LANDAU THEORY OF A QED PLASMA*

By Ch. G. van Weert and M. C. J. Leermakers

Institute for Theoretical Physics, University of Amsterdam**

(Received November 5, 1984)

The statistical thermodynamics of hot dense matter is developed as a covariant extension of the Landau quasi-particle theory of Fermi liquids. The basic postulate is that the thermodynamic potential is a functional of one-particle distribution functions. The formalism is given a microscopic foundation by a study of perturbative QED at finite temperature and density. It is shown that a QED plasma can be pictured as a collection of stable quasi-electrons and positrons plus damped collective modes. The transverse plasmon effect is discussed briefly.

PACS numbers: 05.30.Fk, 67.90.+z, 11.15.Bt

1. Introduction

The Landau theory of Fermi liquids gives a comprehensive semi-phenomenological description of degenerate quantum fluids, like ^{3}He and nuclear matter [1, 2]. Central to the theory is the quasi-particle concept which provides an illuminating framework for understanding dense fermion systems at low temperature. It might be of great value if this theory could be extended to apply to hot dense matter systems such as high-temperature astrophysical plasmas and, especially, quark matter in the plasma phase.

The present paper reports on our efforts to construct a covariant extension of the Landau theory [3] and to give a microscopic foundation by perturbative methods [4, 5]. The macroscopic theory is developed from a single postulate, namely that the thermodynamic potential (pressure) is a functional of the one-particle distribution functions of the different particle species present in the system. When this postulate is combined with relativistic thermodynamics, which topic we shall review briefly in the next section, it has as a consequence that the entropy density of the system has the same form as for an ideal gas, but of elementary excitations instead of bare particles. The possible values of the excitation energies are determined by the functional derivative of the pressure with respect to the

* Based on the talk given by the first author at the XXIV Cracow School of Theoretical Physics, Zakopane, June 6–19, 1984.

** Address: Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018XE Amsterdam, The Netherlands.

distribution functions and this holds true irrespective of the temperature. What statistical thermodynamics cannot tell is whether the elementary excitations are long-lived so that they may be interpreted as quasi-particles (dressed particles). Such information may be provided by a microscopic analysis.

In the early sixties Luttinger and Ward [6] constructed the thermodynamic potential of an electron gas as a functional of the full many-body fermion propagator. Later it was shown that such functionals can also be obtained for other systems, notably those described by QED and QCD [7]. Thus, the formal structure of euclidean field theory at finite temperature suggests that the elementary excitations are associated with the poles of the full fermion propagator. Unfortunately, this propagator always has an appreciable imaginary part, except close to the Fermi surface where the attenuation is suppressed by kinematical and statistical factors. This seems to preclude any meaningful quasi-particle interpretation at arbitrary temperature [8].

Nonetheless, further progress is possible if it is observed that, on the one hand, field theory yields the pressure as a functional of propagators, while, on the other hand, the Landau theory requires the pressure as a functional of distribution functions. In principle, the latter representation is related to the former by an analytic continuation of the euclidean propagators, but there are subtleties in going from the one to the other. This will be discussed in detail for a QED plasma in a neutralizing background. However, the reasoning also applies to the physically interesting case of a QCD plasma.

The major difficulty is that in finite-temperature field theory the energies are complex and that the rule for transporting them to the real axis is not at all evident. One could deal with this problem by employing the real-time method, also known as the time-path formalism [9] or thermofield dynamics [10]. This method automatically yields Feynman rules involving real energies. However, the drawback is that the propagators now have a matrix structure. This feature is essential for avoiding ambiguities in the treatment of propagators with coinciding four-momenta, but it considerably complicates actual calculations because the mechanism hinges on intricate cancellations amongst the potentially dangerous terms [11].

As we shall show, it is profitable, before analytic continuation, to first classify closed diagrams in terms of cycles [12], that is, according to the occurrence of products of coinciding bare propagators. As far as the fermions are concerned, such products may be treated as distinct entities and analytically continued separately. In this way, no pathologies like powers of δ-functions ever arise, and, as a further benefit, the ensuing pressure functional has a remarkably transparent form. Surprisingly enough, it allows for a true quasi-particle interpretation of the fermions. These quasi-fermions have an effective self energy which is represented by the same set of diagrams as the ordinary many-body self energy, but the rules for its calculation are such that it has no imaginary part.

This fundamental difference with the ordinary self energy is the result of the particular sequence we have chosen to perform the discrete energy summations. This amounts to a reallocation of fermionic and bosonic degrees of freedom such that the fermion self energy is real. However, the analysis is not complete without a discussion of the boson excitations. We confine ourselves to the ring approximation and show that the appropriate

self energy determining these excitations is the well-known second-order polarization function. We close with some remarks on the transverse plasmon effect which is found to be non-vanishing [5, 13] contrary to common belief.

2. *Relativistic thermodynamics*

The relativistic formulation of the laws of thermodynamics goes back to Planck and Einstein, and is based on the observation that in the rest frame of a system in equilibrium the first and second law may be stated in their usual form. This implies that the rules of thermodynamics do not rely on non-relativistic conditions for their validity and that thermodynamic theory has as wide a range of applicability in relativistic physics as it has in classical physics. Below the principal thermodynamic formulae are assembled in so far as they are pertinent to the subject matter of this paper.

An arbitrary macroscopic state of a relativistic fluid is phenomenologically characterized by a symmetric conserved energy-momentum tensor $T^{\mu\nu}(x)$, and a number of conserved currents $J^{\mu}_{A}(x)$, $A = 1, 2, \ldots$[1]. If the system is in thermodynamic equilibrium and moving with a constant four-velocity U^{μ}, $U_{\mu}U^{\mu} = 1$, J^{μ}_{A} and $T^{\mu\nu}$ are space-time independent and of the form,

$$J^{\mu}_{A} = Q_{A}U^{\mu}, \tag{2.1}$$

$$T^{\mu\nu} = EU^{\mu}U^{\nu} - P\Delta^{\mu\nu}, \tag{2.2}$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - U^{\mu}U^{\nu}$ is the projector on space-like directions. The quantities Q_{A}, E, and P are Lorentz-scalars which may be identified with the charge densities, the energy density, and the pressure, respectively, all defined with respect to the rest frame.

