indicated that the result depends on the integration path (52).

Different integration paths correspond to different renormalization schemes. The difference between ω^0_{D+1}'s taken along two different paths is a total derivative by Poincaré's lemma since d acting on the integral along a closed path is $\mathrm{tr}\,(F^{n+1}-F^{n+1}) = 0$. We shall derive explicit expressions below and show it is in fact exact. In contrast Ω_{D+2} is independent of the renormalization scheme.

Let us reserve the name $\omega^0_{D+1}(A_\mathrm{L}, A_\mathrm{R})$ (with no path indicated) for the integral along the straight line (Fig. 1)

Fig. 1. Integration path for the generating functional $\omega^0_{D+1}(A_\mathrm{L}, A_\mathrm{R})$ for anomalies in the vector gauge invariant scheme (*A*)

$$\omega^0_{D+1}(A_\mathrm{L}, A_\mathrm{R}) \equiv (n+1) \int_{-1}^{1} dt\, \mathrm{tr}\,(AF^n(t)), \tag{54}$$

where $A(t) = V+tA$, V and A being the vector and axial vector fields

$$V = \tfrac{1}{2}(A_\mathrm{R}+A_\mathrm{L}), \quad A = \tfrac{1}{2}(A_\mathrm{R}-A_\mathrm{L}).$$

As pointed out by Kawai and Tye [19] this corresponds to the vector gauge invariant renormalization scheme (*A*). Indeed under a pure vector transformation ($v_\mathrm{L} = v_\mathrm{R} = v$)

$$\delta_V V = dv+[V, v], \quad \delta_V A = [A, v], \quad \delta_V A(t) = dv+[A(t), v] \equiv D_t v,$$

so that $\delta_V F(t) = [F(t), v]$, whence

$$\delta_V \omega^0_{D+1}(A_\mathrm{L}, A_\mathrm{R}) = (n+1) \int_{-1}^{1} dt\, \mathrm{tr}\,\{[A, v]F^n(t)+A[F^n(t), v]\}$$

$$= (n+1) \int_{-1}^{1} dt\, \mathrm{tr}\,[AF^n(t), v] = 0.$$

Next let us consider a pure axial transformation $v_\mathrm{R} = -v_\mathrm{L} \equiv v$, and let us calculate the anomaly for $A = 0$ where $A_\mathrm{L} = A_\mathrm{R}$ and the integration over t becomes trivial. In fact

$$\delta A = dv+[V, v] = Dv \quad \text{for} \quad A = 0,$$

so that

$$\delta_A \omega^0_{D+1}(V, V) = 2(n+1)\, \mathrm{tr}\,[DvF^n] = 2(n+1)d\, \mathrm{tr}\,[vF^n],$$

where we used $DF = 0$ and $\mathrm{tr}\,[V, vF^n] = 0$. Then

$$\omega^1_D(-v, +v; V, V) = 2(n+1)\, \mathrm{tr}\,[vF^n]$$

and

$$G(-v, v; V, V) = c_n 2(n+1) \int_{S^D} \mathrm{tr}\,[vF^n]. \tag{55}$$

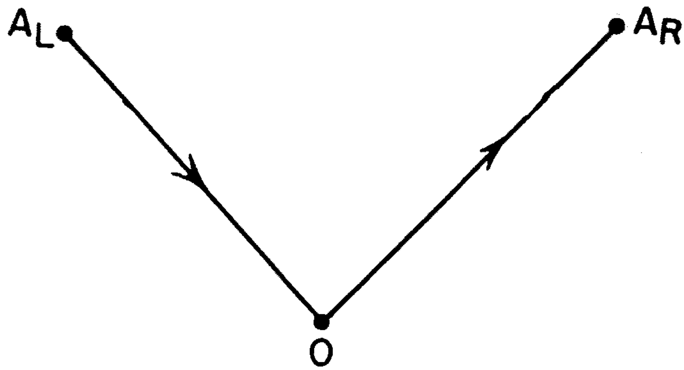

Fig. 2. Integration path for the generating functional $\omega^0_{D+1}(0, A_R) - \omega^0_{D+1}(0, A_L)$ for anomalies in the LR-scheme

This agrees with Eq. (33) and determines the normalization to be[7]

$$c_n = (-i)^{n+1} \frac{N_c}{(2\pi)^n (n+1)!} \tag{56}$$

remembering $F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$.

It is now easy to see that the extra terms

$$\operatorname{tr}(F_R^p - F_L^p) \operatorname{tr}(F_R^{n+1-p} + F_L^{n+1-p}) \text{ etc. in } \Omega_{D+2}$$

must have vanishing coefficients (cf. the discussion after Eq. (51)).

In the LR-*scheme* of Ref. [17] where the effective action separates in terms depending only on A_L and A_R respectively, we have to integrate through the zero field as indicated in Fig. 2, and we evidently have

$$\omega^0_{D+1}(A_L, A_R; \mathrm{LR}) = \omega^0_{D+1}(A_L, 0) + \omega^0_{D+1}(0, A_R)$$

or

$$\omega^0_{D+1}(A_L, A_R; \mathrm{LR}) = \omega^0_{D+1}(0, A_R) - \omega^0_{D+1}(0, A_L) \tag{57}$$

explaining the sign in Eq. (22).

