ON THE EFFECTIVE PAIRING-INTERACTION STRENGTH IN NUCLEI*

BY K. BÖNING, A. SOBICZEWSKI

Institute for Nuclear Studies, Warsaw**

AND K. POMORSKI

Institute of Physics, The Maria Skłodowska-Curie University, Lublin***

(Received August 29, 1984)

An explicit formula expressing the pairing-interaction strength G by the pairing-energy gap parameter $\tilde{\Delta}$, averaged over shell effects, is found in a uniform level-distribution model based on the harmonic oscillator potential. The dependence of G on the proton and neutron numbers, obtained from the formula in case of the phenomenological $\tilde{\Delta}$ ($\tilde{\Delta} = 12A^{-1/2}$ MeV), is discussed and compared with that assumed in various microscopic analyses in which G is numerically fitted to empirical odd-even mass differences.

PACS numbers: 21.30.+y

1. Introduction

The pairing interactions have been used in nuclear physics for more than twenty five years [1–3]. Their strength G is determined from empirical odd-even mass differences.

The usual procedure is to solve numerically the pairing equations (which connect the pairing energy-gap parameter Δ, that is assumed to be equal to the odd-even mass difference) for each nuclide, separately for neutrons and protons. The solution is based on a definite single-particle energy spectrum, obtained with a realistic single-particle potential. The empirical odd-even mass differences, i.e. the values of the gap parameter Δ, are the input of the calculation, while the strength G is treated as an adjustable parameter. The calculations are performed for a given, larger or smaller, region of nuclei (cf. e.g. Refs. [4–7]).

In some papers (e.g. Refs. [8, 9]), however, the energy-gap parameter $\tilde{\Delta}$, averaged over shell effects (e.g. $\tilde{\Delta} = 12\,A^{-1/2}$ MeV), is taken as a starting point. The strength G is

* Supported in part by the Polish-US Maria Skłodowska-Curie Fund, Grant No. P-F7F037P.

** Address: Instytut Problemów Jądrowych, Hoża 69, 00-681 Warszawa, Poland.

*** Instytut Fizyki, UMCS, Nowotki 10, 20-031 Lublin, Poland.

then calculated from $\tilde{\Delta}$ by the use of a single-particle spectrum which is also averaged over the shell effects. The calculation is performed numerically using a definite averaging procedure, e.g. the Strutinski procedure.

In the present paper, we also base on the average gap parameter $\tilde{\Delta}$. However, instead of calculating G numerically for separate nuclei, we aim at finding a simple, explicit formula for G, which will show us the dependence of G on the neutron, N, and proton, Z, numbers.

It is interesting to check if such formula predicts the dependence on N and Z which is usually assumed in the numerical fits of G to the odd-even mass differences. It is also interesting to see how close are the values of G obtained from such formula (which may be considered as a general formula for all, not too light, nuclei in the nuclear chart) to the values obtained in the numerical fits. More generally, it is interesting to see if the formula can replace the fits.

We base on the harmonic oscillator spectrum, averaged in a simple way.

2. *Explicit formula for the strength*

Let us consider the pairing equation in the model of uniformly distributed energy levels around the Fermi level λ. The equation is (cf. e.g. Refs. [6, 8, 9])

$$e^{\frac{1}{G_l \tilde{\varrho}_l}} = \frac{S_l}{\tilde{\Delta}_l} + \left[1 + \left(\frac{S_l}{\tilde{\Delta}_l}\right)^2\right]^{1/2}, \tag{2.1}$$

where G_l is the pairing-interaction strength for one kind of nucleons (neutrons, $l = N$, or protons, $l = Z$), $\tilde{\varrho}_l$ is the single-particle level density for these nucleons, $2S_l$ is an energy interval (taken symmetric with respect to λ) in which the pairing interaction is assumed to be active (i.e. its matrix elements are assumed to be different from zero) and $\tilde{\Delta}_l$ is the pairing gap parameter. For

$$S_l/\tilde{\Delta}_l \gg 1,$$

which we always fulfil, Eq. (2.1) becomes

$$e^{\frac{1}{G_l \tilde{\varrho}_l}} = \frac{2S_l}{\tilde{\Delta}_l}. \tag{2.2}$$

