MASS RENORMALIZATION SCHEME INVARIANT QUARKONIUM SUM RULES*

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We perform a mass renormalization scheme invariant analysis of the ITEP heavy quarkonium sum rules in terms of the scale invariant mass $\hat{m}$, thus relating in an unambiguous manner the quarkonium data to the basic QCD parameters $\hat{m}$ and $\Lambda_{\overline{\rm MS}}$. Satisfactory predictions are obtained for all low-lying charmonium states, except the scalar and pseudoscalar ones, for 150 MeV $< \Lambda_{\overline{\rm MS}} <$ 250 MeV. The gluon condensate comes out in average a factor of 3 smaller than the ITEP value. Both expanded and unexpanded ratios are used, yielding similar results. Bottonium sum rules in the vector channel are shown to be consistent with the results of the charmonium sector.

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The interesting sum rule approach developed [1] by Shifman, Vainshtein and Zakharov (SVZ) to relate low energy resonance parameters to fundamental QCD parameters such as α_s and the gluon condensate $G = \frac{\alpha_s}{\pi} \langle 0|F^2|0\rangle$ was first applied to heavy quarkonium, where the dominant power correction to the SVZ sum rules can be parametrized in term of the gluon condensate alone. Comparing experimental data in the $^3S_1$ channel with the purely perturbative contribution to the sum rules, evidence was found [1] for existence of the gluon condensate, with a value $G_{\rm SVZ} \simeq 0.012\ {\rm GeV}^4$, a non-zero value of G being necessary to reduce the systematic discrepancy between data and the purely perturbative contribution. It is clear however that in such an approach one has to be confident that the perturbative contribution has been properly treated, in particular with respect to inclusion of higher order radiative corrections. Now the well-known problem arises that higher order perturbative corrections depend upon the definition of the heavy

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quark mass. In this note, we suggest a mass renormalization scheme (RS) invariant formulation, which yields rather different results from the standard treatment [1]. To set up the problem, consider the SVZ moments:

$$M_n = \frac{1}{n!}\left(-\frac{d}{dQ^2}\right)^n \pi(Q^2)\Big|_{Q^2=0} = \frac{1}{\pi}\int \frac{ds}{s^{n+1}}\,\mathrm{Im}\,\pi(s), \tag{1}$$

where $\pi(Q^2)$ is the charmed vector current hadronic vacuum polarization:

$$i\int d^4x e^{-iqx}\langle 0|T(J_\mu(x)J_\nu(0)|0\rangle = (\eta_{\mu\nu}q^2 - q_\mu q_\nu)\pi(Q^2),$$

$$J_\mu(x) = \bar{c}(x)\gamma_\mu c(x), \quad (Q^2 = -q^2).$$

Using a heavy quark mass expansion one obtains, taking into account only the perturbative contribution:

$$M_n = \frac{a_n}{(\overline{m}^2)^n}[1 + b_n\varrho + O(\varrho^2)], \tag{2}$$

where $\overline{m}$ = pole mass of charmed quark and $\varrho \equiv \frac{\alpha_s}{4\pi}$. Changing the quark mass definition changes the value of b_n, since two masses definitions $\overline{m}_1$ and $\overline{m}_2$ are related by $\overline{m}_2 = \overline{m}_1[1+O(\varrho)]$. There are two problems here: (*i*) one would like to find an "optimum" definition of the mass, in order to decrease the magnitude of the $O(\varrho)$ corrections (this motivated SVZ to adopt the so-called "euclidean mass" definition [1] instead of pole mass). (*ii*) For a given definition of quark mass, one would like to be able to relate the effective coupling ϱ in Eq. (2) to the fundamental QCD scale parameter Λ, as well as to have some control over its n-dependence. This task in principle requires the knowledge of 3-loop radiative corrections with masses, a very involved calculation not undertaken yet. Fortunately, this problem can be bypassed. Suppose one uses instead of $\overline{m}$ the running mass m in any mass independent RS (for definiteness, we shall use the $\overline{\text{MS}}$ scheme). Then, one easily obtains from Eq. (2) and the one loop relation [2] between $\overline{m}$ and m:

$$M_n = \frac{a_n}{(m^2)^n}\left\{1 + \left[b_n - n\left(-8\ln\frac{m^2}{\mu^2} + \tfrac{32}{3}\right)\right]\varrho + O(\varrho^2)\right\}, \tag{3}$$

where μ is the renormalization point.

