PHYSICALLY REALISTIC MODEL OF INSTANTANEOUS PREDICTIVE RELATIVISTIC DYNAMICS

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The Lorentz Invariance conditions in Instantaneous Predictive Relativistic Dynamics are integrated in the case of one dimensional motion of two particles. The physically realistic models expressed in terms of non-relativistic potential are constructed in implicit form. The iteration procedure giving an explicit form of dynamics describing the unbound motions is presented and an asymptotic explicit form of dynamics describing the case of weak interaction between particles is found. Trajectories of particles are presented in parametric form. Some new simple solutions of Currie–Hill equations are found.

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1. Introduction

In Instantaneous Predictive Relativistic Dynamics (IPRD) [1–5] the system of two point-like particles is described by the Newtonian-like differential equations of motion

$$\frac{d^2x_i^n}{dt^2} \equiv \ddot{x}_i^n = a_i^n(\vec{x}, \dot{\vec{x}}^1, \dot{\vec{x}}^2, m_1, m_2), \tag{1.1}$$

$n = 1, 2$, $i = 1, 2, 3$, where $\vec{x}^n$ is the n-particle position, while $\vec{x} = \vec{x}^1 - \vec{x}^2$, m_n is its mass and a_i^n are the particle accelerations expressed as functions of particle positions $\vec{x}^n$ and velocities $\dot{\vec{x}}^n$ ("forces").

Currie [1] and Hill [2] have given independently the differential conditions which ensure the Lorentz invariance of IPRD. They form a set of non-linear partial differential equations. In the case of one dimensional motion of two particles they are:

$$(1-\dot{x}^n\dot{x}^n)\frac{\partial a^n}{\partial \dot{x}^n} + [1-\dot{x}^m\dot{x}^m + (x^n - x^m)a^m]\frac{\partial a^n}{\partial \dot{x}^m}$$

$$-\dot{x}^m x\frac{\partial a^n}{\partial x} + 3\dot{x}^n a^n = 0, \tag{1.2}$$

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$m \neq n$, $n = 1, 2$, no summation on n. We call them C-H equations. These equations were integrated by Hill [6]. His solutions, given in implicit form are presented in very unconvenient form for further physical discussion. In the present paper we obtain a class of solutions of Lorentz invariance conditions whose form allows us to obtain physically realistic models of IPRD. We say that a model is physically realistic if it has the following characteristics (they have been already formulated in our previous paper [3]):

A. The system of equations of motion (1.2) has two independent constants of motion E and P which transform like energy and momentum if one changes the inertial frame.

B. The constants of motion E and P should reduce to the free particle form

$$\mathring{E} = \frac{m_1}{\sqrt{1-\dot{x}^1\dot{x}^1}} + \frac{m_2}{\sqrt{1-\dot{x}^2\dot{x}^2}}, \qquad \mathring{P} = \frac{m_1\dot{x}^1}{\sqrt{1-\dot{x}\dot{x}^1}} + \frac{m_2\dot{x}^2}{\sqrt{1-\dot{x}^2\dot{x}^2}} \tag{1.3}$$

either when $|\vec{x}| \to \infty$ or when one switches off the interaction whose magnitude would be characterized by some coupling constant. Note that all the explicit solutions known until now do not have this characteristics [7, 8]. It seems that the solutions given in Ref. [6] also have this drawback since the transition from unphysical variables ϕ, η, ζ to physical variables becomes singular in the free particle limit.

C. The dynamics has the correct non-relativistic limit.

D. The "forces" never permit the particle velocities to exceed the velocity of light.

E. The dynamics is symmetric with respect to the interchange of the particles.

F. The dynamics is invariant with respect to the reflections $x_n \to -x_n$, which correspond to the rotations by angle π with respect to the axis perpendicular to $0x$ axis. This requirement is related to the rotation invariance of the three dimensional model.

G. The functions $a_n(x, \dot{x}^1, \dot{x}^2)$, $E(x, \dot{x}^1, \dot{x}^2)$ and $P(x, \dot{x}^1, \dot{x}^2)$ should be regular functions of all physically admissible values of variables.

All known until now explicit solutions of C-H equations (1.2) do not satisfy some of these requirements. It is difficult to select the physically realistic model from the solutions presented in implicit form in Ref. [6], because of unconvenient choice of variables. In the present paper we obtain a class of explicit solutions of Lorentz invariance conditions (1.2) described by a convergent iteration procedure, which have the characteristics A-F and describe the unbound motions of two particles in one space dimension. The case of bound motion is more difficult to handle and may easily lead to the situation in which the requirement G is not satisfied.

Instead of solving the C-H equations directly, we first assume, following Kerner [9], that the equations of motion (1.1) are "half integrated", i.e. that the motion is determined by the equations

$$\dot{x}^n = \varphi_n(x, c_1, c_2), \qquad n = 1, 2, \tag{1.4}$$

where c_1, c_2 are some first constants of motion, which have simple transformation properties. We choose $c_i = V_i$ where V_i are constants of motion, which transform like free particle velocity when one changes the inertial frame. They are related to the energy E and momentum P of the whole system by the equations

$$E = \sum_{i=1}^{2} \frac{m_i}{\sqrt{1-V_i^2}}, \quad P = \sum_{i=1}^{2} \frac{m_i V_i}{\sqrt{1-V_i^2}}. \tag{1.5}$$

Thus the energy and momentum transform correctly and the requirement B is equivalent to

$$\varphi_n = V_n \tag{1.6}$$

for $|x| \to \infty$ or when one switches off the interaction. One easily checks that the Lorentz invariance of dynamics leads to the equations [9]:

$$\varphi_n^2 - 1 = x\varphi_m \frac{\partial \varphi_n}{\partial x} + \sum_{s=1}^{2} (1-V_s^2) \frac{\partial \varphi_n}{\partial V_s}, \tag{1.7}$$

where $m, n = 1, 2$ and $n \neq m$. More precisely, equations (1.7) lead to C-H equations provided the dynamics is non-degenerate, i.e. $\det\left(\frac{\partial \varphi_n}{\partial V_n}\right) \neq 0$. This condition allows us to calculate the functions $V_n = V_n(x, \dot{x}^1, \dot{x}^2)$ from the system $\dot{x}^n = \varphi_n(x, V_1, V_2)$ and then one finds energy, momentum E, P and "forces" a_n expressed as functions of relative particle position x and velocities $\dot{x}^n$.

All examples found by Kerner are degenerate in that sense [9]. In Sections 2 and 3 we find the general solution of the system (1.7) satisfying the requirements A-F which are given in implicit form, except the harmonic oscillator case and we propose in Section 6 an iteration procedure which gives the explicit expressions for the functions $\varphi_n(x, \eta, \zeta)$, $E(x, \dot{x}^1, \dot{x}^2)$, $P(x, \dot{x}^1, \dot{x}^2)$ and $a_n(x, \dot{x}^1, \dot{x}^2)$. If these functions are found, we say that the dynamics is given explicitly. In Section 3 we present the parametric description of world lines of the particles.

Summarizing, one may say that our work constitutes a complement to the papers [6] and [9].