ω^0_{D+1} in the (A)-scheme differs from ω^0_{D+1} in the LR-scheme by a renormalization term which we shall denote [19] $d\varrho_D(0, A_L, A_R)$ and which is given by an integral along the triangle $0A_RA_L$ in *A-space*. This object turns out to be very useful for all our remaining purposes so we consider the slightly more general situation depicted in Fig. 3

$$d\varrho(A_0, A_1, A_2) = (n+1) \oint d\tau \operatorname{tr}\{A'(\tau)F^n(\tau)\}, \tag{58}$$

where τ is a parameter on the triangle and $A'(\tau)$ denotes the rate of change in $A(\tau)$ along the perimeter. We evaluate this using a slight generalization of a trick given by Zumino [17] in a special case.

[7] Going from euclidean to Minkowski space ($-W \to iW$, $i\gamma_{D+1} \to \gamma_{D+1}$) we pick up an extra sign.

Define a field inside the triangle by

$$A(s,t) = A_0 + sA_1 + tA_2, \qquad s,t \in [0,1]$$

and

$$F(s,t) = dA(s,t) + A^2(s,t).$$

The integral in Eq. (58) involves the circulation of the two-component vector field

$$(\mathrm{tr}\,(A_1F^n(s,t)), \mathrm{tr}\,(A_2F^n(s,t)))$$

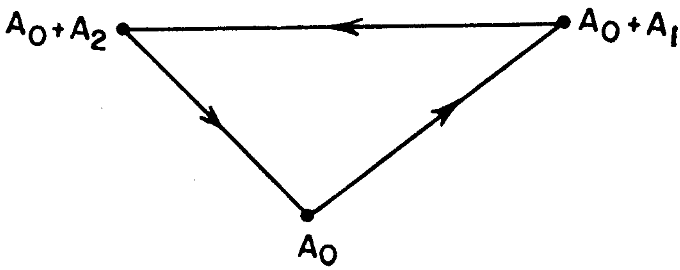

Fig. 3. Integration path for the renormalization contribution $\varrho(A_0, A_1, A_2)$

and hence we may evaluate Eq. (58) using the elementary 2-dimensional Stoke's theorem:

$$d\varrho_D(A_0, A_1, A_2) = (n+1)\int_\Delta ds dt\, \mathrm{tr}\left\{A_2 \frac{\partial}{\partial s} F^n(s,t) - A_1 \frac{\partial}{\partial t} F^n(s,t)\right\}$$

$$= (n+1)\sum_{p=0}^{n-1}\int_0^1 ds \int_0^{1-s} dt\, \mathrm{tr}\,\{A_2F^p(s,t)\,(dA_1 + A_1A(s,t) + A(s,t)A_1)F^{n-1-p}(s,t)$$

$$-A_1F^{n-1-p}(s,t)\,(dA_2 + A_2A(s,t) + A(s,t)A_2)F^p(s,t)\}.$$

Using the cyclic property of the trace as well as the Bianchi identity

$$[A(s,t), F^p(s,t)] = -dF^p(s,t)$$

repeatedly gives

$$\varrho_D(A_0, A_1, A_2) = -(n+1)\sum_{p=0}^{n-1}\int_0^1 ds \int_0^{1-s} dt\, \mathrm{tr}\,\{A_2F^p(s,t)A_1F^{n-p-1}(s,t)\} \tag{59}$$

and these integrals are elementary.

Of course ϱ_D is determined only up to a total derivative, but that does not contribute to the integrals over S^D in which we shall be interested, due to the boundary conditions we have assumed on the fields.

The renormalization part involved in the difference between the A- and the LR-scheme is $\varrho_D(0, A_R, A_L)$.

The expression, Eq. (59), will provide an easy way of obtaining the anomaly. However, one advantage of the present approach is that we can construct directly the Wess-Zumino action already from what we have obtained. This we do now.

We use Eq. (20) in the vector gauge invariant scheme

$$WZ(g_L, g_R; A_L, A_R) \equiv WZ(U; A_L, A_R) = W(A_L, A_R) - W(A_L^U, A_R)$$
$$= W^0(A_L, A_R) - W^0(A_L^U, A_R), \tag{60}$$

where $U(x) = g_L(x)g_R^{-1}(x)$ (Eq. (21)), and where (Eq. (46))

$$W^0(A_L, A_R) = c_n \int_{B_{D+1}} \omega^0_{D+1}(A_L, A_R)$$

is a generating functional of anomalies in the A-scheme. Thus, although W^0 cannot be considered a sensible model of $W(A_L, A_R)$ (it depends explicitly on the way A_L and A_R are extended into $D+1$-dimensional space), the difference, Eq. (60), does represent the integrated anomaly[8].

First, let us evaluate WZ for vanishing gauge fields. Following Witten [5] we call that action

$$N_c \Gamma_{D+1}(U) \equiv WZ(U; 0, 0) = -W^0(U^{-1}dU, 0). \tag{61}$$

From Eq. (53) we find

$$N_c \Gamma_{D+1}(U) = + \int_{B_{D+1}} c_n(n+1) \int_0^1 dt \operatorname{tr}(U^{-1}dUF^n(t)),$$

where

$$A(t) = (1-t)U^{-1}dU, \quad F(t) = dA(t) + A^2(t) = -t(1-t)(U^{-1}dU)^2.$$

Using

$$\int_0^1 dt t^n (1-t)^n = \frac{(n!)^2}{(2n+1)!}$$

and the expression for c_n, Eq. (56), we get

$$\Gamma_{D+1}(U) = i^{n-1} \frac{2\pi}{A(S^{D+1})} \frac{1}{(D+1)!} \int_{B_{D+1}} \operatorname{tr}(U^{-1}dU)^{D+1}, \tag{62}$$

where $A(S^{D+1}) = (2\pi)^{n+1}/n!$ is the "area" of the D-dimensional unit sphere S^D (say in $D+1$ dimensions).