For $\tilde{\varrho}_l$, we use the level density of the spherical harmonic oscillator, averaged over one oscillator shell, i.e.

$$\tilde{\varrho}_l = \frac{\frac{1}{2}(n_l+1)(n_l+2)}{\hbar\omega_{0l}} \approx \frac{(3l)^{2/3}}{2\hbar\omega_{0l}}, \tag{2.3}$$

where n_l is the total oscillator quantum number of the last filled shell and $\hbar\omega_0$ is the oscillator energy. With this, Eq. (2.2) reads

$$l^{2/3} \cdot G_l = \frac{2}{3^{2/3} \ln(2S_l/\tilde{\Delta}_l)} \hbar\omega_{0l}. \tag{2.4}$$

Thus, the value of G_l depends on the cut-off energy S_l. We adopt here the same S_l as in Ref. [6] (cases of other S_l are discussed in Sect. 3), where $2S_l$ is the energy interval including $\sqrt{15l}$ single-particle levels below the Fermi level and the same number of levels above it. For the smooth pairing energy-gap parameter $\tilde{\Delta}_l$, we take the widely used phenomenological formula

$$\tilde{\Delta}_l = 12A^{-1/2}\,\mathrm{MeV}, \tag{2.5}$$

obtained from a fit to all odd-even mass differences available (cf. e.g. Ref. [10]). Thus,

$$\frac{2S_l}{\tilde{\Delta}_l} = \frac{2\sqrt{15l}}{\tilde{\varrho}_l}\cdot\frac{A^{1/2}}{12} \approx \frac{2\sqrt{15l}}{(3l)^{2/3}}\,2\hbar\omega_{0l}\,\frac{A^{1/2}}{12}. \tag{2.6}$$

To express the oscillator energy $\hbar\omega_{0l}$ by the number of particles l, we express the total single-particle energy of the oscillator by the average value of the potential, on one hand,

$$\sum_i e_i = M\omega_{0l}^2\langle r^2\rangle_l \cdot l \tag{2.7a}$$

(where e_i is the single-particle energy in a state i and M is the mass of a nucleon) and by the number of particles, on the other. For the latter case, we calculate the energy explicitly, assuming n_l oscillator shells to be filled

$$\sum_i e_i = \hbar\omega_{0l}\sum_{n=0}^{n_l}(n+\tfrac{3}{2})(n+1)(n+2)$$

$$= \hbar\omega_{0l}\cdot\tfrac{3}{4}\,l(n_l+2) \approx \tfrac{3}{4}\,\hbar\omega_{0l}\cdot l(3l)^{1/3}. \tag{2.7b}$$

In Eq. (2.7a), we have used the virial theorem for each oscillator state i. Comparing Eq. (2.7a) with Eq. (2.7b) one gets

$$\hbar\omega_{0l} \approx \tfrac{3}{4}\,\frac{\hbar^2}{M\langle r^2\rangle_l}(3l)^{1/3}, \tag{2.8}$$

which, with the assumption of the same value of $\langle r^2\rangle$ for neutrons and protons and of the uniform distribution of them in a nucleus

$$\langle r^2\rangle_N = \langle r^2\rangle_Z = \tfrac{3}{5}\,R_0^2 = \tfrac{3}{5}\,r_0^2A^{2/3}, \tag{2.9}$$

where R_0 is the radius of the nucleus, leads to

$$\hbar\omega_{0l} = \tfrac{5}{4}\,\frac{\hbar^2}{Mr_0^2}\,\frac{(3l)^{1/3}}{A^{2/3}}. \tag{2.10}$$

Using the value $r_0 = 1.2$ fm and introducing the relative neutron excess $I = (N-Z)/A$, Eq. (2.10) becomes

$$\hbar\omega_{0l} = \frac{41.0}{A^{1/3}}\left(\frac{2l}{A}\right)^{1/3} = \frac{41.0}{A^{1/3}}(1\pm I)^{1/3}\,\mathrm{MeV}. \tag{2.10a}$$

To obtain this, we have used the relation

$$2l = A(1 \pm I), \tag{2.10b}$$

where sign plus holds for neutrons ($l = N$) and minus for protons ($l = Z$). An expansion of this expression up to the first order in I,

$$\hbar\omega_{0l} \approx \frac{41.0}{A^{1/3}} (1 \pm \tfrac{1}{3} I) \text{ MeV}, \tag{2.10c}$$

is usually used in the harmonic oscillator calculations [6].

