AN IMPROVED MEAN-FIELD CALCULATION FOR THE LATTICE ABELIAN HIGGS MODEL*

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We consider the lattice Abelian Higgs model with frozen radial degrees of freedom using the mean-field approximation with corrections. The free energy corrections contain essential $O(1)$ and $O(1/d)$ terms. In the weak coupling region the behaviour of the frequencies arising in the expansion of action allows one to distinguish the Higgs from the Coulomb phases. Analytical results are presented for all phase transition lines. The phase structure obtained is in qualitative agreement with Monte-Carlo calculations for Higgs charges $q = 1$, 2 and 6.

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1. *Introduction*

One of the analytical methods to study the phase structure of lattice gauge theories is the mean-field approach corresponding to a saddle-point evaluation of a properly rewritten path integral of the theory [1, 2, 3]. It is known that the simplest approximation does not always reproduce the phase structure predicted from Monte-Carlo simulations.

The mean-field approach can be systematically improved including quantum fluctuations around the saddle-point values of the fields. One expands the effective action in the rewritten path integral in terms of these fluctuations. Integrating them out in the partition function one obtains corrections to the lowest order free energy. The successful calculation of the phase structure of the U(1) [4] and the $Z(N)$ [5] models using the mean-field method including corrections indicates that also the correct phase structure of more complicated models could be found.

The purpose of the present paper is to discuss the phase structure of the lattice Abelian Higgs model in this mean-field approximation with corrections. It was shown in a paper by Ranft, Kripfganz and Ranft [6] that the simple mean-field method does not reproduce the correct phase structure of the Abelian Higgs model as found in Monte-Carlo studies. In particular for a Higgs charge $q = 1$ no end-point appears for the confinement-Higgs transition and for $q = 6$ the Coulomb phase is absent for increasing Higgs coupling κ.

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The paper is organized as follows. In Section 2 we rewrite the path integral and discuss the effective action in terms of the new variables. Section 3 deals with the saddle-point method and the expansion in terms of quantum fluctuations. A strong coupling expansion is performed in the strong coupling region. We consider the problems of zero- and $1/d$-frequencies in the weak coupling region. In Section 5 we discuss the phase structure for different Higgs charges and compare with Monte-Carlo data of Ref. [6]. Section 6 contains our conclusions. Some calculational remarks are given in the Appendices.

2. *The effective action for the lattice Abelian Higgs model*

We work with a hypercubical lattice in the Euclidean metric. We assume the space-time dimension d to be a large quantity and use $1/d$ as expansion parameter. This is the general assumption of the mean-field approximation with corrections. Furthermore we assume that $d = 4$ is large enough.

We denote the lattice sites by x, the links by (x, μ) or L, where μ is a lattice unit vector, the site variables of the Higgs field by σ and the link variable of the gauge field by U. N is the number of lattice sites. The action of the model is given by

$$S[U, \sigma] = S_{G}[U]+S_{H}[U, \sigma], \tag{2.1}$$

where S_G is the Wilson action

$$S_{G}[U] = \beta \sum_{\substack{x,\mu,\nu \\ \nu \neq \mu}} (U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\mu+\nu)U_\nu^\dagger(x+\nu)+\text{h.c.}), \tag{2.2}$$

and S_H is the Higgs action

$$S_{H}[U, \sigma] = \kappa \sum_{x,\mu} (\sigma(x)U_\mu^{q\dagger}(x)\sigma^\dagger(x+\mu)+\text{h.c.}), \tag{2.3}$$

describing the interaction between Higgs and Abelian gauge fields. The sum in Eq. (2.2) runs over all plaquettes of the lattice, β is the inverse square of the gauge coupling g

$$\beta = \frac{1}{g^2}\,.$$

The sum in Eq. (2.3) runs over all links, κ denotes the Higgs coupling. In our model the radial modes of the Higgs field are frozen. The coupling κ can be interpreted as a mean-value of the square of this mode. For the group variables we use the following notations

$$U_\mu(x) = \exp\{i\theta_\mu(x)\}, \quad \sigma(x) = \exp\{i\chi(x)\}, \tag{2.4}$$

where $\theta_\mu(x)$ and $\chi(x)$ are angle variables. The usual partition function Z of the model is defined as

$$Z = \int d\mu[U]d\mu[\sigma]\exp\{S[U, \sigma]\}, \tag{2.5}$$

where $d\mu[U]$ and $d\mu[\sigma]$ are the invariant measures of the corresponding fields.

Inserting the group variables into larger manifolds one obtains a decoupling of the genuine variables. The interaction in the plaquettes and in the Higgs action is then replaced by an interaction of each group variable with an external random field. For the Abelian Higgs model we replace the original variables U_L and σ_x by V_L and H_x acting in the full complex plane. Furthermore we introduce the following additional fields at each link

$$C_\mu(x) = -i2H(x)H^*(x+\mu), \qquad W_\mu(x) = V_\mu^q(x). \tag{2.6}$$

To rewrite the path integral (2.5) we multiply its integrand by

$$1 = \int d[V]d[H]d[C]\delta(U-V)\delta(\sigma-H)\delta(V^q-W)\delta(-i2H^*H-C). \tag{2.7}$$

We define a scalar product $(\cdot)$ for fields with link components by the expression

$$l^{(1)}\cdot l^{(2)} = \sum_L (l_L^{(1)\dagger} l_L^{(2)} + \text{h.c.}), \qquad l = \{l_L\}_{L=1}^{Nd} \tag{2.8}$$

and for fields with site components by

$$s^{(1)}\cdot s^{(2)} = \sum_x (s_x^{(1)\dagger} s_x^{(2)} + \text{h.c.}), \qquad s = \{s_x\}_{x=1}^{N}. \tag{2.9}$$

Introducing the external random fields A at each link and B at each site we express two of the δ-functions of Eq. (2.7) in form of

$$\delta(U-V) = \iint_C \prod_L \frac{dA_L^*}{4\pi}\frac{dA_L}{4\pi}\exp\left\{\frac{i}{2}A\cdot(U-V)\right\},$$

$$\delta(\sigma-H) = \iint_C \prod_x \frac{dB_x^*}{4\pi}\frac{dB_x}{4\pi}\exp\left\{\frac{i}{2}B\cdot(\sigma-H)\right\}. \tag{2.10}$$

The integration measures are defined as follows

$$d[V] = N_V\prod_L dV_L dV_L^*, \qquad d[W] = N_W\prod_L dW_L dW_L^*,$$

$$d[H] = N_H\prod_x dH_x dH_x^*, \qquad d[C] = N_C\prod_L dC_L dC_L^*, \tag{2.11}$$

where the N_i are normalization constants and V_L, W_L, C_L and H_x act in the full complex plane. The invariant measures of Eq. (2.5) are given by

$$d\mu[U] = \prod_L\left(\frac{d\theta_L}{2\pi}\right) \quad\text{and}\quad d\mu[\sigma] = \prod_x\left(\frac{d\chi_x}{2\pi}\right); \quad \theta_L, \chi_x \in [0, 2\pi]. \tag{2.12}$$

Defining the one-link integral

$$f_L^{(q)}(A, C) = \int_0^{2\pi}\frac{d\theta_L}{2\pi}\exp\left\{\frac{i}{2}(A_L^* U_L + C_L^* U_L^q + \text{h.c.})\right\} \tag{2.13}$$

and the one-site integral

$$f_x(B) = \int_0^{2\pi} \frac{d\chi_x}{2\pi} \exp\left\{\frac{i}{2}(B_x^*\sigma_x + \text{h.c.})\right\} \tag{2.14}$$

as well as the expressions

$$\omega_L^{(q)}(A, C) = \log f_L^{(q)}(A, C), \quad \omega_x(B) = \log f_x(B) \tag{2.15}$$

we obtain the rewritten path integral of the theory

$$Z = \int d[A]d[B]d[C]\delta(-i2H^*H - C)d[V]d[H]d[W]\delta(V^q - W)e^{S_{\text{eff}}}. \tag{2.16}$$