In addition to these hydrodynamic state variables, one may introduce an entropy flux

$$S^{\mu} = SU^{\mu}, \tag{2.3}$$

where S is the entropy density, again defined with respect to the rest frame. From ordinary thermodynamics we know that infinitesimal variations of the entropy density are related to changes in the hydrodynamic variables through the Gibbs relation

$$\delta S = \beta\delta E + \sum_{A} \alpha_{A}\delta Q_{A}. \tag{2.4}$$

The thermodynamic parameters entering here are the inverse temperature $\beta = T^{-1}$ and the independent chemical potentials μ_{A} in the combination $\alpha_{A} = -\beta\mu_{A}$.

A second fundamental thermodynamic relationship is the Euler equation

$$S = \beta P + \beta E + \sum_{A} \alpha_{A}Q_{A}, \tag{2.5}$$

which expresses the additivity of the entropy and the other extensive state variables. Alternatively, one may regard (2.5) as the definition of the grand-canonical potential $\Omega = -P$

[1] Notation and conventions: $x = x^{\mu} = (t, \vec{x})$, $\mu = 0, 1, 2, 3$, $\partial_{\mu} = (\partial_{t}, \nabla)$; metric tensor $g^{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$; units such that $\hbar = c = k_{B} = 1$.

which is the Legendre transform of E such that entropy as an independent variable is replaced by temperature, and charge density by chemical potential. When we take the variation of (2.5) and subtract (2.4) we obtain

$$\delta P = S\delta T + \sum_A Q_A \delta \mu_A, \tag{2.6}$$

which is usually called the Gibbs-Duhem relationship. It shows that Ω, or P, regarded as a function of T and μ_A, is a characteristic thermodynamic function from which the other thermodynamic quantities can be obtained by differentiation.

At this point we note that in the case of gravitational long-range forces the equality $\Omega = -P$ may be violated, and that therefore the Euler and Gibbs-Duhem relationships may not hold [14]. This case is explicitly excluded from our considerations here.

The covariant formulation [15] of the various thermodynamic identities stated above is achieved by treating the four-velocity U_μ as a thermodynamic variable on a par with the temperature T, and the chemical potentials μ_A. Multiplying (2.5) by U^μ and taking (2.1)–(2.3) into account, we get for the entropy flow

$$S^\mu = \beta^\mu P + \beta_\nu T^{\nu\mu} + \sum_A \alpha_A J_A^\mu, \tag{2.7}$$

where β_μ is the time-like vector βU_μ. The variation of this expression may be written as

$$\delta S^\mu = \beta_\nu \delta T^{\nu\mu} + \sum_A \alpha_A \delta J_A^\mu, \tag{2.8}$$

which is the covariant Gibbs relation. It embodies both the ordinary Gibbs relation (2.4) and the Euler relation (2.5), and has the same physical content.

3. *Functional ansatz*

Let us now consider an interacting particle system consisting of a number of particle species $k = 1, 2, \ldots$. According to the rules of statistical mechanics all thermodynamic properties of this system can be determined by calculating a partition function. For applications in quantum physics the grand-canonical partition function $Z(\beta, \alpha_A, V)$ is the most convenient. It depends on the volume of the system in the rest frame, and in the thermodynamic limit furnishes the pressure

$$P(T, \mu_A) = \lim_{V\to\infty} \frac{1}{\beta V} \log Z(\beta, \alpha_A, V), \tag{3.1}$$

on account of the fact that the logarithm of the partition function is extensive.

To make further progress we now assume that by way of the partition function it is possible, in principle, to determine the pressure as a functional

$$P(T, \mu_A) = P[n_k] \tag{3.2}$$

of the equilibrium distribution functions

$$n_k = \frac{1}{e^{y_k} - \theta_k} \tag{3.3}$$

of the different fermions ($\theta_k = -1$) and bosons ($\theta_k = 1$) in the system. The exponent is a linear combination of the conserved quantum numbers q_{Ak} carried by particles of species k, and their four-momenta,

$$y_k = \sum_A q_{Ak}\alpha_A + \beta_\mu p^\mu, \tag{3.4}$$

with coefficients $\alpha_A = -\beta\mu_A$, and $\beta_\mu = \beta U_\mu$. The functional form (3.2) will be justified later on, but we first like to explore some of its consequences.

It is important to notice that no mass-shell restriction has been imposed: all four components of the four-momentum p^μ are treated as independent variables. This is essential for what follows because it will enable us to introduce the concept of an elementary excitation. For this purpose we consider the change of the pressure due to an infinitesimal change δn_k in the distribution functions:

$$\delta P = \sum_k \int \frac{d^4p}{(2\pi)^3} H_k(p)\delta n_k(p). \tag{3.5}$$

This defines the quantity $H_k(p)$, which we shall call the spectral function, as the functional derivative

$$H_k(p) = \frac{\delta P}{\delta n_k(p)} \tag{3.6}$$

of the pressure with respect to the distribution functions. Like the pressure it is a functional of the distribution functions and, therefore, depends on the state of the system, that is, the distribution of all particles.