The renormalization group equations for the $\overline{\text{MS}}$ scheme parameters are:

$$\mu\frac{dm}{d\mu} = -m\gamma(\varrho) = -m(\gamma_1\varrho + \gamma_2\varrho^2 + \ldots), \tag{4a}$$

$$\mu^2\frac{\partial\varrho}{\partial\mu^2} = \beta(\varrho) = -\beta_1\varrho^2 - \beta_2\varrho^3 + \ldots, \tag{4b}$$

with $\beta_1 = 11 - \frac{2f}{3}$, $\beta_2 = 102 - \frac{38f}{3}$, $\gamma_1 = 8$, and [2] $\gamma_2 = 16\left(\frac{101}{12} - \frac{5f}{18}\right)$ and f = number of flavors ($f = 4$ for charmonium). Integrating Eq. (4), we obtain:

$$m^2 = \hat{m}^2[\beta_1 \varrho]^{\gamma_1/\beta_1}\left[1+\left(\frac{\gamma_2}{\beta_1}-\frac{\beta_2}{\beta_1}\frac{\gamma_1}{\beta_1}\right)\varrho+O(\varrho^2)\right], \tag{5}$$

where $\hat{m}$ is the "scale invariant" mass [3]. We stress that $\hat{m}$ *so defined* (as well as γ_1) is *universal* [4], i.e. the same for all mass independent RS, as follows immediately from the relation $m_1^2 = m_2^2\,[1+O(\varrho)]$ between two masses definitions. We suggest to use $\hat{m}$ as a fundamental quark mass parameter in the sum rules. We then obtain:

$$M_n = \frac{a_n}{(\hat{m}^2)^n}\,[\beta_1 \bar{\varrho}_n(m)]^{-n\gamma_1/\beta_1} \tag{6}$$

with the "effective charge" [4] $\bar{\varrho}_n$, defined to all orders as a power series in ϱ, known up to next to leading order:

$$\bar{\varrho}_n(m) \equiv \varrho\left[1+\varrho\left(-\beta_1 \ln\frac{m^2}{\mu^2}+K_n\right)+O(\varrho^2)\right] \tag{7}$$

with $K_n = \frac{\beta_1}{\gamma_1}\left(\frac{32}{3}-\frac{b_n}{n}\right)+\frac{\gamma_2}{\gamma_1}-\frac{\beta_2}{\beta_1}$.

The formalism for the SVZ moments set up in Eq. (6) and (7) is quite similar to the standard formalism for the structure function moments in deep inelastic scattering (DIS), the usual non-perturbative normalization factor of the DIS moments being here replaced by the factor $\frac{a_n}{(\hat{m}^2)^n}$, and the $\bar{\varrho}_n$ (analogue of the perturbatively calculable part of the DIS moments) being known up to next to leading order in ϱ, thus allowing to relate $\bar{\varrho}_n$ to $\Lambda_{\overline{MS}}$ (note that in Eq. (2), ϱ appears only in leading order). The latter step requires some further explanation, since $\bar{\varrho}_n$ apparently depends upon a running μ dependent parameter m, instead of a μ-independent, non-renormalized variable like the momentum transfer in DIS. To clarify this issue, let us write Eq. (6) as:

$$M_n = \frac{a_n}{(\bar{m}_n^2)^n} \tag{8a}$$

with

$$\bar{m}_n^2 \equiv \hat{m}^2[\beta_1 \bar{\varrho}_n]^{\gamma_1/\beta_1}, \tag{8b}$$

where we introduce an n-dependent "effective fixed mass" $\bar{m}_n$, all higher order perturbative corrections to M_n being absorbed into the definition of $\bar{m}_n$ (this is the best one can achieve, if one naively attempts to minimize, in the SVZ spirit, the $O(\varrho)$ corrections in each moment). The question now is how to relate together the various fixed masses $\bar{m}_n$, since there is only one free heavy mass parameter in the theory. In general, to any fixed (i.e., non--running) mass $\bar{m}$ one can associate an effective charge [4] $\bar{\varrho}$ which relates it to $\hat{m}$ just as in

Eq. (8b):

$$\overline{m}^2 = \hat{m}^2[\beta_1\bar{\varrho}(m)]^{\gamma_1/\beta_1} \tag{9}$$

with

$$\bar{\varrho}(m) = \varrho\left[1+\varrho\left(-\beta_1 \ln \frac{m^2}{\mu^2} + K\right) + O(\varrho^2)\right].$$

Now, $\bar{\varrho}$ satisfies a renormalization group (RG) equation:

$$m^2 \left.\frac{\partial\bar{\varrho}}{\partial m^2}\right|_{\varrho,\mu} = \hat{\beta}(\bar{\varrho}) = -\hat{\beta}_1\bar{\varrho}^2 - \hat{\beta}_2\bar{\varrho}^3 + O(\bar{\varrho}^4). \tag{10a}$$