Here the effective action of the lattice Abelian Higgs model has the form

$$S_{\text{eff}} = S_{\text{G}}[V] + S_{\text{H}}[W, H] - \frac{i}{2}(A \cdot V + C \cdot W + B \cdot H)$$
$$+ \sum_L \omega_L^{(q)}(A, C) + \sum_x \omega_x(B). \tag{2.17}$$

Instead of the pure gauge field interactions in the plaquettes an interaction of the gauge fields U_L with the external random field A_L appears. The interactions of the Higgs fields σ_x in the Higgs action are replaced by their interactions with the external random fields B_x at each site. Formally we regard also the C_L field as an external random field. However, since after the integration over this C_L-field a coupling to the Higgs field H_x remains, the C_L is not really an external random field. Furthermore the C_L-integration leads to a cancellation of the terms $i/2\ (C \cdot W)$ and $S_H[W, H]$ in the effective action so that S_{eff} depends on the Higgs field H only via the one-link integral $\omega_L^{(q)}$ and the term $(B \cdot H)$.

3. *The mean-field method with corrections*

Performing a saddle-point approximation of the partition function (2.16) we obtain the following saddle-point equations

$$-\frac{i}{2}V_L + \frac{\partial}{\partial A_L^*}\omega_L^{(q)}(A, C) = 0, \quad -\frac{i}{2}V_L^* + \frac{\partial}{\partial A_L}\omega_L^{(q)}(A, C) = 0,$$

$$-\frac{i}{2}H_x + \frac{\partial}{\partial B_x^*}\omega_x(B) = 0, \quad -\frac{i}{2}H_x^* + \frac{\partial}{\partial B_x}\omega_x(B) = 0,$$

$$-\frac{i}{2}A_L^* + \frac{\partial}{\partial V_L}S_{\text{G}}[V] = 0, \quad -\frac{i}{2}A_L + \frac{\partial}{\partial V_L^*}S_{\text{G}}[V] = 0,$$

$$-\frac{i}{2}B_x^* + \frac{\partial}{\partial H_x}S_{\text{H}}[W, H] = 0, \quad -\frac{i}{2}B_x + \frac{\partial}{\partial H_x^*}S_{\text{H}}[W, H] = 0. \tag{3.1}$$

Additionally to (3.1) we formally obtain the following two equations

$$-\frac{i}{2}C_L^*+\frac{\partial}{\partial W_L}S_{\rm H}[W,H]=0,\qquad -\frac{i}{2}C_L+\frac{\partial}{\partial W_L^*}S_{\rm H}[W,H]=0, \tag{3.2}$$

$$-\frac{i}{2}W_L^*+\frac{\partial}{\partial C_L}\omega_L^{(q)}(A,C)=0,\qquad -\frac{i}{2}W_L+\frac{\partial}{\partial C_L^*}\omega_L^{(q)}(A,C)=0, \tag{3.3}$$

which are redundant due to conditions (2.6). In the simple mean-field approximation one assumes, that all fields are identical to their mean values

$$\begin{aligned} &V_L=m, \quad A_L=-iz_1\,\forall L, \quad \text{where with Eq. (2.6)}\\ &H_x=M, \quad B_x=-iz_3\,\forall x, \quad K=m^q,\\ &W_L=K, \quad C_L=-iz_2\,\forall L, \quad z_2=2\kappa M^2. \end{aligned} \tag{3.4}$$

The mean values m, M, ..., z_3 are real numbers and independent on the space-time positions. Using the ansätze (3.4) we obtain the one-link and one-site integrals

$$f_L^{(q)}(A,C)=f^{(q)}(z_1,z_2)=\int_0^{2\pi}\frac{d\theta}{2\pi}\exp\{z_1\cos\theta+z_2\cos q\theta\},$$

$$f_x(B)=I_0(z_3)\equiv\int_0^{2\pi}\frac{d\chi}{2\pi}\exp\{z_3\cos\chi\}. \tag{3.5}$$

With relations (3.4) and (3.5) we find from Eq. (3.1) the usual mean-field equations

$$\begin{aligned} &m=\frac{\partial}{\partial z_1}\log f^{(q)}(z_1,z_2), \quad z_1=4\bar\beta m^3,\\ &M=\frac{d}{dz_3}\log I_0(z_3), \quad z_2=2\kappa M^2,\\ &\bar\beta=\beta(d-1), \quad z_3=4d\kappa m^q M. \end{aligned} \tag{3.6}$$

The $I_n(z)$ are modified Bessel functions. We interpret the three leading solutions of Eq. (3.6) as the three different phases of the Higgs model. We approximate them by

$m=0,\quad M=0$ confinement phase (strong coupling),

$m\approx 1-\dfrac{1}{8\bar\beta},\quad M=0$ Coulomb phase (weak coupling) $\bar\beta$ large enough, κ small,

$m\approx 1-\dfrac{1}{8\bar\beta+4q^2\kappa},\quad M\approx 1-\dfrac{1}{8d\kappa}$ Higgs phase (weak coupling) $\bar\beta$ and κ large. (3.7)

The gauge fixing does not lead to the general solution of the mean-field equations in the weak coupling region. It is easy to see that in the case of continuous gauge symmetry all gauge transformations

$$(m_g)_\mu(x) = g^{-1}(x) m g(x+\mu) \tag{3.8}$$

are possible solutions for the saddle-point of the gauge field. The U(1) gauge transformation $g(x)$ at the site x is given by the phase

$$g(x) = e^{i\phi(x)}, \quad \text{where} \quad \phi(x) \in [0, 2\pi]. \tag{3.9}$$

The same is true for the Higgs saddle-point M. If an integration over the full gauge orbit of the saddle points in the case of continuous gauge symmetry is performed, the mean-field method can be reconciled with Elitzers theorem [7]. To determine the phase structure we only need the saddle-points as given in Eq. (3.6). We introduçe the zeroth order free energy per link f_0 of the model using the saddle-point approximation of Z

$$Z \approx Z_0 = e^{-Ndf_0}, \tag{3.10}$$

where

$$f_0 = -\frac{1}{Nd} S_{\text{eff}}^{(0)}.$$

We obtain the following free energies of the simple mean-field approximation

$$f_0^{\text{CONF}} = 0, \quad \text{confinement}$$

$$f_0^{\text{COUL}} = -\beta m^4 + z_1 m - \log I_0(z_1), \quad \text{Coulomb}$$

$$f_0^{\text{HIGGS}} = -\bar{\beta} m^4 + z_1 m - \log f^{(q)}(z_1, z_2) + \frac{1}{d}[z_3 M - \log I_0(z_3)], \quad \text{Higgs}$$

$$f_0 = f_0^{\text{HIGGS}} \text{ for general values of } m, \bar{\beta}, M \text{ and } \kappa \quad \text{general case.} \tag{3.11}$$

The mean values correspond to a first derivative of the free energy. So the jumps of these mean values define the phase transition points. To obtain these jumps using the approximate solutions (3.7) we compare the free energies of the two adequate phases i and j, $f^i - f^j = 0$, to calculate the phase transition lines. We remark that f_0^{CONF} and f_0^{COUL} are identical to the U(1) model in Ref. [4]. The free energy in the Higgs phase contains $1/d$-terms which indicates that a simple $1/d$-expansion to find corrections is invalid.