2. *Integration of Lorentz invariance conditions*

In our approach the Lorentz invariance of the dynamics is ensured by equations (1.7). It is convenient to introduce the global and relative rapidity variables η and ζ defined by the equations

$$\eta = \alpha_1 W_1 + (1-\alpha_1) W_2 \tag{2.1}$$

$$\zeta = W_1 - W_2, \quad V_n = \operatorname{th} W_n, \tag{2.2}$$

where $\alpha_1(\zeta)$ may be an arbitrary function of ζ having the property: $\alpha_1 \cong m_1/M$ for $|\zeta| \ll 1$. This condition ensures that in the non-relativistic limit η is simply the speed of center of mass system. Now, the system (1.7) takes on the form

$$1-\varphi_n^2 = \frac{\partial \varphi_n}{\partial \eta} - x\varphi_m \frac{\partial \varphi_n}{\partial x} \quad \text{for} \quad m \neq n. \tag{2.3}$$

This system of equations has the following particular solutions which do not depend on variable x:

$$\varphi_n = \operatorname{th}(\eta+\delta_n(\zeta)), \tag{2.4}$$

where $\delta_n(\zeta)$ is an arbitrary function. Equation (2.4) describes free particles if

$$\delta_1 = (1-\alpha_1)\zeta, \quad \delta_2 = -\alpha_1\zeta. \tag{2.5}$$

This result suggests the following substitutions for a general solution of equation (2.3)

$$\delta_n = \delta_n(\zeta, \eta, x), \quad u_n = u_n(x, \eta, \zeta) \equiv \operatorname{th}\delta_n$$

which according to (2.4) lead to

$$\varphi_n = \operatorname{th}(\eta+\delta_n) = \frac{\operatorname{th}\eta + u_n}{1+\operatorname{th}\eta\, u_n}, \quad |u_n| \leqslant 1. \tag{2.6}$$

These substitutions lead to the following form of equation (2.3):

$$(1+\operatorname{th}\eta\, u_m)\frac{\partial u_n}{\partial \eta} - x(\operatorname{th}\eta + u_m)\frac{\partial u_n}{\partial x} = 0, \quad n \neq m \tag{2.7}$$

which after a change of variables η, x into

$$\alpha = \frac{x}{l_0}\operatorname{ch}\eta \quad \text{and} \quad \beta = \frac{x}{l_0}\operatorname{sh}\eta, \tag{2.8}$$

leads to the system

$$\frac{\partial u_n}{\partial \beta} - u_m \frac{\partial u_n}{\partial \alpha} = 0, \quad n \neq m, \tag{2.9}$$

where l_0 is an arbitrary constant having the dimension of length. The system (2.9) is integrated in Appendix A. One may verify directly that the following system of equations defines implicitly solutions of (2.9):

$$u_n = -f_n'(s_n, \zeta), \tag{2.10}$$

$$\alpha = f_1(s_1, \zeta) + f_2(s_2, \zeta), \tag{2.11a}$$

$$\beta = s_1 + s_2, \tag{2.11b}$$

where $f_n(s_n, \zeta)$ are arbitrary functions and $f_n'(s_n, \zeta) \equiv \dfrac{\partial f_n}{\partial s_n}$ for $n = 1, 2$ (no summation over n!). The functions $u_n(x, \eta, \zeta)$ are found after elimination of the variables s_n from (2.10) i.e. one should calculate the functions $s_n = s_n(x, \eta, \zeta)$ from the system (2.11). This can be done provided the following condition holds

$$f_1' - f_2' = u_2 - u_1 \neq 0.$$

Following this procedure, the functions $\varphi_n(x, \eta, \zeta)$ are determined after using equations (2.6). Some cases, when the functions φ_n may be found explicitly, are presented in Appendix B and in Section 5.

The "forces" a_n may now be found from the equations

$$a_n = \frac{d\varphi_n}{dt} = \frac{\partial \varphi_n}{\partial s_n} \frac{\partial s_n}{\partial x} (\dot{x}^1 - \dot{x}^2). \tag{2.1-}$$

The derivation of (2.11) with respect to the variable x produces

$$\frac{\partial s_1}{\partial x} = [l_0 \operatorname{ch} \eta (\varphi_2 - \varphi_1)(1 + \operatorname{th} \eta\, u_1)]^{-1},$$

$$\frac{\partial s_2}{\partial x} = -[l_0 \operatorname{ch} \eta (\varphi_2 - \varphi_1)(1 + \operatorname{th} \eta\, u_2)]^{-1} \tag{2.13}$$

and thus

$$a_n = \frac{(-1)^{n+1}}{l_0} f_n''[1 + \operatorname{th} \eta\, u_n]^{-3} \operatorname{ch}^{-3} \eta \tag{2.14a}$$

or in another form, obtained by applying Eq. (2.6)

$$a_n = \frac{(-1)^{n+1}}{l_0} f_n''[1 + \operatorname{th} \eta\, \varphi_n]^3 \operatorname{ch}^3 \eta, \quad \text{where} \quad f_n'' \equiv \frac{\partial^2 f_n}{\partial s_n^2}. \tag{2.14b}$$

The functions $s(x, \eta, \zeta)$ and next the functions $\eta(x, \dot{x}^1, \dot{x}^2)$, $\zeta(x, \dot{x}^1, x^2)$, obtained from the system $\dot{x}^n = \varphi^n$ may be found explicitly only for some special choices of the functions f_n. These cases are presented in Appendix B. They provide some new solutions of C-H equations (1.2) and thus constitute a good test of our approach, but they are not physically realistic, since they lead to a dynamics which does not have the characteristics B, C, D, G.

3. Parametric description of world lines

It is interesting to note that in IPRD it is easier to find the trajectories than the "forces" a_n. The description of trajectories becomes particularly simple when one tries to parametrize them with the help of our parameters s_n.

Let us rewrite equations (2.6) in the form

$$\frac{dx^n}{ds^n} = G_n \left(\operatorname{th} \eta - \frac{df_n}{ds_n} \right), \tag{3.1}$$

$$\frac{dt^n}{ds^n} = G_n \left(1 - \operatorname{th} \eta \frac{df_n}{ds_n} \right), \tag{3.2}$$

where the functions G_n are chosen in such a form that after the integration of the equations (3.1), (3.2) with respect to the variable s_n one gets the expressions which for $t_1 = t_2$ do not contradict the system (2.11). One checks that the choice $G_1 = -l_0 \operatorname{ch}\eta$, $G_2 = l_0 \operatorname{ch}\eta$ satisfies this requirement. Thus after the integration of (2.18) of equations (3.1), (3.2) with respect to the variable s_n one gets

$$x^n(s_n) = (-1)^{n+1} l_0[\operatorname{ch}\eta f_n - \operatorname{sh}\eta\, s_n] + d_0,$$

$$t^n(s_n) = (-1)^{n+1} l_0[\operatorname{sh}\eta f_n - \operatorname{ch}\eta\, s_n] + c_0, \tag{3.3}$$

where d_0, c_0 are the integration constants. The system (3.3) leads to a 4-parameter family of trajectories $x^n(t_n, d_0, f_0, \eta, \zeta)$ whose Lorentz invariance is ensured by the following transformation properties of the parameters ζ, η, c_0, d_0:

$$\zeta' = \zeta, \quad \eta' = \eta - W, \quad d_0' = \operatorname{ch} W d_0 - \operatorname{sh} W c_0,$$

$$c_0' = c_0 \operatorname{ch} W - d_0 \operatorname{sh} W, \tag{3.4}$$

where $V = \operatorname{th}(W)$ is the relative velocity of two different inertial frames and the parameters s_n are Lorentz scalars. Transformation (3.4) ensures that x^n and t^n transform properly.