The right hand side of Eq. (10a) is RS invariant and μ independent, since in any mass independent RS we have:

$$m^2 \left.\frac{\partial\bar{\varrho}}{\partial m^2}\right|_{\varrho,\mu} = \hat{m}^2 \left.\frac{\partial\bar{\varrho}}{\partial \hat{m}^2}\right|_{\varrho,\mu}, \tag{10b}$$

m^2 being proportional to $\hat{m}^2$ at fixed ϱ (see Eq. (5)). Eq. (10) allows us to think of $\bar{\varrho}$ as "running with $\hat{m}$", giving its physical meaning to the scale variable upon which $\bar{\varrho}$ depends[1]. In other words, $\bar{\varrho}$ satisfies the modified RG equation pointed out for zero momentum observables in Ref. [4]. The first two $\hat{\beta}$ function coefficients are universal [4] and given by:

$$\hat{\beta}_1 = \beta_1, \quad \hat{\beta}_2 = \beta_2 - \beta_1\gamma_1.$$

Integration of Eq. (10) yields:

$$\hat{m}^2 = \Lambda^2_{\overline{\rm MS}} \exp\left[\frac{1}{\beta_1\bar{\varrho}} + \frac{\hat{\beta}_2}{\beta_1^2}\ln(\beta_1\bar{\varrho}) + \frac{K}{\beta_1}\right][1+O(\bar{\varrho})] \tag{11}$$

which relates $\bar{\varrho}$ to $\hat{m}$ and $\Lambda_{\overline{\rm MS}}$. For latter use, we note that from Eq. (9) and (11) one can equivalently write:

$$\overline{m}^2 = \Lambda^2_{\overline{\rm MS}} \exp\left[\frac{1}{\beta_1\bar{\varrho}} + \frac{\beta_2}{\beta_1^2}\ln(\beta_1\bar{\varrho}) + \frac{K}{\beta_1}\right][1+O(\bar{\varrho})], \tag{12}$$

where the usual 2 loop β function appears. Furthermore, we can now give the RG improved relation between two fixed masses definitions $\overline{m}_i$ and $\overline{m}_j$. Since $\overline{m}^2_{i,j} = \hat{m}^2[\beta_1\bar{\varrho}_{i,j}]^{\gamma_1/\beta_1}$, we have:

$$\frac{\overline{m}_i^2}{\overline{m}_j^2} = \left[\frac{\bar{\varrho}_i}{\bar{\varrho}_j}\right]^{\gamma_1/\beta_1} \tag{13a}$$

[1] We note the analogy between Eq. (9) and Eq. (5) for the running mass, i.e. "$\overline{m}$ runs with $\hat{m}$". At the differential level, the analogue of Eq. (4a) is: $\hat{m}\dfrac{d\overline{m}}{d\hat{m}} = m\left.\dfrac{\partial\overline{m}}{\partial m}\right|_{\mu,\varrho} \equiv (1+\delta)\overline{m}$, with $\delta = \dfrac{\gamma_1}{\beta_1}\dfrac{\hat{\beta}(\bar{\varrho})}{\bar{\varrho}}$. The latter equation generalizes a well known QED result [5] for the derivative of the physical electron mass with respect to the bare mass at fixed cut-off.

and, taking the ratio of Eq. (11) applied to $\bar{\varrho}_i$ and $\bar{\varrho}_j$ to eliminate $\hat{m}/\Lambda_{\overline{\rm MS}}$, we get up to 2 loop:

$$\frac{1}{\bar{\varrho}_i} - \frac{1}{\bar{\varrho}_j} + \frac{\hat{\beta}_2}{\beta_1} \ln\left(\frac{\bar{\varrho}_i}{\bar{\varrho}_j}\right) = -(K_i - K_j). \tag{13b}$$

Applying this formalism to the effective masses of Eq. (8), and relating for convenience $\overline{m}_n$ to $\overline{m}_1$, we obtain the one loop RG improved formula (we neglect for simplicity the 2-loop contribution proportional to $\hat{\beta}_2$ in Eq. (13b)):

$$M_n = \frac{a_n}{(\overline{m}_n^2)^n} = \frac{a_n}{(\overline{m}_1^2)^n}(1 - d_n\bar{\varrho}_1)^{n\gamma_1/\beta_1} \tag{14a}$$

with

$$\overline{m}_1^2 = \Lambda_{\overline{\rm MS}}^2 \exp\left[\frac{1}{\beta_1\bar{\varrho}_1} + \frac{\beta_2}{\beta_1^2}\ln(\beta_1\bar{\varrho}_1) + \frac{K_1}{\beta_1}\right] \tag{14b}$$

and

$$d_n = K_n - K_1 = \frac{\beta_1}{\gamma_1}\left(b_1 - \frac{b_n}{n}\right). \tag{15}$$

We stress that Eqs (14) represent a mass RS invariant formulation, and it is only for convenience that we use $\overline{m}_1$ as mass parameter instead of $\hat{m}^2$.