Substitution of Eq. (2.10a) into Eq. (2.6), with a use of Eq. (2.10b), gives

$$2S_l/\tilde{A}_l = 28.56(1 \pm I)^{1/6}, \tag{2.6a}$$

or

$$\ln (2S_l/\tilde{A}_l) = 3.352 + \tfrac{1}{6} \ln (1 \pm I) \approx 3.352 \pm \tfrac{1}{6} I. \tag{2.6b}$$

This leads to the final form of Eq. (2.4)

$$l^{2/3} \cdot G_l = 0.287(1 \mp 0.050I)\hbar\omega_{0l}, \tag{2.11}$$

where sign minus holds for neutrons and plus for protons. One can see that G_N, (G_Z), when expressed in units of $\hbar\omega_{0N}$ ($\hbar\omega_{0Z}$), depends essentially only on the number of neutrons N (protons Z) and merely very weakly on the number of protons Z (neutrons N). The latter dependence enters through the small term involving the neutron excess I and may be accounted for, with a high accuracy, by taking an average value of I in Eq. (2.11), e.g. $I_0 = 0.185$ for rare earths and $I_0 = 0.217$ for actinides. With these values, Eq. (2.11) gives for both the regions of nuclei

$$\begin{aligned} N^{2/3} \cdot G_N &= 0.284\,\hbar\omega_{0N}, \\ Z^{2/3} \cdot G_Z &= 0.290\,\hbar\omega_{0Z}. \end{aligned} \tag{2.11a}$$

The strength G_l of Eq. (2.11), expressed in units of MeV, is

$$l^{2/3} \cdot G_l = \frac{11.76}{A^{1/3}} (1 \mp 0.050I)(1 \pm I)^{1/3} \text{ MeV}, \tag{2.12}$$

according to Eq. (2.10a).

3. Relation with the strength obtained numerically

3.1. Relation with the strength of Ref. [6]

The relation with the strength of Ref. [6] is most direct, because we use the same energy interval $2S_l$, Eq. (2.2), as in that paper.

The pairing strength G_l is assumed in Ref. [6] in the form

$$A \cdot G_l = g_0 \mp g_1 I, \tag{3.1}$$

where minus sign holds for neutrons and plus for protons. A fit of the gap parameter Δ_l, calculated microscopically in Ref. [6] with that G_l, to empirical odd-even mass differences, leads to the values

$$A \cdot G_l = (19.2 \mp 7.4I)\ \text{MeV}. \tag{3.2}$$

The fit has been performed for a wide region of A, from $A \approx 150$ to $A \approx 250$, using the energy interval $2S_l$ corresponding to $2\sqrt{15l}$ Nilsson levels nearest to the Fermi level.

Our formula for G_l, Eq. (2.12), transformed to a form similar to that of Eq. (3.2) is

$$A \cdot G_l = 18.67(1 \mp 0.050I)(1 \pm I)^{-1/3}\ \text{MeV}, \tag{3.3}$$

where we have only made use of relation (2.10b). Up to the first order in I, this gives

$$A \cdot G_l = (18.67 \mp 7.15I)\ \text{MeV}. \tag{3.3a}$$

Thus, our values of both g_0 and g_1 differ from those of Eq. (3.2) only by about 3%, i.e. by less than the accuracy with which they are obtained from the odd-even mass differences [6] (we think this accuracy to be about $\pm 5\%$). One could expect such good agreement as the fit of Ref. [6] has been performed for large region of nuclei. For such region, the uniform model used by us, which washes out the shell effects in the energy spectrum, should already work sufficiently well.

The presence of higher-order terms in I, in Eq. (3.3), which are omitted in Eq. (3.1), suggests that the form of G_l, as given by Eqs. (2.11) or (2.12), is more proper for the fit than that assumed in Eq. (3.1). These terms are significant for heavy nuclei, for which I is rather large (around 0.2).

3.2. Relation with the strength of Ref. [5]

It is instructive to look at the relation with the strength obtained in an analysis in which another energy interval $2S_l$, than that of our formula, was used.