Compared to Monte-Carlo data in Ref. [6] the simple mean-field approximation leads to a false phase structure (see Figures). Corrections to the lowest order of the mean--field approximation should improve the results. To correct the free energies per link in the form $f = f_0 + \Delta f$ we consider small quantum fluctuations around the saddle-points

$$V_L = m + v_L, \quad A_L = -iz_1 + a_L, \quad \text{from Eq. (2.6) follows}$$

$$H_x = M + h_x, \quad B_x = -iz_3 + b_x, \quad K = m^q,$$
$$W_L = K + w_L, \quad C_L = -iz_2 + c_L, \quad w_L = qm^{q-1} v_L. \tag{3.12}$$

We perform the expansion in terms of the real and imaginary part of the fluctuations

$$v_L = \lambda_L + i\xi_L, \quad h_x = \alpha_x + i\gamma_x,$$
$$a_L = x_L + iy_L, \quad b_x = n_x + iP_x. \tag{3.13}$$

Using definition (2.6) we obtain for the C-field the following expression

$$c_L = Q_L + Q_L^{(2)} + i(R_L + R_L^{(2)}) \tag{3.14}$$

where

$$Q_\mu(x) = -i2\kappa M(\alpha(x+\mu) + \alpha(x)), \quad R_\mu(x) = -i2\kappa M(\gamma(x+\mu) - \gamma(x)),$$
$$Q_\mu^{(2)}(x) = 2\kappa(\alpha(x)\alpha(x+\mu) + \gamma(x)\gamma(x+\mu)), \quad R_\mu^{(2)}(x) = 2\kappa(\alpha(x)\gamma(x+\mu) - \gamma(x)\alpha(x+\mu)).$$

We expand the effective action (2.17) in terms of these quantum fluctuations up to the second order. All the terms linear in the fluctuations vanish. We write the effective action as a sum of a zeroth order term and a second order one

$$S_{\text{eff}} = S_{\text{eff}}^{(0)} + S_{\text{eff}}^{(2)}, \tag{3.15}$$

where

$$S_{\text{eff}}^{(0)} = -Ndf_0, \tag{3.15a}$$

$$S_{\text{eff}}^{(2)} = S_G^{(2)}[v] + \sum_L m\, Q_L^{(2)} - \frac{i}{2}(x \cdot \lambda + y \cdot \xi + n \cdot \alpha + P \cdot \gamma)$$
$$+ \sum_L \omega_L^{(q)(2)} + \sum_x \omega_x^{(2)}. \tag{3.15b}$$

The quadratic terms for the one-site and one-link integral are given by

$$\omega_x^{(2)} = -\tfrac{1}{2}\left\{\frac{I_0''(z_3)}{I_0(z_3)} - \left(\frac{I_0'(z_3)}{I_0(z_3)}\right)^2\right\} n_x^2 - \tfrac{1}{2}\,\frac{1}{z_3}\,\frac{I_0'(z_3)}{I_0(z_3)}\,P_x^2$$
$$= -\tfrac{1}{2}(1 - M/z_3 - M^2)n_x^2 - \tfrac{1}{2}\,\frac{M}{z_3}\,P_x^2 \tag{3.16}$$

and

$$\omega_L^{(q)(2)} = -\tfrac{1}{2}E_{xx}x_L^2 - \tfrac{1}{2}E_{yy}y_L^2 - \tfrac{1}{2}E_{QQ}Q_L^2 - \tfrac{1}{2}E_{RR}R_L^2 - E_{Qx}x_LQ_L - E_{Ry}y_LR_L \tag{3.17}$$

with the frequencies

$$E_{xx} = (f^{(q)})^{-1}\frac{\partial^2}{\partial z_1^2}f^{(q)} - (f^{(q)})^{-2}\left(\frac{\partial}{\partial z_1}f^{(q)}\right)^2,$$

$$E_{yy} = 1 - (f^{(q)})^{-1}\frac{\partial^2}{\partial z_1^2}f^{(q)},$$

$$E_{QQ} = (f^{(q)})^{-1}\frac{\partial^2}{\partial z_2^2}f^{(q)} - (f^{(q)})^{-2}\left(\frac{\partial}{\partial z_2}f^{(q)}\right)^2,$$

$$E_{RR} = 1-(f^{(q)})^{-1}\frac{\partial^2}{\partial z_2^2}f^{(q)},$$

$$E_{Qx} = (f^{(q)})^{-1}\frac{\partial^2}{\partial z_1 \partial z_2}f^{(q)} - (f^{(q)})^{-2}\left(\frac{\partial}{\partial z_1}f^{(q)}\right)\left(\frac{\partial}{\partial z_2}f^{(q)}\right),$$

$$E_{Ry} = \frac{qz_2}{z_1}\left\{\frac{1}{z_2}(f^{(q)})^{-1}\frac{\partial}{\partial z_1}f^{(q)} - E_{RR}\right\}. \tag{3.18}$$

In the Coulomb phase defined by the simple mean field approximation ($M = z_2 = z_3 = 0$, $m \neq 0$) only the frequencies E_{xx} and E_{yy} remain and are identical to the U(1) result of Ref. [4]. The expansion (3.15) is valid in the weak coupling region. In the case of the trivial saddle-points in the strong coupling region one can calculate the corrections to the free energy by a strong coupling expansion.

4. The calculation of the mean-field corrections to the free energy per link

In the strong coupling region the integration leads to the original form of the partition function Z given by Eq. (2.5). Performing a Fourier expansion of the exponential term in Eq. (2.5) and considering only the zeroth order of this expansion we obtain the following correction to the free energy in the confinement phase

$$\Delta f^{\text{CONF}} = -\frac{d-1}{2}\log I_0\left(\frac{2\bar{\beta}}{d-1}\right) - \log I_0(2\kappa). \tag{4.1}$$

The effective gauge coupling is small enough in this region so that we can approximate the free energy correction by

$$\Delta f^{\text{CONF}} \simeq -\frac{\bar{\beta}^2}{2d} - \log I_0(2\kappa). \tag{4.2}$$

The $\bar{\beta}$-dependent term is identical to the U(1) result of Ref. [4]. The κ-dependent term can be neglected if the first term dominates. However, when $\bar{\beta}$ goes to zero and κ is large enough this term dominates and, as we will see later, it gives the analytical connection between the Higgs and the confinement phases for the Higgs charge $q = 1$. We notice that the correction Δf^{CONF} contains not only $1/d$-terms.

In the weak coupling region we expect two phases. The first one is a Coulomb phase with U(1) gauge symmetry. The second phase is a Higgs phase in which the continuous gauge symmetry is broken. Up to the gauge degrees of freedom (see Appendix A) we can perform the integrations in a unique way. For the partition function we have to deal with the integral

$$Z = e^{-Ndf_0}\int d[a, b, v, h]\mathcal{F}[m, v, M, h]\exp\{S_{\text{eff}}^{(2)}\}. \tag{4.3}$$

The $\mathscr{F}$ denotes the gauge fixing terms in the case of continuous gauge symmetry, otherwise it is equal to unity. Performing the integration over the external fields and transforming the fluctuations into the momentum space the following action contribution remains

$$\tilde{S}^{(2)} = -N\int_{-\pi}^{+\pi} \frac{d^d p}{(2\pi)^d} (\tilde{\alpha}^*(p)A^\alpha(p)\tilde{\alpha}(p)+\tilde{\gamma}^*(p)A^\gamma(p)\tilde{\gamma}(p)$$

$$+\sum_{\mu,\nu}\{\lambda_\mu^*(p)\Omega_{\mu\nu}^\lambda(p)\lambda_\nu(p)+\xi_\mu^*(p)\Omega_{\mu\nu}^\xi(p)\xi_\nu(p)\}). \tag{4.4}$$