The variation of the distribution functions can be induced by a variation of the thermodynamic parameters. Hence, we have from the Gibbs-Duhem relation (2.6)

$$Q_A = \frac{\partial P}{\partial \mu_A} = \sum_k \int \frac{d^4p}{(2\pi)^3} H_k \frac{\partial n_k}{\partial \mu_A}, \tag{3.7}$$

$$S = \frac{\partial P}{\partial T} = \sum_k \int \frac{d^4p}{(2\pi)^3} H_k \frac{\partial n_k}{\partial T}. \tag{3.8}$$

The derivatives of the distribution function (3.3) herein may be replaced by one with respect to p_μ. It then follows with an integration by parts that the current density (2.1) and the

entropy flow (2.3) can be represented in the form

$$J_A^\mu = \sum_k q_{Ak} \int \frac{d^4p}{(2\pi)^3} v_k^\mu n_k, \tag{3.9}$$

$$S^\mu = -\sum_k \int \frac{d^4p}{(2\pi)^3} v_k^\mu [n_k \log n_k - \theta_k(1+\theta_k n_k) \log (1+\theta_k n_k)]. \tag{3.10}$$

The generalized velocity appearing here has been defined as the derivative

$$v_k^\mu(p) = \frac{\partial H_k(p)}{\partial p_\mu}, \tag{3.11}$$

like in the hamiltonian formulation of mechanics. Together with the spectral function, it will play an important role in the following because the connection (3.11) not only determines the relation between velocity and momentum, but also the energy spectrum of the elementary excitations, as one will see.

The two expressions (3.9) and (3.10) resemble the corresponding expressions for an ideal quantum gas [16]. All effects of the interaction are hidden in the generalized velocity which will be further analyzed in subsequent sections. In the energy-momentum tensor, on the other hand, the interaction manifests itself in a more explicit manner. In fact, it can be easily verified that the energy-momentum tensor implied by (3.9), (3.10), and the covariant Euler relation (2.7) must have the form

$$T^{\mu\nu} = \sum_k \int \frac{d^4p}{(2\pi)^3} p^\mu v_k^\nu n_k - g^{\mu\nu} \left(P - \sum_k \frac{d^4p}{(2\pi)^3} H_k n_k \right). \tag{3.12}$$

The first term looks like the energy-momentum tensor of a free gas, whereas the second term should be considered as an interaction density.

It would seem that the energy-momentum tensor (3.12) has an antisymmetric part

$$T_a^{\mu\nu} = \tfrac{1}{2} \sum_k \int \frac{d^4p}{(2\pi)^3} (p^\mu v_k^\nu - p^\nu v_k^\mu). \tag{3.13}$$

However, one may argue that this antisymmetric part vanishes identically. The argument is based on the observation that the pressure is invariant under a Lorentz transformation of the integration variable $p_\mu \to p_\mu + \delta p_\mu$, $\delta p_\mu = \varepsilon_{\mu\nu} p^\nu$ with $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$. The corresponding variation of the distribution function equals

$$\delta n_k(p) = \delta p_\mu \frac{\partial}{\partial p_\mu} n_k(p). \tag{3.14}$$

Inserting this into (3.5) and putting δP equal to zero, we see that the following identity holds true

$$\varepsilon_{\mu\nu} \sum_k \int \frac{d^4p}{(2\pi)^3} H_k \frac{\partial n_k}{\partial p_\mu} p^\nu = 0, \tag{3.15}$$

which after a partial integration yields $T_a^{\mu\nu} = 0$. This argument is independent of the equilibrium form of the distribution function.

In passing we mention that the expressions (3.9), (3.10), and (3.12) for the primary hydrodynamic and thermodynamic state variables may be used to construct a transport theory of relativistic quantum fluids [3]. A discussion of the nonequilibrium aspect lies, however, outside the scope of the present paper.

4. Spectral density

It was mentioned above, that the spectral function and the generalized velocity carry information about the energy spectrum of the system. The simple example of an ideal quantum gas may serve to illustrate this idea. For the pressure of an ideal gas we have the well-known expression

$$P_0 = \beta^{-1} \sum_k \theta_k g_k \int \frac{d^3p_k}{(2\pi)^3 p_k^0} p_k \cdot U \log(1+\theta_k n_k), \tag{4.1}$$

where g_k is the number of internal degrees of freedom and $p_k^0 = (\vec{p}_k^2 + m_k^2)^{\frac{1}{2}}$ the kinetic energy of the particles. Ohe may notice that this expression is not of the form assumed in (3.2) because p_k^μ is on mass-shell. Moreover, the dependence on the temperature is not entirely contained in the distribution function. However, this is easily repaired by insertion of a mass-shell δ-function written as the derivative of a step-function

$$2p \cdot U \delta(p^2 - m^2) = U \cdot \frac{\partial}{\partial p} \theta(p^2 - m^2). \tag{4.2}$$

After an integration by parts we obtain for the pressure the alternative expression

$$P_0 = \sum_k g_k \int \frac{d^4p}{(2\pi)^3} \theta(p_0)\theta(p^2 - m_k^2) n_k(p), \tag{4.3}$$

which is of the desired form.

Functional differentiation yields the spectral function

$$H_k(p) = g_k \theta(p_0)\theta(p^2 - m_k^2), \tag{4.4}$$

and the associated generalized velocity

$$v_k^\mu(p) = 2 g_k p^\mu \theta(p_0)\delta(p^2 - m_k^2). \tag{4.5}$$

This explicitly shows how the mass-shell restriction is contained in both H_k and v_k^μ. We can also write (4.5) as

$$v_k^\mu(p) = v^\mu \varrho_k(p), \tag{4.6}$$

where v^μ is the ordinary velocity p^μ/p_0 and

$$\varrho_k(p) = \frac{\partial H_k(p)}{\partial p_0} = g_k \delta(p^0 - p_k^0) \tag{4.7}$$

the spectral density (defined by the first member of the equation) of a free particle system.

The spectral density (4.7) expresses the fact that the elementary excitations of a free system are stable particles possessing a fixed mass. The question is, what do these excitations look like after the interaction is switched on? In the Landau theory of Fermi liquids it is assumed that the excitation spectrum has a structure which is to some extent similar to that of an ideal Fermi gas and can be discussed in terms of quasi-particles (dressed particles) having a definite energy $\varepsilon(\vec{p})$.