The description of trajectories by formulae (3.3) and (3.4) is interesting since various approaches to Predictive Relativistic Dynamics meet here. IPRD is defined by the relation between s_2 and s_1 described by equations (2.11) which ensure that $t_1 = t_2$. The covariant relation

$$f_1 + f_2 = s_1 + s_2 \tag{3.5}$$

means that $x_\mu x_\mu = 0$ which leads to the dynamics discussed in Ref. [12] and the relation $s_1 = -s_2$ (3.5) leads to the covariant relation $x_\mu W_\mu = 0$ where W_μ is the "two-vector" having a space component $M(\zeta) \operatorname{sh}\eta$ and a time component $M(\zeta) \operatorname{ch}\eta$ where $M(\zeta)$ is an arbitrary function of ζ. Thus relation (3.5) defines the dynamics in which $t_1 = t_2$ in a dynamically determined frame, for which $\eta = 0$. If one chooses α_1 from equation (2.1) and $M(\zeta)$ defined by the relations

$$\operatorname{sh}(\alpha_1\zeta) = \frac{m_2}{M(\zeta)} \operatorname{sh}\zeta,$$

$$M(\zeta) = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \operatorname{ch}\zeta} \tag{3.6}$$

one obtains from (1.5), (2.1), (2.2) that $W_\mu = P_\mu$ where P_μ is the energy-momentum two--vector of the whole system. Thus relations (3.5) and (3.6) lead to the dynamics described by constraint formalism in which $x_\mu P_\mu = 0$ [11].

The relation between our formulae (3.3) and formulae (89) and (90) from Ref. [6] has the form

$$\frac{l_0}{2}(s_1 - f_1) = \frac{\partial f_H}{\partial \xi_H}, \quad \frac{l_0}{2}(s_2 - f_2) = \frac{\partial g_H}{\partial \eta_H},$$

$$\frac{l_0}{2}(s_1+f_1) = \frac{\partial f_H}{\partial \xi_H}(\xi_H-1), \quad \frac{l_0}{2}(s_2+f_2) = \frac{\partial g_H}{\partial \eta_H}(\eta_H-1),$$

$$\xi_H = \zeta, \tag{3.7}$$

where the symbols with index H are those from Ref. [6].

Note that the parametric description given in Ref. [6] is singular when passing to the free particle limit when one should have $\frac{\partial^2 f_H}{\partial \xi_H} \to \infty$, $\frac{\partial^2 g_H}{\partial \eta_H} \to \infty$ and thus the transition to physical variables $\dot{x}^1$, $\dot{x}^2$, x becomes singular. Our parametrization does not have this drawback and is simpler since the derivatives of $f_n(s_n, \zeta)$ do not appear in (3.3) and the constants of motion V_n have a simple physical meaning of asymptotic particle velocities.

4. *Construction of the realistic model of IPRD in implicit form*

Now, the problem consists in finding a class of functions $f_n(s_n, \zeta)$ which describe the dynamics satisfying our requirements A-G.

Let us first consider the requirement C, i.e. let us look for a dynamics which is a relativistic extension of Newtonian dynamics described by the potential $V(x)$. The non-relativistic limit in our notation is described by the following relations $|\varphi_n| \ll 1$, $|\eta| \ll 1$, $|\delta_n| \ll 1$, $\zeta \cong V_1 - V_2$, $\alpha_1 \cong \frac{m_1}{M}$, $M = m_1 + m_2$. Eq. (2.6) now is $\varphi_n \cong \eta + u_n$ and one easily establishes that one should have in this limit

$$u_1 \cong (1-\alpha_1)\zeta\sqrt{1-\frac{2V(x)}{\mu\zeta^2}}$$

$$u_2 \cong -\alpha_1\zeta\sqrt{1-\frac{2V(x)}{\mu\zeta^2}}, \quad \text{where} \quad \mu = \frac{m_1 m_2}{M} \tag{4.1}$$

since these formulae lead to correct non-relativistic expressions for energy and momentum.

Let us introduce the function

$$U(s, \zeta) \equiv l_0[f_1(s, \zeta)+f_2(-s, \zeta)]. \tag{4.2}$$

Now, Eq. (2.11a) acquires either the form

$$x \operatorname{ch} \eta = U(s_1, \zeta)+l_0[f_2(\beta-s_1)-f_2(-s_1)] \tag{4.3}$$

or the form

$$x \operatorname{ch} \eta = U(-s_2, \zeta)+l_0[f_1(\beta-s_2)-f_1(-s_2)], \tag{4.4}$$

where according to Eq. (2.8) $\beta = \frac{x}{l_0} \operatorname{sh} \eta$. We see that in the non-relativistic limit characterized by the condition $|\eta| \ll 1$ we will have

$$U(s_1, \zeta) \cong U(-s_2, \zeta) = U(s_1-\beta_1, \zeta) \cong x$$

so that relativistic extension of Newtonian dynamics may be made by using in formulae (4.1) $V(U(s_1))$ and $V(U(-s_2))$ instead of $V(x)$. Speaking more precisely, this extension can be done with the following procedure: Choose any functions $u_n = F_n(U, \zeta; m_1, m_2)$ having the property

$$F_1 \cong \frac{m_2}{M}\zeta\sqrt{1-\frac{2V(U(s_1))}{\mu\zeta^2}},$$

$$F_2 \cong -\frac{m_1}{M}\zeta\sqrt{1-\frac{2V(U(-s_2))}{\mu\zeta^2}} \tag{4.5}$$

for $|\zeta| \ll 1$, where $\mu = m_1 m_2/M$. The functions $f_n(s_n, \zeta)$ will then be found with the help of the following integrations:

$$f_1(s_1, \zeta) = -\int \frac{F_1 dU}{l_0[F_2-F_1]}\bigg|_{U=U(s_1,\zeta)},$$

$$f_2(s_2, \zeta) = -\int \frac{F_2 dU}{l_0[F_2-F_1]}\bigg|_{U=U(-s_2,\zeta)}, \tag{4.6}$$

where the function $U(s, \zeta)$ is determined by the differential equation

$$\frac{\partial U(s,\zeta)}{\partial s} = l_0[F_2(U,\zeta)-F_1(U,\zeta)]. \tag{4.7}$$

Now, the "forces" a_n may be expressed by the functions $F_n(U, \zeta)$. Using Eq. (2.14b) we get

$$a_1 = -\frac{\partial F_1}{\partial U}(F_2-F_1)(1+\operatorname{th}\eta\,\varphi_1)^3\operatorname{ch}^3\eta\bigg|_{U=U(s_1,\zeta)},$$

$$a_2 = -\frac{\partial F_2}{\partial U}(F_2-F_1)(1+\operatorname{th}\eta\,\varphi_2)^3\operatorname{ch}^3\eta\bigg|_{U=U(-s_2,\zeta)}. \tag{4.8}$$

Using condition (4.5) we easily check that in non-relativistic limit the system (4.8) becomes the system of Newtonian equations of motion

$$a_1 = -\frac{V'(x)}{m_1}, \quad a_2 = \frac{V'(x)}{m_2}$$

since $U(s_1) \cong U(-s_2) \cong x$ in that limit.