The $\lambda_\mu(p)$ and the $\xi_\mu(p)$ are the transformed real and imaginary gauge field fluctuations, respectively. The $\tilde{\alpha}(p)$ and $\tilde{\gamma}(p)$ denote the effective real and imaginary Higgs fluctuations. The $\tilde{\alpha}(p)$ ($\tilde{\gamma}(p)$) are given by a linear combination of the genuine Higgs fluctuations $\alpha(p)$ ($\gamma(p)$) and the gauge fluctuations $\lambda_\mu(p)$ ($\xi_\mu(p)$) (see Appendix A). In this way the coupling between both kinds of fluctuations is realized. The definitions of the operators $A^{\alpha,\gamma}(p)$ and $\Omega_{\mu\nu}^{\lambda,\xi}(p)$ are given in Appendix A. To calculate the integral (4.3) the operators Ω have to be diagonalized. In the case of the operator Ω^λ we find $(d-1)$-approximately degenerate eigenvalues

$$\omega_R(p) = \tfrac{1}{2}\frac{z_1}{m(d-1)}\sum_{\varrho=0}^{d-1}(1-\cos p_\varrho)+\Lambda \tag{4.5a}$$

and one non-degenerate eigenvalue

$$\tilde{\omega}_R(p) = -\frac{z_1}{m(d-1)}\sum_{\varrho=0}^{d-1}\cos p_\varrho+\Lambda-\left(\frac{z_2E_{QX}}{ME_{xx}}\right)^2\frac{\sum_{\varrho=0}^{d-1}(1+\cos p_\varrho)}{2A^\alpha(p)}. \tag{4.5b}$$

The Λ is the mass of the real excitations (4.5a)

$$\Lambda = \tfrac{1}{2}\left(\frac{1}{E_{xx}}-\frac{z_1}{m}\right). \tag{4.5c}$$

From the diagonalization of the operator Ω^ξ for the imaginary gauge fluctuations we obtain $(d-1)$ degenerate eigenvalues

$$\omega_I(p) = \tfrac{1}{2}\frac{z_1}{m(d-1)}\sum_{\varrho=0}^{d-1}(1-\cos p_\varrho)+\Xi \tag{4.6a}$$

and one non-degenerate one

$$\tilde{\omega}_I(p) = \Xi-\left(\frac{z_2R_{Ry}}{ME_{RR}}\right)^2\frac{\sum_{\varrho=0}^{d-1}(1-\cos p_\varrho)}{2A^\gamma(p)}. \tag{4.6b}$$

The Ξ can be interpreted as the mass of the photons (4.6a)

$$\Xi = \tfrac{1}{2}\left(\frac{1}{E_{yy}} - \frac{z_1}{m}\right). \tag{4.6c}$$

Zero-frequency or $1/d$-modes occur among the imaginary fluctuations (see Ref. [4]). Therefore we have to investigate their frequencies in detail.

In the Coulomb like region for the simple mean-field approximation (3.7) the mean values and the one-link itegral are given by

$$m \approx 1 - \frac{1}{8\beta}, \quad M = 0, \quad f^{(q)}(z_1, z_2) = I_0(z_1). \tag{4.7}$$

We find a non-trivial saddle point value m for the gauge field but a trivial one ($M = 0$) for the Higgs field. Therefore we expect a zero-frequency mode in the gauge field part as in the U(1)-model discussed in Ref. [4]. Indeed, we obtain

$$\Xi = 0, \quad E_{Ry} = 0 \tag{4.8a}$$

so that

$$\tilde{\omega}_I(p) = 0, \quad A^\gamma(p) > 0. \tag{4.8b}$$

The characteristic feature of this region is the non-trivial U(1)-symmetry. We call this phase a static Coulomb phase, since the Higgs field saddle-point value is trivial.

In the Higgs-like region defined by the simple mean-field result (3.7) we obtain the one-link integral (see Appendix B)

$$f^{(q)}(z_1, z_2) \simeq I_0(z_1)I_0(z_2)\left(1 + 2\sum_{k=1}^{\infty} \exp\{-t^{(q)}(z_1, z_2)k^2\}\right) \tag{4.9a}$$

where

$$t^{(q)}(z_1, z_2) = \frac{q^2}{2z_1} + \frac{1}{2z_2}. \tag{4.9b}$$

Two different cases have to be distinguished:

(*i*) The couplings $\bar{\beta}$ and κ are such that $t^{(q)}$ is very large, so that the sum can be neglected in the one-link integral.

(*ii*) The couplings $\bar{\beta}$ and κ are large enough so that $t^{(q)}$ is a small quantity, and the sum in Eq. (4.9a) is essential.

To find the critical coupling constant values distinguishing the two cases discussed we use $1/d$ as expansion parameter and consider at most $1/d$-terms. Furthermore it is easy to see that we need to consider only the first term of the sum ($k = 1$). So, if $\exp\{-t^{(q)}\}$ is less than $1/d$, the sum can be neglected and is essential otherwise. From this criterion we find

$$\frac{1}{d} = \exp\{-t^{(q)}(z_1, z_2)\} \quad \text{or} \quad t^{(q)}(z_1, z_2) = \ln d \tag{4.10}$$

and derive from Eq. (4.9b)

$$z_2^{(H)} = \frac{z_1^{(H)}}{2 \ln d(z_1^{(H)} - q^2/2 \ln d)}, \tag{4.10a}$$

where the index H denotes the critical values. Using the approximation $Z_1 \approx 4\bar{\beta}$, $Z_2 \approx 2\kappa$ we obtain the following relation between the critical couplings

$$\kappa_H = \frac{\bar{\beta}_H}{4 \ln d\,(\bar{\beta}_H - q^2/8 \ln d)}. \tag{4.11}$$

The one-link integral, the saddle-point values (m, M), the mass Ξ and the frequency E_{Ry} are given in Table I. In case (*ii*) we have approximated the sum of Eq. (4.9a) by a Gaussian integral. For case (*i*) follows

$$\Xi \approx 0, \qquad E_{Ry} \approx 0 \tag{4.12a}$$

TABLE I

The one-link integral, the saddle-point values, the photon mass Ξ and the frequency E_{Ry} in the cases (*i*) and (*ii*) of the Higgs like region defined by the simple mean field calculation

Case	(*i*)	(*ii*)
$t^{(q)}(z_1, z_2)$	$\gg \ln d$	$\ll \ln d$
$f^{(q)}(z_1, z_2)$	$I_0(z_1)I_0(z_2)$	$I_0(z_1)I_0(z_2)\,(\pi/t^{(q)}(z_1, z_2))^{1/2}$
m	$1-1/8\bar{\beta}$	$1-1/(8\bar{\beta}+4q^2\kappa)$
M	$1-1/8d\kappa$	$1-1/8d\kappa$
Ξ	0	$1/d \cdot (z_3 M q^2)/4m^{q+1}$
E_{Ry}	0	$qm^q/(z_1+q^2 z_2)$

so that

$$\tilde{\omega}_I(p) \approx 0, \qquad A^{\gamma}(p) \approx 0. \tag{4.12b}$$

The non-trivial U(1)-symmetry indicates a Coulomb like behaviour. The existence of a zero frequency for the Higgs field $A_{\gamma}(p)$ corresponds to the non-trivial Higgs mean value ($M \neq 0$). Therefore we call this region a dynamical Coulomb phase. The case (*ii*) leads to

$$\Xi = \frac{1}{d}\,\frac{z_3 M q^2}{4m^{q+1}}, \qquad E_{Ry} = \frac{qm^q}{z_1+q^2 z_2} \tag{4.13a}$$

so that

$$\tilde{\omega}_I(p) \approx \Xi(1-0.29\, m^{q-1}),$$

$$A^{\gamma}(p) \approx \frac{1}{2d}\,\frac{z_3}{M}\left(1-\frac{q^2 z_2}{q^2 z_2+z_1}(1-m^{q-1})\right)\sum_{\varrho=0}^{d-1}(1-\cos p_{\varrho}). \tag{4.13b}$$

Instead of zero-frequencies we find frequencies of order $O(1/d)$. Since the continuous symmetry is broken, we call this region a Higgs phase. We interprete the relation (4.11) between the critical couplings as the phase transition line between the Coulomb and the Higgs phases.