In the present theory the quasi-particle picture emerges naturally if we assume that for an interacting Fermi system the spectral function (4.4) can be generalized to

$$H_k(p) = g_k \theta(p_0)\theta[p^2 - M_k^2(p)], \tag{4.8}$$

where $M_k(p)$ is some effective mass. Such a spectral function would imply a sharply defined spectral density

$$\varrho_k(p) = g_k \theta(p_0)\delta(p_0 - \varepsilon_k(\vec{p})), \tag{4.9}$$

where $\varepsilon_k(\vec{p})$ is the quasi-particle energy determined by the solution (assumed to be unique) of the dispersion relation

$$p^2 - M_k^2(p) = 0. \tag{4.10}$$

For the generalized velocity we find

$$v_k^0 = \varrho_k, \quad \vec{v}_k = \frac{\partial \varepsilon(\vec{p})}{\partial \vec{p}} \varrho_k. \tag{4.11}$$

The first equation is simply the definition of the spectral density. The second one follows immediately from (4.8) if the argument of the step-function is read as $p_0 - \varepsilon_k(\vec{p})$.

Unlike the other formulae we have derived so far, equations (4.9) and (4.11) have a non-covariant appearance. This has come about because, for the purpose of illustration, we have chosen the energy in the observer's frame of reference p_0 as the variable for which the dispersion relation (4.10) is to be solved. Another possibility would be to select the energy in the rest frame $p \cdot U$. The latter choice leads to a formally covariant theory and will be further adopted here. The former choice furnishes a link with the non-covariant formulation of the relativistic Landau theory as given by Baym and Chin [2]. To verify this last statement one only needs to insert (4.11) into the expressions for the macroscopic currents, given in Sect. 3, and to integrate out the energy.

The Landau theory has been successful as a phenomenological description of normal Fermi systems, but according to conventional wisdom its validity is confined to low temperatures. From the point of view of many-body theory this is only too understandable, if one identifies quasi-particles with the poles of the full many-body propagator [6]. Indeed, its imaginary part, which is only small close to the Fermi surface, signifies rapid decay of these "dynamical" quasi-particles. Nevertheless, there exist calculations indicating that a different kind of non-decaying quasi-particle, called "statistical", can be obtained as an exact result of many-body theory at any temperature [17]. This would imply that the Landau theory is exact at arbitrary temperatures, and not limited by the life-time of the quasiparticles.

At this stage we shall not take sides in this matter. Instead, we shall continue and apply the theory of the last three sections to the particular case of a QED plasma. The programme we shall follow is dictated by Eqs. (3.2), (3.6), and is quite straightforward in principle: (*i*) express the pressure as a functional of the distribution functions, and (*ii*) calculate the spectral function by functional differentiation. The result should speak for itself and reveal what type of elementary excitations we are dealing with, and whether they are long-lived or not.

5. *Feynman rules*

We shall study quantum electrodynamics at finite temperature describing an interacting system of electrons, positrons and photons in a neutralizing background [18–20]. The euclidean fermion and photon propagators, when Fourier transformed, have the form

$$S_0(p) = \frac{1}{\gamma \cdot p - m}, \qquad D_0(k) = \frac{g^{\mu\nu}}{k^2}. \tag{5.1}$$

The γ's are the ordinary Dirac matrices and we adopt the Feynman gauge. The difference with vacuum propagators is that the energies take discrete and complex values:

$$p_0 = i(2n+1)\pi\beta^{-1} + \mu, \qquad k_0 = i2n\pi\beta^{-1}, \tag{5.2}$$

with n integer. The single chemical potential μ corresponds to the conserved charge carried by the fermions.

The advantage of the euclidean formalism is that the Feynman rules for calculating the generating functional, i.e. the partition function, closely resemble those of the vacuum theory. The principal statement is that the interaction pressure $P_I = P - P_0$ can be equated to the sum of all closed linked diagrams (bubbles) each multiplied by a combinatorial factor depending on the order and topology of the diagram [21]. For QED the first few diagrams have been depicted in Fig. 1.

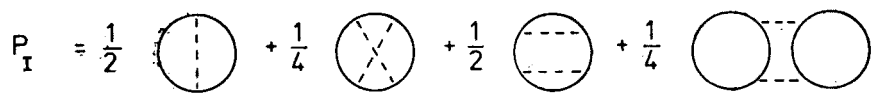

Fig. 1. Lowest-order QED diagrams contributing to P_I

(Tadpole diagrams have been left out because they do not contribute in a charge-neutral system.)

The Feynman rules for finite-temperature QED [22] may now be stated as follows:

1. A bubble diagram consists of $2N$ vertices, $2N$ directed fermion lines, and N photon lines.

2. Each bubble is weighted with a factor $S/(2N)!$ where S is the number of different ways the vertices can be connected with the same topological result [23].

3. Propagators (5.1) are associated with each fermion and photon line.

4. A factor $e\gamma^\mu$ is inserted for each vertex.

5. Energy and momentum are conserved at each vertex; momentum by a three-dimensional δ-function, and energy by a Kronecker delta, multiplied by the factor $\beta(2\pi)^3$. [One arbitrary vertex is excluded since energy and momentum conservation would yield a factor βV which has already been divided out, cf. (3.1).]

6. All discrete spinor and Lorentz indices are contracted at each vertex. All momenta and energies are summed and integrated over according to $\beta^{-1}\Sigma \int d^3p/(2\pi)^3$.

7. Each closed fermion loop carries an additional minus sign.

In formulating these rules we have tried to minimize the nuisance value of the factors i and 2π.