As discussed by Witten [5] this expression is consistent for a quantum theory of the chiral Goldstone boson field $U(x)$, despite the fact that it involves a seemingly arbitrary extension from the physical space (equivalent to) S^D, to $(D+1)$-dimensional space. There are several points to be observed.

---

[8] Strictly speaking, only that part which is odd under $U \leftrightarrow U^{-1}$, $A_L \leftrightarrow A_R$ (cf. Sect. 3).

(*i*) $\Gamma(U)$ only depends on those values of the fields $U(x)$ for which $x \in S^D$. In fact, varying $U(x)$ is equivalent to a chiral variation and we have seen that the result is a total derivative $d\omega_D^1$. Thus, two U-fields agreeing on S^D produces the same $\Gamma(U)$.
(*ii*) The above argument is valid only for a definite choice of B_{D+1}. Now in the situation we have in mind where the chiral group is spontaneously broken down to flavour $SU(N_f)$, $U(x)$ is an element in $SU(N_f)$. These groups have trivial homotopy groups π_D for D even (provided $N_f > n$, cf. [5, 17] and references therein). This means that the image of S^D in group space (i.e. the set of field values) may be considered the equator of a $(D+1)$-dimensional hemisphere (equivalent to the image of) B_{D+1} in just two ways, but there would be no way of preferring one to the other. Thus the path integral-weight exp $(iN_c\Gamma)$ should agree for the two. This means that the difference between Γ's evaluated for the two hemispheres should be a multiple of 2π. Notice that in a coordinate system in $(D+1)$-dimensional Minkowski space, the integrand in Eq. (62) is proportional to

$$\varepsilon^{\mu_1 \cdots \mu_{D+1}} \operatorname{tr}(U^+ \partial\mu_1 U \ldots U^+ \partial\mu_{D+1} U).$$

Thus the phase in Eq. (62) makes Γ_{D+1} real (cf. Eq. (4)).
(*iii*) The *difference* between Γ's evaluated for the two $(D+1)$-dimensional hemispheres is equal to the integral in Eq. (62) taken over S^{D+1}, since one orientation in S^D induces opposite orientations on the two hemispheres. To see that

$$A(S^{D+1})^{-1} \cdot 1/(D+1)! \int_{S^{D+1}} \operatorname{tr}(U^{-1}dU)^{D+1} \tag{63}$$

is an integer is non-trivial. However, we may make the following remarks. First, the integral is a topological invariant. This follows because locally it may be written as a total derivative. That in turn follows because (in $D+2$-dimensions) $d\omega_{D+1}^0(U^{-1}dU, 0) = \Omega_{D+2}(U^{-1}dU, 0) = 0$ since $A_L = U^{-1}dU$, $A_R = 0$ are pure gauges. The fact that $\operatorname{tr}(U^{-1}dU)^{D+1}$ may be written as a total derivative means that we really might write $N_c\Gamma$ as an integral directly in D-dimensional space S^D. However, we would then be forced to introduce fields like log U which are *not* defined on compact space, S^D [10, 5, 6]. It is the singular behaviour of these fields which prevents the integral over S^{D+1} from being zero.

It is also very plausible that (63) is a winding number, since it is expressed as a volume form (with correct Haar-invariance properties), normalized by $A(S^{D+1})$. Nevertheless there are global subtleties which must be dealt with (see Refs [5] and [34]). The fact that field configurations $U(x)$, $x \in S^{D+1}$, may be characterized by an integer winding number, is related to the fact that the odd homotopy groups π_{D+1} of the groups in question (like $SU(N_f)$), is Z (the set of integers).

Next we consider the case of general non-vanishing A_L and A_R. In principle we might just use Eq. (60) for WZ and Eq. (53) for ω_{D+1}^0. However, the consistency of the result follows from the fact that it may be written

$$WZ(U; A_L, A_R) = N_c\Gamma(U) + \int_{B_{D+1}} dwz_D(U; A_L, A_R) = N_c\Gamma(U) + \int_{S^D} wz_D(U; A_L, A_R). \tag{64}$$

Such a form, however, is far from obvious from Eq. (53).