In Ref. [5] (cf. also Ref. [11]), the pairing strength G_l has been assumed in the form

$$A \cdot G_l = g_{0l} \tag{3.4}$$

and adjusted to the odd-even mass differences. The Nilsson potential has been used and 42 energy levels have been taken for the pairing interaction. The analysis has been performed for a wide region of nuclei, $150 \leqslant A < 256$, leading to the result

$$A \cdot G_N = (22.5 \pm 0.5)\ \text{MeV}, \quad A \cdot G_Z = (26.5 \pm 0.5)\ \text{MeV}. \tag{3.5}$$

Our Eq. (2.4), rewritten to the form useful for the comparison, is

$$A \cdot G_l = 62.58\,[\ln(2S_l/\tilde{\Delta}_l)]^{-1}(1 \pm I)^{-1/3}\ \text{MeV}, \tag{3.6}$$

where we have used Eqs. (2.10a) and (2.10b). To calculate G_l, we should relate $2S_l$ taken in Ref. [5] with that of our analysis. The relation is

$$(2S_l/\tilde{\Delta}_l)_{[5]} = \frac{42}{2\sqrt{15l}}(2S_l/\tilde{\Delta}_l)_{\text{our}}, \tag{3.7}$$

where our value is given by Eq. (2.6a). To get only one value for the whole region, $150 \leqslant A < 256$, we take a nucleus around the center of the region, e.g. $^{203}_{81}\mathrm{Tl}_{122}$, and obtain from Eq. (3.6)

$$A \cdot G_N = 22.0 \text{ MeV}, \quad A \cdot G_Z = 24.0 \text{ MeV}. \tag{3.8}$$

Thus, our value for AG_N is only by about 2% smaller than that of Eq. (3.5) (22.5 MeV), i.e. by less than expected accuracy of the latter, while our value for AG_Z is by about 9% smaller than the value given in Eq. (3.5) (26.5 MeV), what is more than one might expect.

3.3. Relation with the strength of Ref. [7]

It is interesting to look at the relation with the strength obtained with the use of another single-particle potential than that of the harmonic oscillator.

In Ref. [7], the strength G_l has been assumed in the form

$$A \cdot G_l = g_{0l} + g_{1l}(N-Z) \tag{3.9}$$

and adjusted to the odd-even mass differences. The Woods-Saxon potential has been used and l levels, starting from the lowest one, have been taken when solving the pairing equations. The result for the rare-earth region (analyzed for nuclei from $^{144}_{60}\mathrm{Nd}$ up to $^{180}_{72}\mathrm{Hf}$),

$$\begin{aligned} A \cdot G_N &= [18.95 - 0.078(N-Z)] \text{ MeV}, \\ A \cdot G_Z &= [17.90 + 0.176(N-Z)] \text{ MeV}, \end{aligned} \tag{3.10a}$$

and for the actinide region (analyzed for nuclei from $^{222}_{88}\mathrm{Ra}$ up to $^{248}_{96}\mathrm{Cm}$),

$$\begin{aligned} A \cdot G_N &= [19.3 - 0.08(N-Z)] \text{ MeV}, \\ A \cdot G_Z &= [13.3 + 0.217(N-Z)] \text{ MeV}, \end{aligned} \tag{3.10b}$$

has been obtained.

To get our values, we again use Eq. (3.6), with the relation

$$(2S_l/\tilde{\Delta}_l)_{[7]} = \frac{l}{2\sqrt{15l}} (2S_l/\tilde{\Delta})_{\text{our}}, \tag{3.11}$$

which gives

$$[\ln (2S_l/\tilde{\Delta}_l)]_{[7]} = 0.9584 + \tfrac{1}{2} \ln A + \tfrac{2}{3} \ln (1 \pm I). \tag{3.12}$$

Then, the right-hand side of Eq. (3.6) is expanded in powers of I, up to the first order, at I_0 corresponding to a nucleus about the center of a region. As such a nucleus, we take $^{162}_{66}\mathrm{Dy}$ for rare earths and $^{235}_{92}\mathrm{U}$ for actinides. The values of A are also taken as corresponding to these nuclei. The results are

$$\begin{aligned} A \cdot G_N &= [17.77 - 0.0501(N-Z)] \text{ MeV}, \\ A \cdot G_Z &= [17.71 + 0.0670(N-Z)] \text{ MeV} \end{aligned} \tag{3.13a}$$

for rare earths, and

$$A \cdot G_N = [16.84 - 0.0315(N-Z)] \text{ MeV},$$

$$A \cdot G_Z = [16.75 + 0.0441(N-Z)] \text{ MeV} \tag{3.13b}$$

for actinides.