Now performing the integration over the fluctuations we present the results for the corrections to the free energy. Details of the calculation are given in Appendix A. For the static (STAT) and dynamical (DYN) Coulomb phase we obtain

$$\Delta f_{\mathrm{STAT}}^{\mathrm{COUL}} = \frac{1}{d} F_{\mathrm{STAT}}^{\mathrm{COUL}}(z_1, m, d) \tag{4.14a}$$

and

$$\Delta f_{\mathrm{DYN}}^{\mathrm{COUL}} = \frac{1}{d} F_{\mathrm{DYN}}^{\mathrm{COUL}}(z_1, z_2, z_3, m, M, d). \tag{4.14b}$$

The expressions for the F's are given by

$$\begin{aligned} F_{\mathrm{STAT}}^{\mathrm{COUL}} &= -\tfrac{1}{2}\log(4\pi d m^2) - \tfrac{1}{2}\log\left(\frac{z_1}{2m - z_1 + z_1 m^2}\right) + \tfrac{3}{8} \\ &\quad -\tfrac{1}{2}[-\tilde{K} + \tfrac{1}{2}\tilde{K}^2] - \log 2 \end{aligned} \tag{4.15a}$$

and

$$\begin{aligned} F_{\mathrm{DYN}}^{\mathrm{COUL}} &= F_{\mathrm{STAT}}^{\mathrm{COUL}} - \tfrac{1}{2}\log M^2 + \left(1 - \frac{2}{N}\right)\tfrac{1}{2}\log 2\pi \\ &\quad -\tfrac{1}{2}\log\left(\frac{z_3}{2M - z_3 + z_3 M^2}\right) - \tfrac{1}{2}[-L + \tfrac{1}{2}L^2] \end{aligned} \tag{4.15b}$$

where

$$\tilde{K} = \frac{z_1 - m - z_1 m^2}{2m}, \qquad L = \frac{2m^{2q} - z_2(1 - m^{2q})}{m^{2q}} \, \frac{z_3 - M - z_3 M^2}{2M}. \tag{4.15c}$$

The first two logarithmic terms in Eq. (4.15a) include all the Jacobians of the gauge field part and the contribution from the non-degenerate eigenvalue $\tilde{\omega}_R(p)$ (Eq. (4.5b)) of the real fluctuations. The constant contribution (3/8) comes from the $(d-1)$ massless photons (4.6a). The bracket term is the contribution of the $(d-1)$ massive excitations (4.5a) whereas the log 2 is a correction coming from the Higgs fluctuations. Up to this log 2 term the result is identical to the U(1) calculation of Ref. [4]. In the expression for the dynamical part of the Coulomb phase (4.15b) the new terms arise from the Higgs part. The first two new terms include the Jacobians. The following terms come from the real Higgs excitation $A^{\alpha}(p)$. We remark that in the Coulomb phase all the corrections are of the order $O(1/d)$.

In the Higgs phase we obtain the following expression for the free energy correction

$$\begin{aligned} \Delta f^{\mathrm{HIGGS}} &= \tfrac{1}{2}\log\left(\frac{2z_1 + q^2 z_2}{2(z_1 + q^2 z_2)}\right) + \tfrac{1}{2}\log\left(\frac{m}{z_1 + q^2 z_2} - \frac{z_1(1 - m^q)}{2z_1 + q^2 z_2}\right) \\ &\quad + \tfrac{1}{2}\log\left(\frac{z_1}{m}\right) + \tfrac{1}{2}\log(1 + 4m\Xi) + \frac{1}{d} F^{\mathrm{HIGGS}}(z_1, z_2, z_3, m, M, d), \end{aligned} \tag{4.16a}$$

where

$$F^{\mathrm{HIGGS}} = -\tfrac{1}{2}\log\left(\frac{z_1}{m}\right) - \tfrac{1}{2}\left[-\tilde{K}+\tfrac{1}{2}\tilde{K}^2\right]+\tfrac{3}{8}+\tfrac{1}{2}\log \Xi - \tfrac{1}{2}\left[0.29\, m^{q-1}+0.04\, m^{2(q-1)}\right]$$

$$-\tfrac{1}{2}\log 2-\tfrac{1}{2}\log\left(\frac{2M}{2M-z_3+z_3M^2}\right)+\tfrac{1}{2}\log\left(\frac{q^2m^{q-1}z_2+z_1}{q^2z_2+z_1}\right)-\tfrac{1}{2}\left[-L+\tfrac{1}{2}L^2\right] \quad (4.16b)$$

and

$$\tilde{K} = \frac{2z_1}{2z_1+q^2z_2}\,\frac{(z_1+q^2z_2)(1-m^2)-m}{2m},$$

$$L = \left(\frac{2q^2z_2+z_1}{q^2z_2+z_1}-\frac{z_2(1-m^{2q})}{m^{2q}}\right)\frac{z_3-M-z_3M^2}{2M},$$

$$\Xi = \frac{1}{d}\,\frac{z_3Mq^2}{4m^{q+1}} \quad \text{photon mass.} \quad (4.16c)$$

As expected, the corrections in the Higgs phase contain essential terms (of order $O(1)$) and $1/d$-terms. The first two essential terms in Eq. (4.16a) arise from the $(d-1)$ real massive excitations $\tilde{\omega}_R(p)$ given by Eq. (4.5a). The other ones are contributions of the $(d-1)$ massive photons (4.6a). Now we consider the $1/d$-terms in Eq. (4.16b). The first two $1/d$-terms include the contributions of the non-degenerate real excitation $\tilde{\omega}_R(p)$ as well as the $(d-1)$ real massive excitations $\omega_R(p)$ (see Eq. 4.5a, b)). The constant term (3/8) comes from the photons (4.6a), the following two are the contributions of the $1/d$-frequency given in Eq. (4.13b). The remaining terms are Higgs contributions. The first logarithmic Higgs term and the last Higgs expression include the contributions of the real Higgs excitation $A^{\alpha}(p)$. The third Higgs term comes from the imaginary excitation $A^{\gamma}(p)$. The second term contains the sum of contributions coming from $A^{\alpha}(p)$ and $A^{\gamma}(p)$. The determination of the free energy including corrections allows one to improve the phase structure study of the lattice Abelian Higgs model.

5. *The phase structure of the lattice Abelian Higgs model*

We have to consider two kinds of phase transitions in the model. The first type describing the transitions from strong to weak coupling corresponds to a jump in the mean values. We determine the phase transition lines comparing the effective free energies of the adequate phases. We define these effective free energies by including all the essential terms coming from the zeroth order free energies f_0 and the corrections Δf. In a phase transition point, not corrected by $1/d$-terms of f, the effective free energies of the two phases i and j satisfy

$$f_{\mathrm{eff}}^{(i)}(\bar{\beta}_{\mathrm{eff}}, \kappa_{\mathrm{eff}})-f_{\mathrm{eff}}^{(j)}(\bar{\beta}_{\mathrm{eff}}, \kappa_{\mathrm{eff}}) = 0. \quad (5.1)$$

To correct the phase transitions we use the effective corrections Δf_{eff} including only the $1/d$-terms and assume that the corrections $\delta\bar{\beta}$ and $\delta\kappa$ to the critical couplings are small

$$\bar{\beta}_{\mathrm{crit}} = \bar{\beta}_{\mathrm{eff}}+\delta\bar{\beta}, \quad \kappa_{\mathrm{crit}} = \kappa_{\mathrm{eff}}+\delta\kappa. \quad (5.2)$$

We determine $\delta\bar{\beta}$ and $\delta\kappa$ by

$$\left.\begin{array}{l}\dfrac{\partial}{\partial\bar{\beta}}\left[f_{\text{eff}}^{(i)}-f_{\text{eff}}^{(j)}\right](\bar{\beta}_{\text{eff}},\kappa_{\text{eff}})\delta\bar{\beta}\\ \dfrac{\partial}{\partial\kappa}\left[f_{\text{eff}}^{(i)}-f_{\text{eff}}^{(j)}\right](\bar{\beta}_{\text{eff}},\kappa_{\text{eff}})\delta\kappa\end{array}\right\} = \left[\Delta f_{\text{eff}}^{(j)}-\Delta f_{\text{eff}}^{(i)}\right](\bar{\beta}_{\text{eff}},\kappa_{\text{eff}}). \tag{5.3}$$

The second type of the phase transitions in the weak coupling region corresponds to a drastic change in the behaviour of the frequencies.