As an example we calculate the functions $f_n(s_n, \zeta)$ for the harmonic oscillator in the next Section.

The requirement B is satisfied provided the following conditions hold:

$$F_1|_{U\to\pm\infty} = \operatorname{th}[(1-\alpha_1)\zeta], \quad F_2|_{U\to\pm\infty} = -\operatorname{th}[\alpha_1\zeta]. \tag{4.9}$$

The interaction may be easily switched off by writing

$$F_1 = \text{th}\,[(1-\alpha_1)\zeta]+\tilde{F}_1, \quad F_2 = -\text{th}\,[\alpha_1\zeta]+\tilde{F}_2 \tag{4.10}$$

and imposing $\tilde{F}_n$ to vanish.

The requirements D, E, F lead to the following conditions imposed on the functions $F_n(U, \zeta)$

$$|F_i| = |U_i| \leqslant 1, \tag{4.11}$$

$$F_i(-U, -\zeta) = -F_i(U, \zeta), \tag{4.12}$$

$$F_i(-U, -\zeta, m_2, m_1) = F_i(U, \zeta, m_1, m_2), \tag{4.13}$$

$$\det\begin{bmatrix} \dfrac{\partial\varphi_1}{\partial\eta}, & \dfrac{\partial\varphi_2}{\partial\eta} \\ \dfrac{\partial\varphi_1}{\partial\zeta}, & \dfrac{\partial\varphi_2}{\partial\zeta} \end{bmatrix} \neq 0, \quad i = 1, 2. \tag{4.14}$$

The last condition ensures that it is possible to obtain the functions $\eta(x, \dot{x}^1, \dot{x}^2)$, $\zeta(x, \dot{x}^1, \dot{x}^2)$ from the system $\dot{x}^n = \varphi_n(x, \eta, \zeta)$. This inversion procedure should be done if one wants to have the dynamics explicitly. Knowing these functions we will know the "forces" $a_n(x, \dot{x}^1, \dot{x}^2)$ from equations (4.8) and the constants of motion $E(x, \dot{x}^1, \dot{x}^2)$ and $P(x, \dot{x}^1, \dot{x}^2)$ will be known from equations (1.5) which now may be written in the form

$$E = m_1\,\text{ch}\,[\eta+(1-\alpha_1)\zeta]+m_2\,\text{ch}\,[\eta-\alpha_1\zeta],$$

$$P = m_1\,\text{sh}\,[\eta+(1-\alpha_1)\zeta]+m_2\,\text{sh}\,[\eta-\alpha_1\zeta]. \tag{4.15}$$

The procedure of elimination of the variables s_n and constants η, ζ from equations (2.6), (4.8) and (4.15) should be made with the help of some numerical methods in most physically interesting cases. In Section **6** we propose an iteration procedure giving the functions $\varphi_n(x, \eta, \zeta)$ and $\eta(x, \dot{x}^1, \dot{x}^2)$, $\zeta(x, \dot{x}^1, \dot{x}^2)$.

Summarizing, formulae (2.6), (2.11), (4.3), (4.7), (4.8) and (4.15) with conditions (4.9)–(4.14) determine implicitly a physically realistic model of IPRD.

5. *Relativistic harmonic oscillator*

For the sake of simplicity we consider now the particles with equal masses and choose the functions F_n in the following form

$$u_1 = F_1(U, \zeta) = \text{th}\left(\frac{\zeta}{2}\right)\sqrt{1-\frac{V(U(s_1))}{2\mu\,\text{th}^2\left(\dfrac{\zeta}{2}\right)}},$$

$$u_2 = F_2(U, \zeta) = -\text{th}\left(\frac{\zeta}{2}\right)\sqrt{1-\frac{V(U(-s_2))}{2\mu\,\text{th}^2\left(\dfrac{\zeta}{2}\right)}}. \tag{5.1}$$

This choice allows us to find the functions $\varphi_n(x, \eta, \zeta)$ explicitly for the harmonic oscillator case when we should put $V(U) = \frac{k}{2} U^2$. Using formulae (4.7) and (4.6) we get

$$U(s) = -\frac{2\,\mathrm{th}\left(\frac{\zeta}{2}\right)}{\omega} \sin(\omega l_0 s), \tag{5.2}$$

where

$$\omega = \sqrt{\frac{k}{\mu}} \tag{5.3}$$

and thus

$$f_1 = (2l_0)^{-1} U(s_1), \quad f_2 = (2l_0)^{-1} U(-s_2). \tag{5.4}$$

The trajectories $x^n = x^n(t, \eta, \zeta, d_0, c_0)$ may already be obtained with the help of equations (3.3).

In this simple case we are able to determine the functions $\varphi_n(x, \eta, \zeta)$ explicitly since equations (2.11) are now

$$x\,\mathrm{ch}\,\eta = -2(\omega)^{-1}\,\mathrm{th}\left(\frac{\zeta}{2}\right) \sin\left[\omega l_0 s_1 - \tfrac{1}{2}\omega l_0 \beta\right] \cos\left(\frac{\omega l_0 \beta}{2}\right) \tag{5.5}$$

and thus

$$U(s_1) \equiv \overline{U}(x, \eta, \zeta) = x\,\mathrm{ch}\,\eta - (\omega)^{-1} \Delta(x, \eta, \zeta) \sin\left(\frac{\omega l_0 \beta}{2}\right)$$
$$\times\,\mathrm{sgn}\left[\cos\left(\omega l_0\left(s_1 - \frac{\beta}{2}\right)\right)\right],$$

$$U(-s_2) \equiv \overline{\overline{U}}(x, \eta, \zeta) = x\,\mathrm{ch}\,\eta + (\omega)^{-1} \Delta(x, \eta, \zeta) \sin\left(\frac{\omega l_0 \beta}{2}\right)$$
$$\times\,\mathrm{sgn}\left[\cos\left(\omega l_0\left(s_1 - \frac{\beta}{2}\right)\right)\right], \tag{5.6a}$$

where

$$\Delta(x, \eta, \zeta) = 2\,\mathrm{th}\frac{\zeta}{2} \sqrt{1 - \frac{x^2\,\mathrm{ch}^2\,\eta\,\omega^2}{4\,\mathrm{th}^2\frac{\zeta}{2}\cos^2\left(\frac{1}{2}\omega x\,\mathrm{sh}\,\eta\right)}} \tag{5.6b}$$

and

$$\mathrm{sgn}(y) = \begin{cases} +1 & \text{if} \quad y > 0 \\ -1 & \text{if} \quad y < 0. \end{cases} \tag{5.7}$$

Note that we were unable to eliminate the parameter s_1 completely, it intervenes in equation (5.6) after the half period of motion is accomplished. Similar situation will probably occur in the case fo any relativistic bound motion. We hope to be able to discuss this problem more thoroughly in a separate note.