Nevertheless the following very simple treatment yields the desired answer directly:

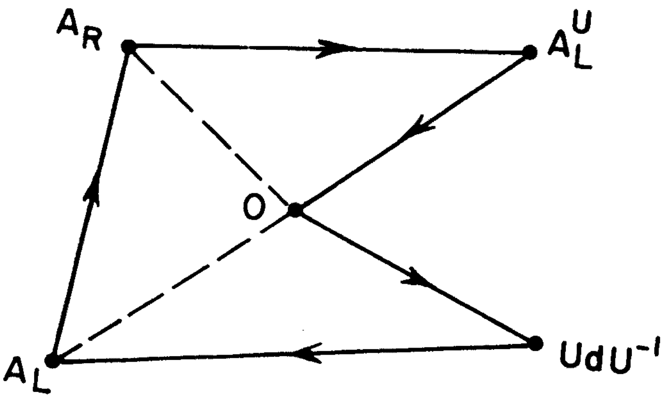

Fig. 4. Integration path giving the Wess-Zumino effective action for the goldstone boson field U interacting with the gauge fields A_L and A_R

From Fig. 4 we may write (cf. Eq. (58) and Fig. 3)

$$\omega^0_{D+1}(A_L, A_R)+\omega^0_{D+1}(A_R, A_L^U)+\omega^0_{D+1}(A_L^U, 0)+\omega^0_{D+1}(0, UdU^{-1})+\omega^0_{D+1}(UdU^{-1}, A_L)$$
$$= d(\varrho_D(0, A_R, A_L^U)+\varrho_D(0, UdU^{-1}, A_L)+\varrho_D(0, A_L, A_R)). \tag{65}$$

On the left hand side we use the antisymmetry and vector gauge invariance of ω^0_{D+1} to obtain

$$\omega^0_{D+1}(A_L^U, 0)+\omega^0_{D+1}(UdU^{-1}, A_L) = \omega^0_{D+1}(A_L^U, 0)-\omega^0_{D+1}(A_L, UdU^{-1}) = 0.$$

Similarly, $\omega^0_{D+1}(0, UdU^{-1}) = \omega^0_{D+1}(U^{-1}dU, 0)$ and $\omega^0_{D+1}(A_R, A_L^U) = -\omega^0_{D+1}(A_L^U, A_R)$. Thus, Eq. (65) implies Eq. (64) upon integration over B_{D+1} and multiplication with c_n, and we find

$$wz_D(U; A_L, A_R) = -c_n(\varrho_D(0, A_L^U, A_R)+\varrho_D(0, A_L, UdU^{-1})-(U = \mathbf{1})) \tag{66}$$

and the ϱ's are obtained from Eq. (59). Explicit expressions for $D = 4$ are given in the next Section.

For completeness we finally obtain a closed expression for the anomaly ω^1_D in the A-scheme. The result for the LR-scheme obtains by putting first $A_L = 0$, and then $A_R = 0$ (cf. Ref. [17]).

Consider first the variation

$$\delta A_R = D_R v \equiv dv+[A_R, v], \quad \delta A_L \equiv 0. \tag{67}$$

We evaluate the corresponding anomaly from

$$\delta\omega^0_{D+1}(A_L, A_R) = \omega^0_{D+1}(A_L, A_R+D_R v)-\omega^0_{D+1}(A_L, A_R)$$
$$= d\varrho_D(A_R, A_L-A_R, D_R v)+\omega^0_{D+1}(A_R, A_R+D_R v) \tag{68}$$

since $d\varrho_D(A_R, A_L-A_R, D_R v)$ represents an integration around a (nearly collapsed) triangle: A_R, A_L, $A_R+D_R v$ (cf. Fig. 3). Then the last term in Eq. (68) is trivially obtained from Eq.

(53) as

$$\omega^0_{D+1}(A_R, A_R+D_R v) = (n+1)\,\mathrm{tr}\,(D_R v F^n_R) = (n+1) d\,\mathrm{tr}\,(v F^n_R),$$

using the Bianchi identity on F^n_R. Thus we have written ω^0_{D+1} explicitly as a total derivative, and from Eq. (48) we find

$$\omega^1_D(0, v; A_L, A_R) = \varrho_D(A_R, A_L-A_R, D_R v)+(n+1)\,\mathrm{tr}\,(v F^n_R). \tag{69}$$

From Eq. (59) we find to first order in v

$$\varrho_D(A_R, -2A, D_R v) = (n+1) \sum_{p=0}^{n-1} \int_{-1}^{1} d\tau\, \tfrac{1}{2}(1+\tau)\,\mathrm{tr}\,\{D_R v F^p(\tau) A F^{n-p-1}(\tau)\}, \tag{70}$$

where $A(\tau) = V+\tau A$ (one integration in Eq. (59) is trivial for the nearly collapsed triangle).

Adding a similar expression for $\omega^1_D(-v, 0; A_L, A_R)$ gives

$$\omega^1_D(-v, v; A_L, A_R) = (n+1)\,\{\mathrm{tr}\,(v[F^n_L+F^n_R])$$

$$+\sum_{p=0}^{n-1}\int_{-1}^{1} d\tau[(1-\tau^2)\,\mathrm{tr}\,([A, v]F^p(\tau)AF^{n-p-1}(\tau))-\tau\,\mathrm{tr}\,(vF^p(\tau)D_\tau AF^{n-p-1}(\tau))],$$

where we used

$$\tfrac{1}{2}(1+\tau)D_R v-\tfrac{1}{2}(1-\tau)D_L v = [A, v]+\tau D_V v = \tau D_\tau v+(1-\tau^2)\,[A, v],$$

$$D_\tau v \equiv dv+[A(\tau), v] \quad \text{and} \quad D_\tau F^p_\tau = 0.$$

Observing $D_\tau A = d/d\tau F(\tau)$ we finally obtain

$$\omega^1_D(-v, v; A_L, A_R) = (n+1) \int_{-1}^{1} d\tau\,\mathrm{tr}\,[v(F^n(\tau)-(1-\tau^2)\sum_{p=0}^{n-1}[A, F^p(\tau)AF^{n-p-1}(\tau)])] \tag{71}$$

where $[A, B] = AB+BA$ when both A and B are odd forms (cf. Sect. 4) as here.