First we investigate the confinement-Coulomb transition (C) which is of the first type. We obtain the critical coupling

$$\bar{\beta}_c^{\text{eff}} = 1.80 \tag{5.4}$$

independent of the Higgs charge q and the dimension d. The correction is of the form

$$d\bar{\beta} = -\tfrac{1}{2}\frac{\log d}{d} - \frac{2.31}{d} + \frac{1.62}{d-1}, \tag{5.5}$$

so the corrected coupling in the case for $d = 4$ is given by

$$\bar{\beta}_c = 1.52. \tag{5.6}$$

Next we regard the Coulomb-Higgs transition (H). The transition line is given by (Eq. (4.11))

$$\kappa_H = \frac{1}{4\ln d}\frac{\bar{\beta}_H}{\bar{\beta}_H-\bar{\beta}_p} \quad \text{where} \quad \bar{\beta}_p = \frac{q^2}{8\ln d}. \tag{5.7}$$

Physically, the pole $\bar{\beta}_p$ of expression (5.7) exists only for

$$\bar{\beta}_p > \bar{\beta}_c = 1.52. \tag{5.8}$$

Then a Coulomb phase exists for all values of the Higgs coupling κ up to infinity. In four dimensions it is easy to see that the Higgs charges $q \geqslant 5$ yield values of $\bar{\beta}_p$ satisfying (5.8). This is in agreement with the Monte-Carlo calculations of Ref. [6]. We remark that the transition is of the second type.

Finally we discuss the confinement-Higgs transition (CH) for a Higgs charge $q = 1$. For small enough $\bar{\beta}$ the effective free energies of the two phases are given by

$$f_{\text{eff}}^{\text{HIGGS}} \approx -\bar{\beta}-\log I_0(2\kappa M^2), \quad f_{\text{eff}}^{\text{CONF}} = -\log I_0(2\kappa). \tag{5.9}$$

For κ large enough and M nearly equal to one we find for

$$\bar{\beta} \leqslant 1 \equiv \bar{\beta}_{\text{eff}}^{*} \tag{5.10}$$

that the coupling constant dependence of the free energies is approximately the same. Therefore we can regard the two phases analytically connected in this region. The $\bar{\beta}_{\text{eff}}^{*}$ denotes the uncorrected $\bar{\beta}$-value for the end-point of the phase transition.

TABLE II

The corrected results for all phase transitions calculated for dimension $d = 4$ and Higgs charges $q = 1$, 2 and 6

q	Confinement-Higgs transition	Confinement-Coulomb transition	Coulomb-Higgs transition
1	end-point: $\bar{\beta}^* = 1.125$ $\bar{\beta}_C \geqslant \bar{\beta}_{CH} \geqslant \bar{\beta}^*$: $\kappa_{CH} = 0.49 - 0.12\bar{\beta}_{CH}$	$\bar{\beta}_C = 1.52$	$\bar{\beta}_p = 0.09$
2	$\kappa_{CH} = \dfrac{\bar{\beta}_{CH}}{4(\bar{\beta}_{CH} - \ln 2)} - 0.16$	$\bar{\beta}_C = 1.52$	$\bar{\beta}_p = 0.36$
6	—	$\bar{\beta}_C = 1.52$	$\bar{\beta}_p = 3.25$

In Table II we summarize the corrected results for all phase transitions calculated for dimension $d = 4$ and Higgs charges $q = 1$, 2 and 6. Our results for the improved mean--field calculation (IMF) compared to Monte-Carlo data (MC) and the simple mean-field results (MF) of Ref. [6] are presented in the Figures. We present the phase transitions in the plane of the couplings $\bar{\beta}$ and κ for Higgs charges $q = 1$, 2 and 6. Note that our definitions for the couplings differ from those in Ref. [6]. The dashed-dotted lines denote the phase transitions obtained by MF, the continuous ones are phase transitions resulting from the IMF. The MC-data are represented by the circles. The bends in the transition lines of IMF are due to different analytical approximations.

In the case of the Higgs charge $q = 1$ (Fig. 1) the end-point of the confinement-Higgs transition in the IMF calculation is in good agreement with the Monte-Carlo result. The simple mean-field calculation does not allow one to find this end-point of the phase transition.

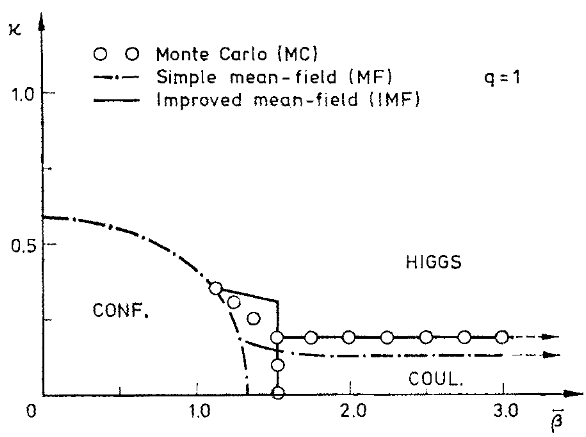

Fig. 1. The phase structure of the model for Higgs charge $q = 1$ and dimension $d = 4$ in the $(\kappa, \bar{\beta})$-coupling constant plane

Fig. 2 shows the phase structure for the Higgs charge $q = 2$. Both simple MF and IMF results are in reasonable agreement with data. In Fig. 3 we present our results for the Higgs charge $q = 6$. Using the improved mean-field calculation we can qualitatively reproduce the correct phase structure as predicted by the Monte-Carlo simulation. For increasing values of coupling κ the Coulomb phase remains opposite to the simple mean-field prediction.

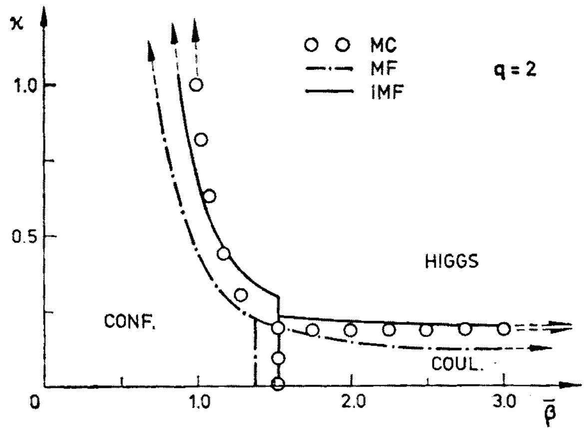

Fig. 2. The phase structure for Higgs charge $q = 2$ and dimension $d = 4$

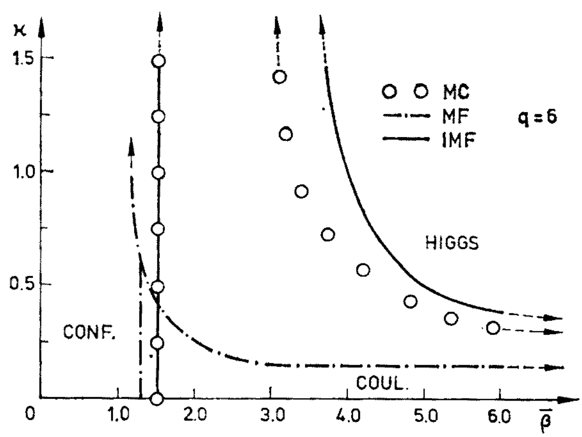

Fig. 3. The phase structure for Higgs charge $q = 6$ and dimension $d = 4$

6. Conclusion and summary

We have found that the improved mean-field calculation qualitatively leads to the true phase structure of the lattice Abelian Higgs model as found in Monte-Carlo studies for all Higgs charges considered. The simple mean-field result is corrected by considering small quantum fluctuations of the fields in a proper way.