One can see that the g_{0l} part of the strength of Eq. (3.13) agrees better with that of the strength of Eq. (3.10) than the g_{1l} part, especially for the rare-earth region, where the g_{0l} part of AG_N differs by about 6% and that of AG_Z by about 1%. A portion of the large discrepancy in the g_{1l} part may come from the difference between the dependence of G on Z and N obtained from our formula and that assumed in the numerical analysis. This conclusion is especially supported by the fact that the total values of G differ less than the corresponding values of separate g_{0l} and g_{1l} parts.

Some portion of the discrepancy in G may also come from the difference in the single-particle level distribution, especially because the large number of levels (all occupied levels and the same number of levels above the Fermi level) taken here for the pairing interaction.

4. Conclusions

An explicit formula expressing the pairing-interaction strength G by the pairing-energy gap parameter $\tilde{\Delta}$, averaged over the shell effects, is found. The formula depends on the number of protons Z and neutrons N in a nucleus, as well as on the single-particle energy interval, in which the pairing forces are assumed to be active. The formula is obtained in a model of uniformly distributed single-particle levels, based on the harmonic oscillator potential.

Our study of the formula and its relation with formulae used in microscopic analyses, in which the strength G is numerically fitted to empirical odd-even mass differences, leads to the following conclusions:

(1) The formula predicts a different dependence of G on Z and N numbers than assumed in the microscopic analyses.

(2) If the difference in the dependence is small, as e.g. in case of the analysis of Ref. [6], the values of G_N and G_Z obtained from the formula agree with those of the microscopic fit inside the accuracy of the latter. This indicates that the microscopic fit is unnecessary when we have the formula at our disposal.

(3) In particular, the formula shows that the strength for neutrons, G_N, is almost independent of the number of protons, Z, in a nucleus and that the strength for protons, G_Z, is almost independent of the number of neutrons, N, if both strengths are expressed in units of the oscillator energy $\hbar\omega_{0l}$. This result is of a practical value as it allows one to solve the pairing equations only once, for a given N, for all isotones considered, and also only once, for a given Z, for all isotopes considered, in an arbitrary microscopic calculation. Without that one would have to solve these equations, both for neutrons and protons, for each nuclide separately.

REFERENCES

[1] A. Bohr, B. R. Mottelson, D. Pines, *Phys. Rev.* **110**, 936 (1958).
[2] S. T. Belyaev, *Mat. Fys. Medd. Dan. Vid. Selsk.* **31**, no. 11 (1959).
[3] V. G. Soloviev, *Nucl. Phys.* **9**, 655 (1958/59).
[4] D. R. Bes, Z. Szymański, *Nucl. Phys.* **28**, 42 (1961).
[5] L. A. Malov, V. G. Soloviev, I. D. Khristov, *Yad. Fiz.* **6**, 1186 (1967).
[6] S. G. Nilsson, C. F. Tsang, A. Sobiczewski, Z. Szymański, S. Wycech, C. Gustafson, I. L. Lamm, P. Möller, B. Nilsson, *Nucl. Phys.* **A131**, 1 (1969).
[7] J. Dudek, A. Majhofer, J. Skalski, *J. Phys.* **G6**, 447 (1980).
[8] M. Bolsterli, E. O. Fiset, J. R. Nix, J. L. Norton, *Phys. Rev.* **C5**, 1050 (1972).
[9] M. Brack, J. Damgaard, A. S. Jensen, H. C. Pauli, V. M. Strutinsky, C. Y. Wong, *Rev. Mod. Phys.* **44**, 320 (1972).
[10] A. Bohr, B. R. Mottelson, *Nuclear structure*, vol. 1, Benjamin, New York 1969, p. 169.
[11] V. G. Soloviev, *Teoria zlozhnykh yader*, Nauka, Moskva 1971, p. 165.