6. Explicit results in 4 dimensions

The generating functional $W^0(A_L, A_R)$ for anomalies in the vector gauge-invariant (A)-scheme is given by Eqs. (46), (53) and (56) as

$$W^0(A_L, A_R) = \frac{iN_c}{24\pi^2}\int_{B_5} \omega^0_5(A_L, A_R)$$

$$= \frac{iN_c}{240\pi^2}\int_{B_5} [5(A_R-A_L)\,(2(F^2_R+F^2_L)+F_R F_L+F_L F_R)$$

$$-5(A_R-A_L)^3\,(F_R+F_L)+(A_R-A_L)^5]$$

$$= \frac{iN_c}{24\pi^2}\left(\tfrac{1}{10}\int_{B_5} [10A_R F^2_R-5A^3_R F_R+A^5_R]-(A_R \leftrightarrow A_L)+\int_{S^4} \varrho_4(0, A_L, A_R)\right), \tag{72}$$

where the last form is the generating functional in the LR-scheme plus the necessary renormalization part ϱ_4, obtained from Eq. (59) as (cf. Ref. [19])

$$\varrho_4(0, A_L, A_R) = \tfrac{1}{2}\,\mathrm{tr}\,[(A_L A_R - A_R A_L)(F_L + F_R) - A_L^3 A_R + A_R^3 A_L + \tfrac{1}{2} A_L A_R A_L A_R]. \quad (73)$$

From Eq. (71) we find the Bardeen anomaly [2]

$$\omega_4^1(-v, v; A_L, A_R) = 6\,\mathrm{tr}\,[v(F_V^2 + \tfrac{1}{3} F_A^2 + \tfrac{8}{3} A^4 - \tfrac{4}{3}(F_V A^2 + A F_V A + A^2 F_V))]. \quad (74)$$

where $F_V \equiv dV + V^2 + A^2$, $F_A = dA + VA + AV$. This agrees with Eq. (34) ($F = \frac{1}{2} dx^\mu \times dx^\nu F_{\mu\nu}$ etc.)

Notice that $W^0(A_L, A_R) = -W^0(A_R, A_L)$ and $\omega_4^1(-v, v; A_L, A_R) = +\omega_4^1(-v, v; A_R, A_L)$ as discussed in Sects. 3 and 5.

Finally, the Wess-Zumino effective action is obtained from Eqs. (60), (61), (62), (64) and (66) as

$$WZ(U; A_L, A_R) = W^0(A_L, A_R) - W^0(A_L^U, A_R)$$

$$= -2\pi \frac{iN_c}{480\pi^3} \int_{B_5} \mathrm{tr}\,(U^{-1}dU)^5 + \int_{S^4} wz_4(U; A_L, A_R) \quad (75)$$

$$wz_4(U; A_L, A_R) = -\frac{iN_c}{24\pi^2}(\varrho_4(0, A_L^U, A_R) + \varrho_4(0, A_L, -dUU^{-1}) - (U = \mathbf{1})),$$

where

$$A_L^U = U^{-1} A_L U + U^{-1} dU. \quad (76)$$

This may be evaluated directly from Eq. (72), but that leads to a very lengthly determination of wz_4. On the other hand wz_4 is immediately written down using Eqs. (76) and (73). Making the antisymmetry under ($L \leftrightarrow R$, $U \leftrightarrow U^{-1}$) manifest takes a few rearrangements (including a partial integration). The result may be written

$$wz_4(U; A_L, A_R) = -\frac{iN_c}{48\pi^2}\,\mathrm{tr}\,[(A_L^U(A_R dA_R + dA_R A_R + A_R^3 - U_R^2 A_R)$$

$$-U^{-1}A_L U(A_R U_R A_R - U_R dA_R) + \tfrac{1}{2} A_L U_L A_L U_L) - (L \leftrightarrow R, U \leftrightarrow U^{-1})$$

$$+ \tfrac{1}{2}(A_L U A_R U^{-1})^2] - (U = \mathbf{1}), \quad (77)$$

where

$$U_L \equiv dUU^{-1},\; U_R \equiv dU^{-1}U = -U^{-1}dU,\; A_L^U = U^{-1}A_L U - U_R.$$

This agrees with Ref. [20] and (apart from the ($U = \mathbf{1}$)-term) with Refs. [18, 19]. However, as discussed in these references there is some disagreement with the original expression given in Ref. [5].

In the case of Goldstone bosons interacting with the photon fields, we put

$$A_L = A_R = -ieQA, \quad A \equiv A_\mu dx^\mu,$$

where $A_\mu(x)$ is the ordinary electromagnetic field and where Q is the $N_f \times N_f$ diagonal matrix of quark-charges. Inserting in Eq. (77) we get, using

$$A^2 = 0, \quad AdA = dAA = FA,$$

$$wz_4(U; -ieQA, -ieQA) = \mathrm{tr}\left\{-eA{*}J_a + i\frac{N_c e^2}{48\pi^2} FA[2(U_R Q^2 - U_L Q^2) + UQU^{-1}QU_L - U^{-1}QUQU_R]\right\} \tag{78}$$

where $*J_a$ is (the dual of) an anomalous electromagnetic current

$$*J_a = \frac{-N_c}{48\pi^2}(U_L^3 - U_R^3)Q, \tag{79}$$

in agreement with Ref. [20], [35].