The energy and momentum expressed by formulae (4.15) now have the form

$$E = 2m\left[1-\mathrm{th}^2\frac{\zeta}{2}\right]^{-1/2}\mathrm{ch}\,\eta, \quad P = 2m\left[1-\mathrm{th}^2\frac{\zeta}{2}\right]^{-1/2}\mathrm{sh}\,\eta \tag{5.8}$$

and the "forces" a_n are now

$$a_1 = -\tfrac{1}{2}\omega^2 U[1+\mathrm{th}\,\eta\varphi_1]^3\,\mathrm{ch}^3\,\eta|_{U=\bar{U}(x,\eta,\zeta)},$$

$$a_2 = \tfrac{1}{2}\omega^2 U[1+\mathrm{th}\,\eta\varphi_2]^3\,\mathrm{ch}^3\,\eta|_{U=\bar{\bar{U}}(x,\eta,\zeta)}. \tag{5.9}$$

We may now find the explicit approximate form of dynamics in some limited area of variables, e.g. let us consider the region defined by $|\eta| \ll 1$. Then formulae (5.7), (5.1) and (2.6) lead to the following approximate relations

$$\bar{U} \cong \bar{\bar{U}} \cong x, \quad \eta \cong \tfrac{1}{2}(w_1+w_2), \tag{5.10}$$

$$\mathrm{th}^2\frac{\zeta}{2} \cong \mathrm{th}^2\frac{w_1-w_2}{2}+\frac{kx^2}{2}, \tag{5.11}$$

where

$$w_n = \mathrm{arth}\,\dot{x}^n, \quad n = 1, 2. \tag{5.12}$$

6. Explicit form of dynamics obtained by iteration procedure

The functions

$$\varphi_n(x,\eta,\zeta) = \frac{\mathrm{th}\,\eta+u_n}{1+\mathrm{th}\,\eta\, u_n} \tag{6.1}$$

will be known after performing the elimination of the variables s_n from the functions u_n by using (2.11b) and (4.3). Let us rewrite the latter in the following, more convenient for iterations form

$$\bar{U} = x\,\mathrm{ch}\,\eta+G(\beta,\bar{U},\zeta), \tag{6.2}$$

where

$$\bar{U} \equiv U(s_1), \quad G(\beta,\bar{U},\zeta) = -l_0[f_2(\beta-s_1(\bar{U}))-f_2(-s_1(\bar{U}))] \tag{6.3}$$

and

$$s_1(\bar{U}) = \int\frac{dU}{l_0[F_2-F_1]}\bigg|_{U=\bar{U}}. \tag{6.4}$$

We propose to determine the unknown function $\bar{U}(x,\eta,\zeta)$ described implicitly by (6.2) with the help of the following iteration procedure

$$\overset{(n+1)}{\bar{U}} = x\,\mathrm{ch}\,\eta+G(\beta,\overset{(n)}{\bar{U}},\zeta), \quad n = 0, 1, 2, \ldots \tag{6.5}$$

The sufficient convergence condition of the series (6.5) is determined by the condition

$$\left|\frac{\partial G(\beta, \overline{U}, \zeta)}{\partial \overline{U}}\right| < 1. \tag{6.6}$$

One finds that

$$\frac{\partial G}{\partial \overline{U}} = \frac{F_2(\overline{\overline{U}}, \zeta) - F_2(\overline{U}, \zeta)}{F_2(\overline{U}, \zeta) - F_1(\overline{U}, \zeta)}, \tag{6.7}$$

where

$$\overline{\overline{U}} = U(s_1(\overline{U}) - \beta). \tag{6.8}$$

For $\beta = \dfrac{x}{l_0}\,\mathrm{sh}\,\eta$ so small that the Taylor expansion in (6.8) and (6.7) are justified one obtains

$$\frac{\partial G}{\partial \overline{U}} \cong x\,\mathrm{sh}\,\eta\,\frac{\partial F_2}{\partial \overline{U}}. \tag{6.9}$$

Thus, we may conclude that our iteration procedure will be well convergent either in some limited area of variables x, η, ζ or in the case of sufficiently "weak" interaction (note that $\partial F_2/\partial U = 0$ for free particles). The latter conclusion will be more clear if one makes some particular choice of functions F_n, e.g. let us take

$$F_1 = \mathrm{th}\left(\frac{m_2}{M}\zeta\right)\sqrt{1 - \frac{2V(U)}{\mu\zeta^2}}\Bigg|_{U=\overline{U}},$$

$$F_2 = -\mathrm{th}\left(\frac{m_1}{M}\zeta\right)\sqrt{1 - \frac{2V(U)}{\mu\zeta^2}}\Bigg|_{U=\overline{\overline{U}}}. \tag{6.10}$$

Now, formulae (6.9) and (6.7) yield the following convergence condition of the series (6.5):

$$\left|\frac{x\,\mathrm{sh}\,\eta\,\mathrm{th}\left(\dfrac{m_1}{M}\zeta\right)\dfrac{\partial V}{\partial \overline{U}}}{\mu\zeta^2\sqrt{1 - \dfrac{2V(\overline{U})}{\mu\zeta^2}}}\right| < 1. \tag{6.11}$$

We conclude that for unbound motions, when $\sqrt{1 - \dfrac{2V(U)}{\eta\zeta^2}} \neq 0$ and sufficiently smooth potentials $V(x)$ our procedure should work well. For η small formula (6.2) suggests starting the series (6.5) by the expressions

$$\overset{0}{\overline{U}} = x[\mathrm{ch}\,\eta + \mathrm{sh}\,\eta\,F_2(x\,\mathrm{ch}\,\eta, \zeta)],$$

$$\overset{0}{\overline{\overline{U}}} = x[\mathrm{ch}\,\eta + \mathrm{sh}\,\eta\,F_1(x\,\mathrm{ch}\,\eta, \zeta)]. \tag{6.12}$$

The series (6.5) should be particularly well convergent in the asymptotic case of weak interaction between particle defined by

$$\left|\frac{V(U)}{\mu\zeta^2}\right| \ll 1, \qquad \left|\frac{\partial V}{\partial U}\ \frac{x \operatorname{sh}\eta}{\mu\zeta^2}\right| \ll 1. \tag{6.13}$$

Then, it seems to be reasonable to start this series by

$$\overset{0}{\overline{U}} = x[\operatorname{ch}\eta - \operatorname{sh}\eta \operatorname{th}(\alpha_1\zeta)] = x\frac{\operatorname{ch} W_2}{\operatorname{ch}(\alpha_1\zeta)},$$

$$\overset{0}{\overline{\overline{U}}} = x[\operatorname{ch}\eta + \operatorname{sh}\eta \operatorname{th}[(1-\alpha_1)\zeta]] = x\frac{\operatorname{ch} W_2}{\operatorname{ch}[(1-\alpha_1)\zeta]} \tag{6.14}$$

and thus

$$\varphi_1 = \left.\frac{\operatorname{th}\eta + F_1(\overline{U},\zeta)}{1+F_1(\overline{U},\zeta)\operatorname{th}\eta}\right|_{\overline{U}=\overset{0}{\overline{U}}}, \qquad \varphi_2 = \left.\frac{\operatorname{th}\eta + F_2(\overline{\overline{U}},\zeta)}{1+F_2(\overline{\overline{U}},\zeta)\operatorname{th}\eta}\right|_{\overline{\overline{U}}=\overset{0}{\overline{\overline{U}}}} \tag{6.15}$$

describe the asymptotic behaviour of the functions φ_n.