Self energies are obtained by opening up fermion or photon lines in the diagrammatic expansion of P_I. For example, if one fermion line is cut in all possible ways, one obtains all diagrams contributing to the reducible self energy Σ^*. This rule may be expressed by the functional equation

$$\frac{\delta P_I}{\delta S_0} = -\Sigma^* = -\sum_{m=1}^{\infty} (\Sigma S_0)^m S_0^{-1}, \tag{5.3}$$

where Σ is the proper self energy. The minus sign appears because of rule 7. Similarly, one has

$$\frac{\delta P_I}{\delta D_0} = \tfrac{1}{2}\Pi^* = \tfrac{1}{2}\sum_{m=1}^{\infty} (\Pi D_0)^m D_0^{-1}, \tag{5.4}$$

where Π^* is the reducible and Π the irreducible photon self energy or polarization function.

By using the Dyson equations for the full fermion propagator $S(p)$ and the full photon propagator $D(p)$, it is not difficult to show [4, 5] that the last two formulae are equivalent to the functional relationships

$$\frac{\delta P}{\delta S_0^{-1}} = S, \qquad \frac{\delta P}{\delta D_0^{-1}} = -\tfrac{1}{2}D, \tag{5.5}$$

derived by Freedman and McLerran [7] from the path-integral representation for the total pressure P.

The sums at the right-hand side of (5.3) and (5.4) can be given an interpretation in terms of an m-cycle of lines [12], that is, a set of m lines in a bubble diagram such that

the diagram separates into two disconnected parts if any two of the lines of the set are cut. Removal of all m lines of the set separates the diagram into m disconnected pieces. On account of energy-momentum conservation all lines of an m-cycle carry the same energy and momentum. They are, thus, associated with an m-fold direct product of propagators with coinciding arguments, for which we shall introduce the notations $S_0^m(p) = [S_0(p)]^m$ and $D_0^m(k) = [D_0(k)]^m$.

The lines in a diagram can be unambiguously classified as belonging to cycles with multiplicities $m = 2, 3, \ldots$, or to no cycle at all, which is called a 1-cycle for convenience.

Let us now cut all fermion lines belonging to the subclass of 1-cycles in the graphical representation of P_I. It is then clear that the self-energy diagrams so obtained cannot be separated into two disconnected parts by cutting one other single fermion line. In other words, these diagrams are one-particle irreducible.

This may be formalized by considering P_I as a functional of cycles of increasing multiplicity:

$$P_I = P_I[S_0^1, S_0^2, \ldots; D_0^1, D_0^2, \ldots]. \tag{5.6}$$

Functional differentiation with respect to one single fermion propagator can then be written as

$$\frac{\delta P_I}{\delta S_0} = \sum_{m=1}^{\infty} \frac{\delta P_I}{\delta S_0^m} \cdot \frac{\delta S_0^m}{\delta S_0}. \tag{5.7}$$

Comparing with the second member of (5.3), we conclude that the functional differentiation with respect to m coinciding propagators is given by

$$\frac{\delta P_I}{\delta S_0^m(p)} = -\frac{1}{m}[\Sigma(p)]^m. \tag{5.8}$$

It expresses the fact that the parts of a diagram connected by cycles are proper self-energy insertions. This topological property will play an important role in the sequel.

6. *Analytic continuation*

The Feynman rules given above yield the interaction pressure as a functional $P_I[S_0, D_0]$ of the complex propagators (5.1). However, to make contact with the Landau theory, we need to reexpress the pressure in terms of real energies and distribution functions. This may be achieved by an analytic extension away from the discrete complex energies down to the real axis. The method is standard and involves the replacement of energy sums by contour integrals. For a fermion sum the rule is

$$\frac{1}{\beta}\sum_n f(p_0) = \int_{C_+} \frac{dz}{2\pi i}[n_-(z)f(z) + n_+(z)f(-z)] + \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z), \tag{6.1}$$

and for a boson sum

$$\frac{1}{\beta}\sum_n f(k_0) = -\int_{C_+}\frac{dz}{2\pi i}N(z)[f(z)+f(-z)]+\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}f(z). \tag{6.2}$$

The formulae are valid for any function $f(z)$ which is analytic in the neighbourhood of the imaginary axis, and the contour C_+ encloses counterclockwise all singularities of the functions $f(z)$ and $f(-z)$ in the right-hand complex plane, but not the poles of the Fermi--Dirac and Bose-Einstein functions

$$n_\pm(z) = \frac{1}{e^{\beta(z\pm\mu)}+1}, \qquad N(z) = \frac{1}{e^{\beta z}-1} \tag{6.3}$$

at the discrete points (5.2).

By this procedure the pressure becomes a functional $P = P[n_\pm, N]$ of the distribution functions of the desired form from which by functional variation

$$\delta P = \sum_{k=\pm}\int\frac{d^4p}{(2\pi)^3}H_k(p)\delta n_k(p)+\int\frac{d^4p}{(2\pi)^3}H_\gamma(k)\delta N(k) \tag{6.4}$$

the spectral functions $H_\pm(p)$ of the electrons and positrons, and $H_\gamma(k)$ of the photons follow.

Although straightforward in principle, the actual analytic continuation of an arbitrary diagram is not without difficulties. Mathematically, there is the problem how to deal with the highly complicated analytic structure of higher-order diagrams. But even apart from this, one is confronted with the problem that the procedure is not unique and generates formally different pressure functionals $P[n_\pm, N]$ depending on the order in which the sums are performed. As far as the pressure is concerned, these functionals are all equivalent on account of identities like

$$N(z-z') = \frac{n\pm(z)\,[1-n\pm(z')]}{n\pm(z')-n\pm(z)}, \tag{6.5}$$

but the spectral densities defined by (6.4) are not.