We emphasize: To distinguish the phases of the model we use the different behaviour of the saddle-point values and the effective action under gauge transformations in these phases. The characteristic feature of the confinement phase is the trivial U(1)-symmetry. This means that a gauge transformation of the trivial saddle-points ($m = 0$, $M = 0$) leads to the saddle-points themselves. The Coulomb phase can be characterized by a non--trivial U(1)-symmetry. The gauge transformed non-trivial saddle-points ($m \neq 0$, $M \neq 0$) differ from the original ones by a phase. However, the effective action is invariant under this transformation. In the Higgs phase the continuous gauge symmetry is broken so that the effective action is not invariant under the U(1) transformation. For the transitions from the confinement to the Higgs or Coulomb phases, corresponding to the strong to weak coupling transition, at least one saddle-point becomes non-trivial. So the jumps in the mean values of the fields are used as a phase transition criterion. In the weak coupling region we have to distinguish the Coulomb from the Higgs phases by their effective action behaviour under the gauge transformation. This leads to different frequencies in the loop expansion of the action. So we obtain a second phase transition criterion in the weak coupling region by inspecting the frequencies. From the mean-field equations it follows that the mean values behave as the first derivatives of the free energy. Therefore we can interpret the transitions with jumps in the mean-values as first order phase transitions. The transitions of the second type are obtained directly from the frequencies. The frequencies are the result of the second variation of the effective action corresponding to a second derivative of the free energy. So we interpret these transitions as second order ones.

We have calculated the corrections to the free energy per link up to one loop. We stress that besides the terms of order $O(1/d)$ also essential terms $O(1)$ arise. Therefore, the simple saddlepoint approximation does not contain all essential terms of the free energy per link. As a consequence the simple mean-field calculation does not reproduce the end--point in the confinement-Higgs transition for the Higgs charge $q = 1$, which we have found including one loop corrections.

Using the second phase transition criterion in the weak coupling region we find for Higgs charge $q = 6$ that with increasing Higgs coupling κ the Coulomb phase remains in a region for non-trivial Higgs mean-value ($M \neq 0$). Analytical approximations have been presented for all phase transition lines.

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APPENDIX A

Remarks on the integration of quantum fluctuations in the weak coupling region

Considering the integral (4.3) we deal with the fluctuations of the external fields A_L and B_x. The well defined Gaussian integration leads to the following contribution to the free energy per link

$$\Delta f_1 = \tfrac{1}{2}\log(E_{xx}E_{yy}) + \frac{1}{2d}\log\left(\frac{M}{z_3}\left[1 - \frac{M}{z_3} - M^2\right]\right). \tag{A1}$$

Additionally quadratic fluctuation terms remain coming from the mixing terms in Eq. (3.15b). These terms we have to add to the remaining action terms of the expression (3.15b).

The definition of the operators in Eq. (4.4) are given by

$$\Omega^{\lambda}_{\mu\nu}(p) = \left[\tfrac{1}{2}\frac{z_1}{m(d-1)}\left(1+2\cos p_\mu - \sum_{\varrho=0}^{d-1}\cos p_\varrho\right) + \frac{1}{2E_{xx}}\right]\delta_{\mu\nu}$$

$$-\tfrac{1}{4}\left[\frac{z_1}{m(d-1)} + \left(\frac{z_2 E_{Qx}}{M E_{xx}}\right)^2 \frac{1}{A^{\alpha}(p)}\right](1+e^{ip_\mu})(1+e^{-ip_\nu}), \tag{A2}$$

$$\Omega^{\xi}_{\mu\nu}(p) = \left[\tfrac{1}{2}\frac{z_1}{m(d-1)}\left(1- \sum_{\varrho=0}^{d-1}\cos p_\varrho\right) + \frac{1}{2E_{yy}}\right]\delta_{\mu\nu}$$

$$-\tfrac{1}{4}\left[\frac{z_1}{m(d-1)} + \left(\frac{z_2 E_{Ry}}{M E_{yy}}\right)^2 \frac{1}{A^{\gamma}(p)}\right](1-e^{ip_\mu})(1-e^{-ip_\nu})$$

and

$$A^{\alpha}(p) = \tfrac{1}{2}\left(1-\frac{M}{z_3}-M^2\right)^{-1} - \frac{z_3}{2dM}\sum_{\varrho=0}^{d-1}\cos p_\varrho$$

$$-\frac{z_2^2}{M^2}\frac{E_{QQ}E_{xx}-E_{Qx}^2}{E_{xx}}\sum_{\varrho=0}^{d-1}(1+\cos p_\varrho),$$

$$A^{\gamma}(p) = \tfrac{1}{2}\frac{z_3}{M} - \frac{z_3}{2dM}\sum_{\varrho=0}^{d-1}\cos p_\varrho - \frac{z_2^2}{M^2}\frac{E_{RR}E_{yy}-E_{Ry}^2}{E_{yy}}\sum_{\varrho=0}^{d-1}(1-\cos p_\varrho). \tag{A3}$$

We remark that the second terms in the non-diagonal part of the operators Ω vanish in the Coulomb phase ($E_{Qx} = E_{Ry} = 0$). The effective Higgs fluctuations $\tilde{\alpha}(p)$ and $\tilde{\gamma}(p)$ corresponding to $A^{\alpha}(p)$ and $A^{\beta}(p)$ are the linear combinations

$$\tilde{\alpha}(p) = \alpha(p) - \frac{1}{2A^{\alpha}(p)}\frac{z_2 E_{Qx}}{M E_{xx}}\sum_{\mu=0}^{d-1}(1+e^{-ip_\mu})\lambda_\mu(p),$$

$$\tilde{\gamma}(p) = \gamma(p) + \frac{1}{2A^{\gamma}(p)}\frac{z_2 E_{Ry}}{M E_{yy}}\sum_{\mu=0}^{d-1}(1-e^{-ip_\mu})\xi_\mu(p) \tag{A4}$$

of the genuine Higgs fluctuations $\alpha(p)$ and $\gamma(p)$ as well as the gauge fluctuations $\lambda_\mu(p)$ and $\xi_\mu(p)$. In the Coulomb phase ($E_{Qx} = E_{Ry} = 0$) the effective and genuine Higgs fluctuations are identical.