The explicit form of dynamics will be known after calculating the functions $\eta(x, \dot{x}^1, \dot{x}^2)$ and $\zeta(x, \dot{x}^1, \dot{x}^2)$ from the system $\dot{x}^n = \varphi_n(x, \eta, \zeta)$ which we will rewrite in a form, convenient for iterations:

$$\zeta = w_1 - w_2 + H_1(\eta, \zeta, x),$$

$$\eta = \alpha_1 w_1 + (1-\alpha_1)w_2 + H_2(\eta, \zeta, x) \tag{6.16}$$

where

$$w_n = \operatorname{arth} \dot{x}^n,$$

$$H_1 = \zeta + \operatorname{arth} F_2 - \operatorname{arth} F_1, \qquad H_2 = -\alpha_1 \operatorname{arth} F_1 - (1-\alpha_1)\operatorname{arth} F_2. \tag{6.17}$$

Let us try to solve the system (6.16) in the following two situations:

1st case. Asymptotic region — weakly interacting particles

We assume again that (6.13) holds, then it seems reasonable to solve (6.16) using the series

$$\overset{(n+1)}{\zeta} = w_1 - w_2 + H_1(\overset{(n)}{\eta}, \overset{(n)}{\zeta}, x),$$

$$\overset{(n+1)}{\eta} = \alpha_1(\overset{(n)}{\zeta})w_1 + [1-\alpha_1(\overset{(n)}{\zeta})]w_2 + H_2(\overset{(n)}{\eta}, \overset{(n)}{\zeta}, x) \tag{6.18}$$

with

$$\overset{(0)}{\zeta} = w_1 - w_2, \qquad \overset{(0)}{\eta} = \alpha_1(\overset{0}{\zeta})w_1 + [1-\alpha_1(\overset{0}{\zeta})]w_2 \tag{6.19}$$

being the solutions of (6.16) for free particles. Thus we already obtain the asymptotic form of "forces" a_n using $\overset{0}{\overline{U}}$, $\overset{0}{\overline{\overline{U}}}$ expressed by (6.14) and putting $\eta = \overset{0}{\eta}$, $\zeta = \overset{(0)}{\zeta}$.

Let us illustrate these results by choosing the functions F_n in a form

$$F_1 = \operatorname{th}\left[\frac{m_2}{M}\varrho_1\right], \quad F_2 = -\operatorname{th}\left[\frac{m_1}{M}\varrho_2\right], \tag{6.20}$$

where

$$\varrho_1 = \zeta\sqrt{1-\frac{2V\left[\bar{U}\operatorname{ch}\left(\frac{m_1}{M}\zeta\right)\right]}{\mu\zeta^2}}, \quad \varrho_2 = \zeta\sqrt{1-\frac{2V\left[\bar{\bar{U}}\operatorname{ch}\left(\frac{m_2}{M}\zeta\right)\right]}{\mu\zeta^2}} \tag{6.21}$$

and $\alpha_1 = m_1/M$, which gives a relatively simple form of dynamics in the asymptotic region. Now formulae (6.17) are

$$H_1 = \zeta - \frac{m_2}{M}\varrho_1 - \frac{m_1}{M}\varrho_2,$$

$$H_2 = -\frac{m_1 m_2}{M}[\varrho_1 - \varrho_2]. \tag{6.22}$$

We put $\overset{(1)}{\eta}, \overset{(1)}{\zeta}$ calculated from (6.18) into (4.15) and make several Taylor expansions justified by (6.13). This way we obtain the approximate expressions for the energy and momentum of the system

$$E = \overset{(0)}{E} + (w_1 - w_2)^{-1}\,[\operatorname{sh} w_1 V(x \operatorname{ch} w_1) - \operatorname{sh} w_2 V(x \operatorname{ch} w_2)],$$

$$P = \overset{(0)}{P} + (w_1 - w_2)^{-1}[\operatorname{ch} w_1 V(x \operatorname{ch} w_1) - \operatorname{ch} w_2 V(x \operatorname{ch} w_2)], \tag{6.23}$$

where $\overset{(0)}{E}, \overset{(0)}{P}$ are free particle energy and momentum given by (1.3) and $w_n = \operatorname{arth} \dot{x}^n$. Remember that these formulae are valid only if

$$\left|\frac{V(x \operatorname{ch} w_n)}{\mu(w_1 - w_2)^2}\right| \ll 1, \quad \left|\frac{V'(x \operatorname{ch} w_2) x \operatorname{sh} \overset{0}{\eta}}{\mu(w_1 - w_2)^2}\right| \ll 1 \tag{6.24}$$

hold. The "forces" a_n for the dynamics described by (6.20), calculated with the help of formulae (4.8), are

$$a_1 = -\frac{V'\left[\bar{U}\operatorname{ch}\left(\frac{m_1}{M}\zeta\right)\right]\operatorname{sh}\varrho_1(1+\operatorname{th}\eta\,\varphi_1)^3\operatorname{ch}^3\eta}{m_1\varrho_1\operatorname{ch}^2\left(\frac{m_2}{M}\varrho_1\right)},$$

$$a_2 = \frac{V'\left[\bar{\bar{U}}\operatorname{ch}\left(\frac{m_2}{M}\zeta\right)\right]\operatorname{sh}\varrho_2(1+\operatorname{th}\eta\,\varphi_2)^3\operatorname{ch}^3\eta}{m_2\varrho_2\operatorname{ch}^2\left(\frac{m_1}{M}\varrho_2\right)}. \tag{6.25}$$

In the asymptotic region defined by conditions (6.24) we get

$$a_1 \cong -\frac{V'(x \operatorname{ch} w_2) \operatorname{sh} \overset{(0)}{\zeta} (1+\operatorname{th} \overset{(0)}{\eta} \dot{x}^1)^3 \operatorname{ch}^3 \overset{(0)}{\eta}}{m_1 \overset{0}{\zeta} \operatorname{ch}^2\left(\frac{m_2}{M} \overset{(0)}{\zeta}\right)},$$

$$a_2 \cong \frac{V'(x \operatorname{ch} w_1) \operatorname{sh} \overset{(0)}{\zeta} (1+\operatorname{th} \overset{(0)}{\eta} \dot{x}^2)^3 \operatorname{ch}^3 \overset{(0)}{\eta}}{m_2 \overset{0}{\zeta} \operatorname{ch}^2\left(\frac{m_1}{M} \overset{(0)}{\zeta}\right)}, \tag{6.26}$$

where $\overset{(0)}{\zeta}$ and $\overset{(0)}{\eta}$ are described by formulae (6.19).

Summarizing, formulae (6.23) and (6.26) describe explicitly a physically realistic model of IPRD in the case of weak interaction between particles.