Our solution [4] to this problem rests on the observation that in closed diagrams as they occur in the perturbation expansion of P_I, the fermion lines are grouped together into closed loops with two or more external photon lines, see Fig. 2. Owing to energy--momentum conservation such loops have only one independent fermion variable, over which must be summed and integrated. The expression associated with a loop with l external lines is of the typical form:

$$L_l(k_1, \ldots k_l) = \frac{1}{\beta}\sum_n\int\frac{d^3p}{(2\pi)^3}\prod_{i=1}^{l}S_0(p_i), \tag{6.6}$$

where irrelevant vertex factors have been suppressed. The internal four-momenta p_i are linear combinations $p_i = p + \varepsilon_{ij} k_j$, $\varepsilon_{ij} = 0, \pm 1$, of the loop variable p and the external photon momenta k_i satisfying $\Sigma k_i = 0$.

The reason for focussing on loops is twofold. The first is that different loops are completely independent with regard to the fermion summation and the order in which these sums are performed is immaterial. Secondly, by doing the fermion sums first we ensure that fermion sums lead to Fermi distribution functions and boson sums to Bose distribution functions without these two getting mixed up.

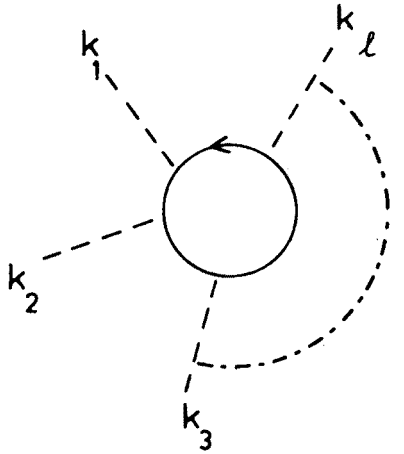

Fig. 2. Fermion loop with external photon lines

The propagators at the right-hand side of (6.6) can be analytically continued away from the discrete energies merely by putting $p_0 = z$. This extension is analytic in the strip $|\mathrm{Re}\, z| < m$, but singularities occur when the internal loop momenta are on their mass shell:

$$z + \varepsilon_{ij} k_j^0 = \pm \sqrt{(\vec{p} + \varepsilon_{ij} \vec{k}_j)^2 + m^2}\,. \tag{6.7}$$

Let us first assume that none of these singularities coincides, that is, the loop does not contain an m-cycle of multiplicity greater than one. The energy sum in (6.6) can be replaced by a contour integral according to the general rule (6.1). The sum of the residues at the points (6.7) may be written as a sum of integrals of the discontinuity across the lines $z = p_0 - \varepsilon_{ij} k_j^0$ with the variable p_0 now real. Shifting also the three-momentum we obtain

$$\int \frac{d^3p}{(2\pi)^3} \int_{C_+} \frac{dz}{2\pi i}\, n_-(z) \prod_{i=1}^{l} S_0(z + \varepsilon_{ij} k_j^0,\ \vec{p} + \varepsilon_{ij} \vec{k}_j)$$

$$= -\int \frac{d^3p}{(2\pi)^3} \int_0^\infty \frac{dp_0}{2\pi i}\, n_-(p_0) \frac{\delta L_l}{\delta S_0(p)} \cdot \mathrm{Disc}\, S_0(p). \tag{6.8}$$

The discontinuity has been defined as Disc $S_0(p) = S_0(p_0+i\varepsilon, \vec{p})-S_0(p_0-i\varepsilon, \vec{p})$, and the functional derivative as

$$\frac{\delta L_l}{\delta S_0(p)} = \sum_{n=1}^{l} \prod_{i\neq n} S_0(p+\varepsilon_{ij}k_j-\varepsilon_{nj}k_j). \tag{6.9}$$

Strictly speaking, this functional derivative only exists for discrete complex values of p_0, but as it appears in (6.8) the energy variable is real. This replacement by a real energy must be understood to take place after the functional differentiation has been performed.

In general, loops contain cycles, i.e. products of fermion propagators with identical arguments. The above reasoning is easily extended to this case by regarding the loop function (6.6) as a functional of m-cycles. The various residues are then found by consecutive functional differentiation with respect to all possible cycles. The result is again an integral expression which contains one single fermion distribution function, like (6.8), corresponding to the one independent fermion-loop energy. Taking the functional derivative with respect to this distribution function, the final result of bringing an entire fermion loop to the real axis may be cast in the compact form

$$\frac{\delta L_l}{\delta n_\pm(p)} = \pm\frac{\theta(p_0)}{2\pi i}\sum_{m=1}^{\infty}\frac{\delta L_l}{\delta S_0^m(\mp p)}\cdot \text{Disc } S_0^m(\mp p). \tag{6.10}$$

This crucial equation links the functional derivative with respect to the Fermi distribution functions, as it occurs in the Landau theory, to functional derivatives, of the kind discussed in the previous section, with respect to propagators.

7. *Effective self energy*

As already mentioned, one of the reasons for isolating fermion loops is that they are completely independent from each other as far as the loop integrations and summations are concerned. Also an m-cycle of fermion lines can never involve more than one loop. It is then immediately clear that the functional differentiation of P_l with respect to $n_\pm$ follows the same rule (6.10) as valid for loops. Combining this insight with Eq. (5.8), we obtain

$$\frac{\delta P_l}{\delta n_\pm(p)} = \pm i\frac{\theta(p_0)}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}[\bar{\Sigma}(\mp p)]^m\cdot \text{Disc } S_0^m(\mp p), \tag{7.1}$$

where the quantity $\bar{\Sigma}$ is the proper self energy, but with an external energy that is real.

In order to clearly understand the nature of $\bar{\Sigma}$, let us first recall the definition of the proper self energy in perturbation theory as a linear combination of one-particle irreducible self-energy graphs with discrete complex external energy $p_0 = \mu+i(2n+1)\pi\beta^{-1}$. Since this self energy is known only at a discrete set of points, its analytic continuation to the complex plane, and ultimately to the real axis, cannot be unique without further delimita-

tions. The customary prescription is to require that the self-energy is an entire function having no singularities except on the real axis and vanishing at infinity. This construction is guaranteed to be unique [24] and yields the many-body self energy $\Sigma(p)$ as an analytic function in the $p_0 = z$ plane cut along the real axis.