In actual applications, WZ is to be added to the ordinary totally (R- and L-) gauge-invariant chiral/Skyrme-Lagrangians (cf. Eqs. (38), (38′)) where the standard replacement

$$\partial_\mu U \to D_\mu U \equiv \partial_\mu + A_{L\mu} U - U A_{R\mu}$$

is made.

Replacing electric charge Q by baryonic charge (of the quarks)

$$1/N_c \cdot \mathbf{1}$$

gives the anomalous baryonic current

$$*J_a^{(\mathrm{baryon})} = \frac{-1}{48\pi^2}\,\mathrm{tr}\,(U_L^3 - U_R^3) = \frac{-1}{24\pi^2}\,\mathrm{tr}\,(U^{-1}dU)^3 \tag{80}$$

for which the space integral over the time-component gives baryon number 1 for a static skyrmion [5].

Notice that for flavour SU(2), $\Gamma_5(U) \equiv 0$ since it involves a 5-dimensional volume element in the 3-dimensional manifold, SU(2). The term wz_4 is nonvanishing, however. (See Refs. [5–7] for a detailed discussion of the SU(2)-case where also $\pi_2(\mathrm{SU}(2)) = Z_2 \neq 0$.)

7. *The case of 2 dimensions. Bosonization*

In two dimensions a complete treatment of the fermionic determinant becomes possible [9, 10]. The result takes the form of a chiral lagrangian plus a Wess-Zumino topological term. This modified chiral theory is conformally invariant and seems very interesting from various points of view [11–15]. We mostly consider the vector gauge invariant (A)-scheme. Other renormalization schemes are readily treated using the techniques developed above.

We have emphasized that the treatment presented so far only gives information on that part of the fermionic determinant $W(A_L, A_R)$, which is odd under $A_L \leftrightarrow A_R$.

However, in two dimensions there is a trivial extra invariance in the problem due to the identity

$$\gamma^{\mu}\gamma_5 = -\varepsilon^{\mu\nu}\gamma_\nu \tag{81}$$

(in Minkowski space; in euclidean space an extra factor i appears; we write γ_5 rather than γ_3).

This means that $W(A_L, A_R)$ is explicitly invariant under the transformations

$$A_L \to {}^*A_L; \quad A_R \to -{}^*A_R, \tag{82}$$

where

$${}^*A_\mu \equiv \varepsilon_{\mu\nu}A^\nu$$

again is a 1-form in 2 dimensions. The invariance under the transformations (82) follows because by Eq. (81) we find

$$A_L\psi_L \to -A_L\gamma_5\psi_L = A_L\psi_L; \quad A_R\psi_R \to A_R\gamma_5\psi_R = A_R\psi_R$$

(cf. Eq (6)).

This additional invariance in 2 dimensions has two implications.

First, any gauge field may be represented by pure chiral gauges:

$$A_L = h_L^{-1}dh_L \sim \tfrac{1}{2}(h_L^{-1}dh_L + {}^*h_L^{-1}dh_L),$$
$$A_R = h_R^{-1}dh_R \sim \tfrac{1}{2}(h_R^{-1}dh_R - {}^*h_R^{-1}dh_R). \tag{83}$$

This means that knowing the response of the determinant to chiral transformations (i.e. knowing the anomaly) is enough to fix the determinant completely (up to a field independent "integration constant", and up to renormalization terms).

It also means that our expression for $W^0(A_L, A_R)$ is a true representation of the *odd* (in $A_L \leftrightarrow A_R$) part of W. In the (A)-scheme we may introduce

$$G = h_L h_R^{-1}.$$

Then from Eq. (62) we find (leaving out the color factor)

$$W^0(A_L, A_R) = W^0(G^{-1}dG, 0) = \Gamma_3(G) = \frac{1}{12\pi}\int_{B_3}(G^{-1}dG)^3. \tag{84}$$

However, due to the symmetry, Eq. (82), we can also fix the *even* part of W under $A_L \leftrightarrow A_R$.

It is simplest to consider the *variation* of $W(A_L, 0)$ where $A_L = G^{-1}dG$ (using Eq. (82) and vector gauge invariance this is no loss of generality).

In the present case of 2 dimensions, calculations are rather simple to perform directly from Eq. (84). But it also follows from our general formula for the anomaly (Eqs. (69), (70), (49), (56)) that under

$$\delta A_L = D_L v \equiv dv + [G^{-1}dG, v]$$

we find

$$\delta W^0 = \frac{1}{4\pi}\int_{S^2} \mathrm{tr}\,(v dA_L). \tag{85}$$

However, δW must be symmetric in A_L and $^*A_L$:

$$\delta W = \frac{1}{4\pi}\int_{S^2} \mathrm{tr}\,(v d(A_L + {}^*A_L)). \tag{86}$$