The diagonalization of the operators Ω leads to the eigenfrequencies given in Eqs. (4.5) and (4.6). We expand the gauge fluctuations $\lambda_\mu(p)$ and $\xi_\mu(p)$ in terms of eigenfunctions of these eigenfrequencies

$$\lambda_\mu(p) = \sum_{n=0}^{d-1} D_n(p)\phi_\mu^{(n)}(p), \qquad \xi_\mu(p) = \sum_{n=0}^{d-1} C_n(p)\psi_\mu^{(n)}(p), \tag{A5}$$

where $D_n(p)$ and $C_n(p)$ are the expansion coefficients and $\phi_\mu^{(n)}(p)$ and $\psi_\mu^{(n)}(p)$ the corresponding eigenfunctions. The eigenvalue $\tilde{\omega}_R(p)$ (Eq. 4.5b) is related to $D_0(p)$ and $\tilde{\omega}_I(p)$ (Eq. 4.6b) to $C_0(p)$. Up to the gauge degrees of freedom the integration over the $D_n(p)$ and $C_n(p)$ is performed in the same way leading to the following free energy correction

$$\Delta f_2 = \int_{-\pi}^{+\pi} \frac{d^d p}{(2\pi)^d}\left[\frac{d-1}{2d}(\log\omega_R(p)+\log\omega_I(p))+\frac{1}{2d}(\log\tilde{\omega}_R(p)+\log A^\alpha(p))\right]. \tag{A6}$$

We have to deal with the remaining fluctuations differently depending on the phases. The zero-frequency modes follow from the existence of degenerate saddle-points in the case of continuous gauge groups (Coulomb phase). This means that fluctuations around a saddle point corresponding to gauge transformations have no restoring forces. This does not allow one to make a naive loop expansion of the effective action. We have performed the perturbation theory only in the Gaussian fluctuations and integrate the collective coordinates [8, 9] arising from the gauge degrees of freedom exactly. In the Higgs phase, where the continuous gauge symmetry is broken instead of the zero frequencies we obtain frequencies of order $O(1/d)$. The corresponding modes can integrate as usual Gaussian fluctuations.

First we deal with the Higgs phase. The integration is Gaussian so that the gauge fixing term $\mathscr{F}$ is given by

$$\mathscr{F}[m, v, M, h] = 1. \tag{A7}$$

The integration leads to the following contribution to the free energy per link

$$\Delta f_3^{\text{HIGGS}} = \int_{-\pi}^{+\pi} \frac{d^d p}{(2\pi)^d}\frac{1}{2d}(\log\tilde{\omega}_I(p)+\log A^\gamma(p)). \tag{A8}$$

Second we consider the Coulomb phase. The genuine Gaussian fluctuations satisfying a background gauge condition are just orthogonal to the gauge degrees of freedom. The background gauge condition for the gauge field is given by

$$m\sum_{\mu=0}^{d-1}(\xi_\mu(x)-\xi_\mu(x-\mu)) = 0. \tag{A9}$$

In the static part of the Coulomb phase the Higgs mean value is zero. Using the standard Fadeev-Popov method [10] we obtain the following gauge fixing factor

$$\mathscr{F}[m, v, M, h] = (2\pi)^N \Delta_{\text{PF}}[m, v]\prod_x \delta\Big[\sum_\mu(\xi_\mu(x)-\xi_\mu(x-\mu))\Big] \equiv \mathscr{F}^{\text{STAT}}. \tag{A10}$$

The first factor in Eq. (A10) comes from the group volume. To perform the integration we write the background gauge condition in terms of $C_n(p)$ and obtain

$$\prod_x \delta\left[\sum_\mu (\xi_\mu(x)-\xi_\mu(x-\mu)\right] = \prod_p \frac{\delta(C_0(p))}{\sqrt{2\sum_\mu (1-\cos p_\mu)}}. \tag{A11}$$

The Jacobian comes from the normalization of the zero-mode eigenfunction

$$\psi_\mu^{(0)}(p) = \frac{1-e^{ip_\mu}}{\sqrt{2\sum_\mu (1-\cos p_\mu)}}. \tag{A12}$$

From the $1/d$ expansion of the Fadeev-Popov determinant we use only the zeroth order term

$$\Delta_{\mathrm{PF}}^{(0)}[m, v] = \exp\left\{N \log m + N \int_{-\pi}^{+\pi} \frac{d^d p}{(2\pi)^d} \log\left[\sum_\mu (2-2\cos p_\mu)\right]\right\}. \tag{A13}$$

The integration of these remaining fluctuations leads to a correction to the free energy

$$\Delta f_{3\,\mathrm{COUL}}^{\mathrm{STAT}} = \frac{1}{d}\left[-\tfrac{1}{2}\log 2\pi - \log m - \tfrac{1}{2}\int_{-\pi}^{+\pi} \frac{d^d p}{(2\pi)^d} \log\left[\sum_\mu (2-2\cos p_\mu)\right]\right.$$
$$\left. + \tfrac{1}{2}\int_{-\pi}^{+\pi} \frac{d^d p}{(2\pi)^d} \log A^\gamma(p)\right]. \tag{A14}$$

The first three terms are the contribution of the Fadeev-Popov determinant, the group volume and the zero-mode integration. The last comes from the Higgs fluctuations. In the dynamical part of the Coulomb phase we find zero-modes for the Higgs field ($A^\gamma(p) = 0$). Choosing a global U(1) symmetry for the Higgs field

$$\gamma(x) = 0 \tag{A15}$$

we obtain

$$\mathscr{F}[m, v, M, h] = 2\pi\Delta[M, h]\prod_x \delta(\gamma(x))\mathscr{F}^{\mathrm{STAT}}. \tag{A16}$$

The first factor is the contribution of the group volume of the global U(1) group. We write the gauge condition (A15) in terms of the transformed fluctuations, so that

$$\prod_x \delta(\gamma(x)) = \prod_p \delta(\gamma(p)). \tag{A17}$$

For the determinant $\Delta[M, h]$ we need only the zeroth order contribution

$$\Delta^{(0)}[M, h] = \exp\{N \log M\}. \tag{A18}$$

Performing the integration we obtain a contribution to the free energy which is identical to the expression (A14) up to the last term which has to be

$$\frac{1}{2d}\left[-\frac{2}{N}\log 2\pi - 2\log M + \log 2\pi\right]. \tag{A19}$$

The first term comes from the group volume, the second is the contribution of the determinant (A18) and the last is the result of the integration of the Higgs zero mode. The corrections to the free energy given in Eqs. (4.14), (4.16) are the sum of corrections $\Delta f_{1,2,3}$.

APPENDIX B

Remarks on the one-link integral in the weak coupling region

The one-link integral of the lattice Abelian Higgs model is defined by

$$f^{(q)}(z_1, z_2) = \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\{z_1 \cos\theta + z_2 \cos q\theta\}. \tag{B1}$$

It is easy to see that in the region of trivial Higgs saddle-point ($M = 0$)

$$f^{(q)}(z_1, z_2) = I_0(z_1) \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\{z_1 \cos\theta\} \tag{B2}$$

since $z_2 = 0$. In the region of non-trivial Higgs saddle-point ($M \approx 1 - 1/8d\kappa$) we perform a Fourier expansion for the exponential term in Eq. (B1)

$$\exp\{z_1 \cos\theta\} = I_0(z_1) + 2\sum_{l=1}^{\infty} I_l(z_1)\cos(l\theta),$$

$$\exp\{z_2 \cos q\theta\} = I_0(z_2) + 2\sum_{k=1}^{\infty} I_k(z_2)\cos(kq\theta), \tag{B3}$$

where the $I_n(z)$ are modified Bessel functions. Integrating out θ we obtain that only terms with $l = kq$ remain. So the result is

$$f^{(q)}(z_1, z_2) = I_0(z_1)I_0(z_2) + 2\sum_{k=1}^{\infty} I_{kq}(z_1)I_k(z_2). \tag{B4}$$

It was shown by Kasperkovitz [11] that for z sufficiently large the following approximation can be used

$$\frac{I_n(z)}{I_0(z)} \approx \exp\left\{-\frac{n^2}{2z}\right\} \tag{B5}$$

independent of n. Using this approximation we obtain the expression for the one-link integral given in Eq. (4.5).

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