2nd case. Particles with mass ratio m_1/m_2 small

We assume that $m_1 \ll m_2$ and that the functions F_n are described by formulae (6.21). It will be convenient to rewrite (6.16) in the form

$$\zeta = F(w_1, w_2, \zeta, \eta) = \sqrt{\left[w_1 - w_2 + \frac{m_1}{M}(\varrho_1 - \varrho_2)\right]^2 + \frac{2V\left[\bar{U} \operatorname{ch}\left(\frac{m_2}{M}\zeta\right)\right]}{\mu}},$$

$$\eta = G(w_1, w_2, \zeta, \eta) = \frac{m_1}{M} w_1 + \frac{m_2}{M} w_2 + \frac{m_1 m_2}{M}(\varrho_2 - \varrho_1), \tag{6.27}$$

where the functions $\bar{U}(x, \eta, \zeta)$, $\bar{\bar{U}}(x, \eta, \zeta)$ should be determined from (6.2) and (6.8).

If $m_1 \ll m_2$ it seems reasonable to look for functions $\eta(x, \dot{x}^1, \dot{x}^2)$, $\zeta(x, \dot{x}^1, \dot{x}^2)$ using the iterations

$$\overset{(n+1)}{\zeta} = F(w_1, w_2, \overset{(n)}{\zeta}, \overset{(n)}{\eta}),$$

$$\overset{(n+1)}{\eta} = G(w_1, w_2, \overset{(n)}{\zeta}, \overset{(n)}{\eta}) \tag{6.28}$$

and to start this series by

$$\overset{(0)}{\zeta} = \sqrt{(w_1 - w_2)^2 + \frac{2V(x \operatorname{ch} \eta)}{\mu}}, \quad \overset{(0)}{\eta} = w_2, \tag{6.29}$$

which is a solution of (6.27) for $m_2 = \infty$. This is the solution for the dynamics of one particle in an external field. Note that the condition $m_1 \ll m_2$ ensures a good convergence of the series (6.5) in a large area of variables, since now $\partial G/\partial U$ is proportional to m_1/M.

7. *Discussion*

In conclusion, we may say that we have constructed a model of IPRD having all characteristics listed in the Introduction, except the last one; our "forces" $a_n(x, \dot{x}^1, \dot{x}^2)$ are not defined for all physically admissible values of variables $x, \dot{x}^1, \dot{x}^2$. They are defined only for values for which the iteration series (6.5) and (6.18) converge. These values of variables roughly coincide with values acquired by particles executing the unbound motions only. The problem of a definite existence of the physically interesting IPRD defined for all physically admissible values of variables $x, \dot{x}^1, \dot{x}^2$ and describing the bound motions remains unresolved.

One may argue that one-dimensional models, formulated in a non-covariant way (though Lorentz invariant) seem to be of little relevance today, since many manifestly covariant models [11–14] involving three space dimensions have been constructed during the past decade, which do not apparently have the above mentioned difficulties. It may be remarked, however, that in all physically interesting manifestly covariant models the "forces" (potentials) are functions of variables constrained to a surface being usually dynamically defined. This surface is given in the $8N$ dimensional phase space and it should impose a symmetric and transitive relations between particle four-vectors (note that this requirement is not satisfied in a covariant model given in Ref. [12]). Thus the initial value problem cannot be formulated independently of dynamics, i.e. the surface on which the initial particle positions and velocities should be chosen by the observer is not known "a priori" in Minkowski space M_4. Of course, one can construct IPRD using a covariant model but the transition from a dynamically defined surface to a surface chosen "a priori" in M_4 (such as one instant surface is) will require the knowledge of trajectories and may easily lead to the situation in which the forces of IPRD are not defined for all physically admissible values of variables. Taking this into account, it seems that it is worthwhile to investigate IPRD more thoroughly in order to see whether it is possible to construct the relativistic theory being the most direct generalization of Newtonian non-relativistic dynamics and having all its characteristics.

APPENDIX A

We explain here how we obtained solutions (2.10), (2.11) of the system (2.9). Let us rewrite it here in a slightly different form

$$\frac{\partial u_1}{\partial \beta} - u_2 \frac{\partial u_1}{\partial \alpha} = 0, \qquad \frac{\partial u_2}{\partial \beta} - u_1 \frac{\partial u_2}{\partial \alpha} = 0. \tag{A1}$$

If one of the functions u_n is known, the other may be found using the method of characteristics, thus one may write $u_1 = u_1(s_1(\alpha, \beta))$, $u_2 = u_2(s_2(\alpha, \beta))$ where $u_n(s_n)$ are arbitrary functions of variables s_n, the equations $s_n(\alpha, \beta) = \text{const}$ define two characteristic lines of the system (A1). Now, the problem consists in finding such new variables s_n that the system (A1) becomes

$$\chi_2^{-1}(s_1, s_2) \frac{\partial u_1}{\partial s_2} = 0, \qquad \chi_1^{-1}(s_1, s_2) \frac{\partial u_2}{\partial s_1} = 0, \tag{A2}$$

where χ_n are arbitrary functions. Let us make in (A2) a change of variables s_n into the variables α, β. Thus one gets

$$\chi_2^{-1}\left(\frac{\partial u_1}{\partial \beta}\frac{\partial \beta}{\partial s_2}+\frac{\partial u_1}{\partial \alpha}\frac{\partial \alpha}{\partial s_2}\right)=0,$$

$$\chi_1^{-1}\left(\frac{\partial u_2}{\partial \beta}\frac{\partial \beta}{\partial s_1}+\frac{\partial u_2}{\partial \alpha}\frac{\partial \alpha}{\partial s_1}\right)=0. \tag{A3}$$

The comparison of (A3) with (Al) yields

$$\chi_n=\frac{\partial \beta}{\partial s_n}, \tag{A4}$$

$$u_1(s_1)\chi_1=-\frac{\partial \alpha}{\partial s_1}, \tag{A5}$$

$$u_2(s_2)\chi_2=-\frac{\partial \alpha}{\partial s_2}. \tag{A6}$$

Let us derive (A5) with respect to s_2 and (A6) with respect to s_1. This way we obtain

$$u_1\frac{\partial^2 \beta}{\partial s_1\partial s_2}=-\frac{\partial^2 \alpha}{\partial s_1\partial s_2},\quad u_2\frac{\partial^2 \beta}{\partial s_1\partial s_2}=-\frac{\partial^2 \alpha}{\partial s_1\partial s_2}. \tag{A7}$$

The difference between two equations of the system (A7) is

$$(u_2-u_1)\frac{\partial^2 \beta}{\partial s_1\partial s_2}=0. \tag{A8}$$

Thus for $u_1 \neq u_2$ one has

$$\frac{\partial^2 \beta}{\partial s_1\partial s_2}=\frac{\partial^2 \alpha}{\partial s_1\partial s_2}=0 \tag{A9}$$

and the solution of (Al) will be described implicitly by the following three equations

$$\alpha=f_1(s_1,\zeta)+f_2(s_2,\zeta), \tag{A10}$$

$$\beta=g_1(s_1,\zeta)+g_2(s_2,\zeta), \tag{A11}$$

$$u_n=-\frac{f_n'}{g_n'}, \tag{A12}$$

where $f_n' \equiv \partial f_n/\partial s_n$ and $g_n' \equiv \partial g_n/\partial s_n$. The redefinition $s_n \to s_n = g_n$ of the variable s_n leads to the simpler system

$$\alpha=f_1(s_1,\zeta)+f_2(s_2,\zeta), \tag{A13}$$

$$\beta=s_1+s_2 \tag{A14}$$

without loosing the generality. Thus the solution of (A1) is defined by the two arbitrary functions $f_n(s_n,\zeta)$.