This is achieved for

$$-W(A_L, A_R) = +S(G) = \frac{1}{8\pi}\int_{S^2} d^2x\, \mathrm{tr}\,(\partial_\mu G \partial^\mu G^{-1}) - \frac{1}{12\pi}\int_{B_3} \mathrm{tr}\,(G^{-1}dG)^3, \tag{87}$$

where A_L, A_R are given by Eq. (83). In fact

$$\int_{S^2} d^2x\, \mathrm{tr}\,(\partial_\mu G \partial^\mu G^{-1}) = -\int_{S^2} d^2x\, \mathrm{tr}\,(G^{-1}\partial_\mu G G^{-1}\partial^\mu G) = \int_{S^2} \mathrm{tr}\,(A_L {}^*A_L).$$

And it is easy to obtain

$$\delta \int_{S^2} \mathrm{tr}\,(A_L {}^*A_L) = -2\int_{S^2} \mathrm{tr}\,(v d({}^*A_L))$$

for the variation above.

The expression, Eq. (87) was first given by Polyakov and Wiegman [9] and (in a different form) by d'Adda, Davis and Di Vecchia and by Alvarez [10].

We emphasize that the "kinetic term" $\mathrm{tr}\,(\partial_\mu G \partial^\mu G^{-1})$ is a renormalization term which is *not* fixed by our renormalization condition, that the fermionic determinant be vector gauge invariant: it is vector gauge invariant by itself since it only depends on $G = h_L h_R^{-1}$. Instead this term is a consequence of the very special 2-dimensional symmetry, Eq. (82).

Next we construct the Wess-Zumino effective action; this time the full one — not just the portion of it which is odd under $A_L \leftrightarrow A_R$, $U \leftrightarrow U^{-1}$. A chiral transformation $A_L \to A_L^{g_L}$, $A_R \to A_R^{g_R}$ may be expressed as (cf. Eq. (83))

$$h_L \to h_L g_L, \quad h_R \to h_R g_R.$$

Hence we get

$$(G \equiv h_L h_R^{-1}; \quad U \equiv g_L g_R^{-1})$$

$$WZ(U; A_L, A_R) = -S(h_L h_R^{-1}) + S(h_L g_L g_R^{-1} h_R^{-1}) = -S(G) + S(h_L U h_R^{-1}). \tag{88}$$

This is not difficult to evaluate directly from Eq. (87). However, we may also use our general formalism, giving (Eqs. (59))

$$\varrho_2(0, A_L, A_R) = +\mathrm{tr}\,(A_L A_R)$$

and (Eqs. (56), (66))

$$wz_2(U; A_L, A_R) = -c_2(\varrho_2(0, A_L^U, A_R)+\varrho_2(0, A_L, UdU^{-1})-(U = \mathbf{1}))$$
$$= -\frac{1}{4\pi}\int_{S^2} \mathrm{tr}\,(A_R U^{-1} A_L U + A_R U^{-1} dU + UdU^{-1} A_L - A_R A_L), \tag{89}$$

which is the odd (in $A_L \leftrightarrow A_R$, $U \leftrightarrow U^{-1}$) part coming in addition to $S(U)$ for $A_{L,R} = 0$. The full expression having the correct symmetry under (82) is immediately written down using light cone coordinates, since

$$\tfrac{1}{2}(A_L + {}^*A_L) \quad \text{only involves} \quad A_+ = \tfrac{1}{2}(A_0+A_1)_L$$

and

$$\tfrac{1}{2}(A_R - {}^*A_R) \quad \text{only involves} \quad A_- = \tfrac{1}{2}(A_0-A_1)_R. \tag{90}$$

The result is [12] (see also Ref. [15])

$$WZ(U; A_+, A_-) = +S(U) - \frac{1}{4\pi}\int_{S^2} d^2x$$
$$\times \mathrm{tr}[A_- U^{-1} A_+ U + A_- U^{-1}\partial_+ U + U\partial_- U^{-1} A_+ - A_- A_+]. \tag{91}$$

We recognize in the last integral a simplified version of some similar terms in the 4-dimensional case, Eq. (77).

The result Eq. (91) has a remarkable interpretation in terms of bosonization[9]. Let us consider the following identity obtained by path-integrating over the field U with a Haar--invariant measure (cf. Eq. (88)):

$$\text{const.} = \int \mathscr{D}U e^{iS(U)} = \int \mathscr{D}U e^{iS(h_L U h_R^{-1})} = e^{iS(G)} \int \mathscr{D}U e^{iWZ(U;A_+,A_-)}. \tag{92}$$

Then we may summarize our findings for the fermionic integration using Eq. (91)

$$\int \mathscr{D}(\overline{\psi}, \psi) e^{i\int d^2x[\overline{\psi}\partial\psi + \mathrm{tr}\,(J_+{}^F A_- + J_-{}^F A_+)]} = e^{-iS(G)}$$
$$= \text{const.} \int \mathscr{D}U e^{+iS(U) + \frac{i}{4\pi}\int d^2x \mathrm{tr}[J_+{}^B A_- + J_-{}^B A_+]} e^{-\frac{i}{4\pi}\int d^2x \mathrm{tr}[A_- U^{-1} A_+ U - A_- A_+]}, \tag{93}$$

where

$$J_\pm^F = -i\psi_\pm \overline{\psi}_\pm$$

and

$$J_\pm^B = \begin{cases} -\dfrac{1}{4\pi} U^{-1}\partial_+ U \\ -\dfrac{1}{4\pi} U\partial_- U^{-1} \end{cases}. \tag{94}$$

---

[9] A treatment similar to the present [12] was given in a very recent paper by Gonzales and Redlich [46].