This many-body self energy must be clearly distinguished from $\bar{\Sigma}(p)$ which is obtained by a different, and almost trivial, prescription. It involves nothing more than a formal replacement of the discrete external energy variable by the continuous variable $p_0 = z$, while keeping all internal boson lines imaginary. The resulting self energy is certainly not analytic, but at the real axis it is well defined. The main difference between the two self energies is that Σ in general has an imaginary part, whereas $\bar{\Sigma}$ is real at the real axis. In virtue of this last property the sum in (7.1) can be written as

$$2i \operatorname{Im} \sum_{m=1}^{\infty} \frac{1}{m} \operatorname{tr} [S_0(\mp p)\bar{\Sigma}(\mp p)]^m = -2i \operatorname{Im} \operatorname{tr} \log S_0 \bar{S}^{-1}, \tag{7.2}$$

where we defined an effective propagator through

$$\bar{S}^{-1}(p) = S_0^{-1}(p) - \bar{\Sigma}(p) + i\gamma_0 \varepsilon, \tag{7.3}$$

which again differs from the ordinary analytic many-body propagator in the fact that it is a real quantity.

Substituting (7.2) into (7.1) and noting that the factor S_0 in the logarithm is cancelled out by the contribution of the free pressure, we finally get for the functional derivative of the pressure, i.e. the spectral function, see (6.4),

$$H_{\pm}(p) = \pm \theta(p_0)/\pi \operatorname{Im} \log \operatorname{Det} \bar{S}^{-1}(\mp p), \tag{7.4}$$

where we also used $\operatorname{tr} \log \bar{S}^{-1} = \log \operatorname{Det} \bar{S}^{-1}$. The last derived formula shows that the interaction is fully incorporated into the propagator $\bar{S}$. The excitation energies of the system are given by the solutions of the dispersion relation

$$\operatorname{Det} \bar{S}^{-1}(\mp p) = 0, \quad p_0 > 0. \tag{7.5}$$

In the absence of an imaginary part these excitations have an infinite life time. Using the terminology of Balian and de Dominicis [17], we may call them "statistical" quasi-particles.

In the "dynamical" approach, which has become standard since the first paper of Luttinger and Ward [6] on the many-body theory of Fermi-systems, the pressure is considered as a functional of the analytic many-body propagator. This functional form then serves as a basis for the formal derivation of the "dynamical" quasi-particle behaviour of an electron gas. Thus, dynamical quasi-particles are defined by functional differentiation with respect to propagators, whereas statistical quasi-particles are defined by functional differentiation with respect to distribution functions as prescribed by the Landau theory. The two prescriptions are quite different, except at temperature zero at the Fermi surface, where they coincide.

8. Ring approximation

In the previous section we saw how the statistical self energy can be used to determine the behaviour of the excited fermion states. However, at finite temperature, there are other excited states that have to be taken into account too, namely the collective modes described by the bosonic spectral function $H_\gamma(k)$ as defined in (6.4). For the purpose of calculating H_γ we now have to carry through the analytic continuation of the bosonic energies.

For simplicity we restrict ourselves to the subclass of ring diagrams, that is, the set of single closed photon loops with an arbitrary number of lowest-order polarization insertions. It is well-known that this selected class of most infra-red divergent diagrams makes the following contribution [18]:

$$P_1^{\text{ring}} = \frac{1}{2\beta} \sum_{m=1}^{\infty} \frac{1}{m} \left\{ \sum_n \int \frac{d^3k}{(2\pi)^3} \operatorname{tr} [\Pi(k) D_0(k)]^m \right\}. \tag{8.1}$$

Here $D_0^{\mu\nu}$ is the bare photon propagator and $\Pi^{\mu\nu}$ the lowest-order (proper) polarization tensor. The trace means a contraction of Lorentz indices, and the m'th power should be understood in the sense of matrix multiplication.

The lowest-order polarization tensor contains one fermion loop which must be brough, to the real axis first. The resulting expression is well-known and can be found in Refs. [18t 20, 5]. It is a functional of the fermion distribution functions $n_\pm$, and explicitly analytic both in the upper and lower complex energy plane. Functional differentiation of the ring pressure with respect to $n_\pm$ and comparison with (7.1) in the truncated form

$$\frac{\delta P_1^{\text{ring}}}{\delta n_\pm(p)} = -\theta(p_0)\delta(p^2 - m^2) \operatorname{tr} (\mp\gamma \cdot p + m) \bar{\Sigma}^{\text{ring}}(\mp p) \tag{8.2}$$

yields the corresponding approximation for the statistical self energy [5].

To obtain the photon spectral function we now have to perform the explicit boson summation in (8.1) with the help of (6.2). Since the product $\Pi(k)D_0(k)$ is analytic everywhere except at the real axis we can do this term by term and sum over m afterwards. The result is the spectral function in the ring approximation [5]

$$H_\gamma^{\text{ring}}(k) = -\frac{\theta(k_0)}{\pi} \operatorname{Im} \left[\log \left(1 - \frac{\Pi_L}{k^2} \right) + 2 \log (-k^2 + \Pi_T) \right], \tag{8.3}$$

where $\Pi_{L,T}$ are the longitudinal and transverse polarization functions defined by the connections

$$\Pi_L = -k^2 \Pi^{00}/\vec{k}^2, \quad \Pi_T = \tfrac{1}{2}(\Pi_\mu^\mu - \Pi_L), \tag{8.4}$$

in terms of the polarization tensor.