APPENDIX B

We obtain here some simple solutions of equation (1.7) and of C-H equations (1.2) by applying the results of the preceding appendix for some simple choice of functions f_n, g_n which allow us to calculate explicitly the functions $\eta(x, \dot{x}^1, \dot{x}^2)$ and $\zeta(x, \dot{x}^1, \dot{x}^2)$ from the system $\dot{x}^n = \varphi_n(x, \eta, \zeta)$.

1st choice

Let us put

$$f_1 = \lambda(\zeta)s_1^2, \quad f_2 = -\lambda(\zeta)s_2^2, g_n = s_n. \tag{B1}$$

Using (A10), (A11) and (2.6) one gets

$$u_1 = -\beta^{-1}(\alpha+\lambda\beta^2), \quad u_2 = \beta^{-1}(\lambda\beta^2-\alpha) \tag{B2}$$

and

$$\dot{x}^1 = \varphi_1 = \frac{1+\lambda \frac{x}{l_0} \operatorname{sh}^2 \eta \operatorname{ch} \eta}{\lambda \frac{x}{l_0}}, \quad \dot{x}^2 = \varphi_2 = -\frac{1-\lambda \frac{x}{l_0} \operatorname{sh}^2 \eta \operatorname{ch} \eta}{\lambda \frac{x}{l_0}}. \tag{B3}$$

The functions $\eta(x, \dot{x}^1, \dot{x}^2)$ and $\zeta(x, \dot{x}^1, \dot{x}^2)$ calculated from (B3) have now the form

$$\operatorname{th} \eta = \frac{2}{\dot{x}^1+\dot{x}^2}, \tag{B4}$$

$$\lambda(\zeta) = 2l_0[x(\dot{x}^1-\dot{x}^2) \operatorname{sh}^3 \eta]^{-1}. \tag{B5}$$

Putting (B4) and (B5) into (2.14b) yields

$$a_1 = -\frac{(\dot{x}^1-\dot{x}^2)^2}{2x}, \quad a_2 = \frac{(\dot{x}^1-\dot{x}^2)^2}{2x}. \tag{B6}$$

This solution of C-H equations is already known in the literature [7, 10]. We do not find any other simple example of dynamics in explicit form being reflection invariant, i.e. satisfying the condition $a_n(-x, -\dot{x}^1, -\dot{x}^2) = -a_n(x, \dot{x}^1, \dot{x}^2)$. Next examples do not have this property.

2nd choice

We put now

$$f_n = \tfrac{1}{2}(F_n(s_n, \zeta)+s_n), \quad g_n = \tfrac{1}{2}(F_n(s_n, \zeta)-s_n) \tag{B7}$$

which according to (A10), (A11) and (2.8) leads to

$$\bar{\alpha} \equiv \frac{x}{l_0} e^{\eta} = F_1(s_1, \zeta)+F_2(s_2, \zeta), \quad \bar{\beta} \equiv \frac{x}{l_0} e^{-\eta} = s_1+s_2. \tag{B8}$$

Now, equations (A12) and (2.6) give

$$u_n = -\frac{F'_n+1}{F'_n-1}, \tag{B9}$$

$$\varphi_n = \frac{e^{2\eta}+F'_n}{e^{2\eta}-F'_n}, \tag{B10}$$

where $F'_n \equiv \partial F_n/\partial s_n$. Using (2.12) we get

$$a_1 = \frac{1}{2l_0} F''_1 e^{-3\eta}(1+\dot{x}^1)^3,$$

$$a_2 = -\frac{1}{2l_0} F''_2 e^{-3\eta}(1+\dot{x}^2)^3. \tag{B11}$$

Let us make now the following choice of the functions F_n:

$$F_1 = w(\zeta)+e^{\lambda s_1}d_1(\zeta), \quad F_2 = d_2(\zeta)e^{-\lambda s_2}. \tag{B12}$$

Equations (B8) now give

$$e^{\lambda s_1} = (\bar{\alpha}-w(\zeta))\,(d_1+d_2e^{-\lambda\bar{\beta}})^{-1}$$

and thus

$$F'_1 = \lambda d_1(\bar{\alpha}-w)\,(d_1+d_2e^{-\lambda\bar{\beta}})^{-1}, \quad F'_2 = \lambda d_2(w-\bar{\alpha})\,(d_1+d_2e^{-\lambda\bar{\beta}})^{-1}. \tag{B13}$$

From (B10) we deduce that

$$\frac{F'_1}{F'_2} = \frac{z_1}{z_2}, \tag{B14}$$

where $z_n \equiv \dfrac{1-\dot{x}^n}{1+\dot{x}^n}$ and thus

$$e^{\lambda\bar{\beta}} = -\frac{d_2}{d_1}\frac{z_1}{z_2}. \tag{B15}$$

The simplest case occurs when $\dfrac{d_2(\zeta)}{d_1(\zeta)} = c_1 = \text{const}$ because then the function $\eta(x, \dot{x}^1, \dot{x}^2)$ is expressed by

$$e^{-\eta} = \frac{l_0}{x\lambda}\ln\left(-c_1\frac{z_1}{z_2}\right) \tag{B16}$$

and this way

$$a_n = -\tfrac{1}{2}\,(1-\dot{x}^n\dot{x}^n)\,(1+\dot{x}^n)\ln\left(-c_1\frac{z_1}{z_2}\right). \tag{B17}$$

More complicated is the case when $w(\zeta) = w = \text{const}$ and $\frac{d_2}{d_1}$ is a function of ζ. Then, putting (B15) into (B13) and using (B10) gives

$$\frac{e^{-\eta}}{l_0} \equiv F(x, z_1, z_2) = 2(z_1 - z_2)\left[-\lambda x \pm \sqrt{\lambda^2 x^2 + L^2(z_1 - z_2)}\right]^{-1}, \tag{B18}$$

where $L^2 \equiv 4\lambda w l_0^2$. This formula defines the following new solution of C-H equations

$$a_n = -\frac{\lambda}{2}(1 - \dot{x}^n \dot{x}^n)(1 + \dot{x}^n) F(x, z_1, z_2), \tag{B19}$$

where $z_n \equiv \frac{1 - \dot{x}^n}{1 + \dot{x}^n}$ and the variables $x, \dot{x}^1, \dot{x}^2$ should be limited to the area in which the function $F(x, \dot{x}^1, \dot{x}^2)$ from (B18) is positive. The solutions (B17) and (B19) are not however reflection invariant, also they do not have characteristics B and D. These results constitute only a good test of our approach since we checked directly that (B17) and (B19) are indeed the solutions of C-H equations (1.2).

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