Consider first the abelian case where $U = e^{i\varphi}$ and φ is a real scalar field. Then

$$WZ(\varphi; A_+, A_-) = \frac{1}{8\pi}\int d^2x \partial_\mu \varphi \partial^\mu \varphi - \frac{i}{4\pi}\int d^2x [A_- \partial_+ \varphi - A_+ \partial_- \varphi]$$

and Eq. (93) expresses an identity between generating functionals of currents in a fermi theory and a certain (free) bose theory

$$\begin{aligned} &\langle J_+^{\mathrm{F}}(x_1) \dots J_+^{\mathrm{F}}(x_n) J_-^{\mathrm{F}}(y_1) \dots J_-^{\mathrm{F}}(y_m)\rangle_{\mathrm{F}} \\ &= \text{const.}\ \langle J_+^{\mathrm{B}}(x_1) \dots J_+^{\mathrm{B}}(x_n) J_-^{\mathrm{B}}(y_1) \dots J_-^{\mathrm{B}}(y_m)\rangle_{\mathrm{B}}, \end{aligned} \tag{95}$$

where $J_\pm^{\mathrm{B}} = \pm i/4\pi \partial_\pm \varphi$, and where the two greens functions correspond to the free fermi and free bose theories respectively. This is the usual famous bosonization rule for currents [36].

In the non-abelian case, Eq. (93) expresses a generalization with the bosonization rules, Eq. (94). These rules were first proposed by Witten [11] using very different arguments (based on the current algebra of Poisson-brackets in the two theories). However, due to the last term in Eq. (93) we see that the bosonic path integral does not seem to be exactly the generating functional of the currents (94). For greens functions (95) containing currents of one chirality only there is no difficulty. These are obtained by functional integrations with respect to A_+ (say) and afterwards putting $A_+ = A_- = 0$ in which case the extra terms disappear.

For greens functions involving currents of both chiralities, the extra terms do contribute. Thus for the 2-point function it is easy to show that [12]

$$\langle (J_+^{\mathrm{F}}(x))_{ij} (J_-^{\mathrm{F}}(y))_{kl}\rangle_{\mathrm{F}} = \frac{1}{4\pi}\,\delta_{il}\delta_{kj}\delta^2(x-y),$$

but

$$\langle J_+^{\mathrm{B}}(x)_{ij} J_-^{\mathrm{B}}(y)_{kl}\rangle_{\mathrm{B}} = \frac{1}{4\pi N_{\mathrm{f}}}\,\delta_{ij}\delta_{kl}\delta^2(x-y), \tag{96}$$

where N_{f} is the size of the matrices and where now $\langle\rangle_{\mathrm{B}}$ refers to the bose theory defined by the action $S(U)$. The different index structure is particularly noteworthy.

Since the extra term in Eq. (93) involves A_+ and A_- taken at the same point we see that even mixed greens functions agree in the fermi and bose theories provided all the arguments are different.

What is the implication of these differences? First notice that the mixed fermionic greens functions depend on the renormalization scheme. In fact, in the LR-scheme they would factorize in two greens functions of currents of one helicity only. Thus the fermionic 2-point function would vanish identically in the LR-scheme. In the A-scheme used here it vanishes only when $x \neq y$. However, such greens functions (where all points are different) are believed to define the operators uniquely. Therefore the operator bosonization rule, Eqs. (94) should be O.K. in the free case ($A_\pm = 0$). Notice that in the Minkowski case our greens functions are vacuum expectation values of time-ordered products since they have been obtained from a path integral formalism continued from euclidean space. It is well known that there is an intrinsic ambiguity in defining time-ordered products at coinciding points [37].

Without the extra terms in Eq. (93) the bosonization could have been immediately generalized to an arbitrary interacting theory if the interaction is built from currents. This is because such an interaction term may be obtained by a suitable number of functional differentiations of the generating functional.

However, due to the extra terms, such a construction would seem to break down giving rise to a far more complicated bose theory than would be obtained using the bosonization rules.

However, due to the fact that the last term in (93) involves U and U^{-1} at the *same* point there is a normal ordering type problem involved in giving a meaning to this term. Arguments [13, 14] have been raised to suggest that we should impose

$$U^{-1}(x)_{il}U(x)_{kj} = \frac{1}{N_f}\,\delta_{ij}\delta_{lk} \tag{97}$$

as an operator identity (the vacuum expectation value of the left hand side *is* equal to the right hand side; this was used in Eq. (96) [12]). If that is true the last term in Eq. (93) may be taken outside the path integral. It may then be absorbed into a different renormalization of the fermi theory; it merely leads to a redefinition of the coupling matrix, if the interaction to be bosonized is of the generalized Thirring form

$$\int d^2x\lambda_{ij,kl}J^F_+(x)_{ij}J^F_-(x)_{kl}.$$

Finally we should mention that apparently the theory defined by S(U) seems to be exactly soluble. This has been suggested by Polyakov and Wiegmann on the basis of Bethe ansatz techniques [15] and from a more general point of view by Zamolodchikov [15] using the recent powerful techniques for conformally invariant 2-dimensional field theories [44].

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