Contrary to the statistical fermion self energy, the polarization functions have imaginary parts. However, one should bear in mind that the longitudinal and transverse parts of the spectral density (8.3) describe two distinct physical phenomena. The former has to

do with the mechanism of dynamical screening by virtual, i.e. space-like, plasma excitations. This screening effect persists even in the static limit where one finds a static screening length κ_L^{-1} given by

$$\kappa_L^2 = \lim_{|\vec{k}|\to 0} \Pi_L(0, \vec{k}). \tag{8.5}$$

In contrast, the transverse polarization vanishes in this limit, implying that a static magnetic field is unscreened. The main effect of Π_T is that it shifts the pole of the free propagator. In this way the photon acquires a temperature and density dependent mass. The imaginary part of the transverse polarization function vanishes in the region $0 < k^2 < 4m^2$, implying that the undamped solution of the dispersion relation may be regarded as a dressed photon with a mass given by

$$\omega_T^2 = \Pi_T(k^2 = 0), \tag{8.6}$$

in accordance with Ueda's prescription [25] that the momentum should be put on the mass-shell of the corresponding bare photon at zero temperature and density.

Elsewhere [5] we have further analyzed the plasmon effect at high temperature, that is, the term of order $\alpha^{3/2}$ in the spectral function (8.3). This is the leading contribution coming from the resummation of the infrared-divergent ring diagrams. This contribution may be isolated by subtracting the zeroth and first-order terms in the fine-structure constant α. The remainder, which is conventionally called the correlation part, takes the form

$$H_\gamma^{\text{corr}} = H_\gamma^{\text{ring}} - \frac{\theta(k_0)}{\pi}\left[2\pi\theta(k^2) + \text{Im}\left(\frac{\Pi_L + 2\Pi_T}{k^2}\right)\right]. \tag{8.7}$$

Our analysis shows that to order α^2 the dominant high-temperature behaviour of the transverse part of this expression is rendered by the spectral function $2\theta(k^2 - \omega_T^2)$, which corresponds to a gas of photons with mass ω_T as defined in (8.6). In the high-temperature limit, where we may use $N(k_0) = (\beta k_0)^{-1}$, the correlation pressure

$$P^{\text{corr}} = \int \frac{d^4k}{(2\pi)^3} H_\gamma^{\text{corr}}(k) N(k_0) \tag{8.8}$$

may be evaluated explicitly. The result is

$$P^{\text{corr}} = \frac{\kappa_L^3}{12\pi\beta} + \frac{\omega_T^3}{6\pi\beta}, \tag{8.9}$$

where κ_L is the inverse screening length and ω_T the mass (plasma frequency) defined in (8.5) and (8.6), respectively.

This answer is easy to understand. The first term is nothing but the well-known Debye-Hückel correction to the equation of state of a classical ionized gas. The second term is the $\alpha^{3/2}$ contribution of the asymptotic expansion [26] of the pressure of an ideal gas of bosons with mass ω_T.

We discussed the plasmon effect in some detail because it illustrates the fact that the summation of the ring diagrams is not a unique operation. Indeed, the usual procedure is to perform this summation directly in (8.1), which leads to the alternative expression for the correlation pressure

$$P^{\text{corr}} = -1/2\beta \sum_n \int \frac{d^3k}{(2\pi)^3} \operatorname{tr} [\log (1 - D_0\Pi) + D_0\Pi], \tag{8.10}$$

for the first time obtained by Akhiezer and Peletminskii [18]. The standard calculation of this expression at high temperature is based on the observation that only the zero--energy term needs to be retained. Subsequently, one replaces $\Pi_{\mu\nu}(0, \vec{k})$ herein by its infra-red limit $\kappa_L^2 g_{\mu 0} g_{\nu 0}$, cf. (8.5), and performs the integration. This yields the first term at the right--hand side of (8.9), but not the second term involving the dynamic photon mass. Hence, we must conclude that the two expressions (8.8) with (8.7), and (8.10) for the correlation pressure, are not equivalent.

The problem may be stated as follows. In (8.10) the ring diagrams are summed to the function $\log (1 - D_0\Pi)$ on the imaginary axis, whereas in our approach, which requires the pressure as a functional of distribution functions, we obtain the imaginary part of this same logarithm at the real axis. According to the contour integral identity (6.2) these two expressions should be equal, apart from a vacuum term which plays no role in the high-temperature limit. Yet, they are not. Thus, there seems to be an internal inconsistency.

The resolution of this paradox lies in the analytic properties of the photon propagator, and its inverse $D^{-1} = D_0^{-1} - \Pi$. Both should be analytic in the neighbourhood of the imaginary axis in order that (8.10) is unambiguous and the identity (6.2) valid. From the spectral representation of Π [20] it can be seen that this is ensured by the positivity conditions

$$|\vec{k}|^2 + \Pi_{L,T}(0, \vec{k}) > 0, \tag{8.11}$$

which must hold for all values of $\vec{k}$. The point is that the transverse inequality is not automatically guaranteed by the analytic behaviour of Π_T and, in fact, is violated by the lowest--order approximation for some values of $\vec{k}$. The implication is that $\log (1 - D_0\Pi)$ is not analytic on the imaginary axis, a fact overlooked in the standard evaluation of the plasmon effect.

In contrast we perform the analytic continuation term by term in (8.1) and then define the logarithm on the domain where it exists as an analytic function, namely, everywhere except on both the real and imaginary axes. This prescription follows naturally from our method of analytic continuation which is aimed at the obtaining of the pressure as a functional of distribution functions. Within this framework physical conclusions about quasi--particles and collective excitations are drawn on the basis of an analysis of the spectral functions, obtained by functional differentiation with respect to these distribution functions. Thereby it is ensured that the thermodynamic contributions of the various elementary excitations are properly accounted for. The example of the plasmon effect gives us confidence that the theory is of value not only as a formal scheme, but also as a tool for practical calculations in relativistic many-body theory.

One of us (Ch. G. van Weert) would like to thank the members of the Institute for Theoretical Physics of the Polish Academy of Sciences in Warsaw, and the organizers of the Cracow School of Theoretical Physics at Zakopane for their warm hospitality and their stimulating interest. This investigation is supported in part by the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)".

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