Inclusion of gluon condensate: with the gluon condensate $G \equiv \frac{-4\beta(\varrho)}{\beta_1\varrho}\langle 0|F^2|0\rangle$, the theoretical expression for the moments becomes

$$M_n = \frac{a_n}{(\overline{m}_n^2)^n} + a_n c_n \frac{G}{(m^2)^{n+2}}[1 + O(\varrho)] \equiv \frac{a_n}{(\overline{m}_n^2)^n} + a_n c_n \frac{G}{(\overline{\overline{m}}_n^2)^{n+2}}, \tag{16}$$

where it is natural to introduce a *separate* fixed mass definition $\overline{\overline{m}}_n$ for the condensate contribution. In practice, since the $O(\varrho)$ corrections for the gluon condensate contribution are not yet known, we will use the *same* fixed mass definition as for the purely perturbative contribution, and assume the resulting corrections are small, i.e. we write:

$$M_n = \frac{a_n}{(\overline{m}_n^2)^n}\left[1 + c_n \frac{G}{\overline{m}_n^4}\right]. \tag{17}$$

Hence, relating $\overline{m}_u$ to $\overline{m}_1$, we use:

$$M_n = M_n^{(0)}\left[1 + c_n \frac{G}{\overline{m}_1^4}(1 - d_n\bar{\varrho}_1)^{2\gamma_1/\beta_1}\right] \tag{18}$$

with $M_n^{(0)}$, the purely perturbative contribution, being given by Eq. (14).

[2] In principle, one could have equally used any other fixed mass instead of $\overline{m}_1$, such as $\overline{m}$ = pole mass. In practice, we cannot use the pole mass for the charm quark because its associated effective charge $\bar{\varrho}$ is too large for RG improved perturbation theory to apply. This fact simply reflects SVZ finding that using the pole mass produces too large perturbative corrections for the moments.

Expanded ratios approach: a similar formalism can be applied if one prefers to deal with the perturbative expansion of the ratios:

$$r_n \equiv \frac{M_n}{M_{n-1}} = \frac{a_n}{a_{n-1}} \frac{1}{m^2} \left\{1+\left[b_n-b_{n-1}-\left(-8\ln\frac{m^2}{\mu^2}+\tfrac{32}{3}\right)\right]\varrho+O(\varrho^2)\right\} \quad (n \geqslant 2). \tag{19}$$

Proceeding as above, we get:

$$r_n = \frac{a_n}{a_{n-1}} \frac{1}{\bar{m}_n^2} = \frac{a_n}{a_{n-1}} \frac{1}{\hat{m}^2} [\beta_1 \bar{\varrho}_n(m)]^{-\gamma_1/\beta_1} \tag{20}$$

with

$$\bar{\varrho}_n(m) = \varrho\left[1+\varrho\left(-\beta_1 \ln\frac{m^2}{\mu^2}+K_n\right)+O(\varrho^2)\right]$$

and

$$K_n = \frac{\beta_1}{\gamma_1}[\tfrac{32}{3}-(b_n-b_{n-1})]+\frac{\gamma_2}{\gamma_1}-\frac{\beta_2}{\beta_1}. \tag{21}$$

Note that the effective masses $\bar{m}_n$ and charges $\bar{\varrho}_n$ introduced here are not the same as in the case of moments, but we keep the same notation when no confusion can arise. Relating $\bar{m}_n$ to $\bar{m}_2$ (the effective mass defined by the $n = 2$ ratio), the one loop RG improved formula for the ratios now reads:

$$r_n = \frac{a_n}{a_{n-1}} \frac{1}{\bar{m}_2^2} (1-d_n\bar{\varrho}_2)^{\gamma_1/\beta_1} \tag{22a}$$

with

$$\bar{m}_2^2 = \Lambda_{\overline{\rm MS}}^2 \exp\left[\frac{1}{\beta_1\bar{\varrho}_2}+\frac{\beta_2}{\beta_1^2}\ln(\beta_1\bar{\varrho}_2)+\frac{K_2}{\beta_1}\right] \tag{22b}$$

and

$$d_n = K_n-K_2 = \frac{\beta_1}{\gamma_1}[(b_2-b_1)-(b_n-b_{n-1})]. \tag{23}$$

Including the gluon condensate, we similarly obtain:

$$r_n = \frac{a_n}{a_{n-1}} \frac{1}{\bar{m}_n^2}\left[1+(c_n-c_{n-1})\frac{G}{\bar{m}_n^4}\right]. \tag{24}$$

Hence we use:

$$r_n = r_n^{(0)}\left[1+(c_n-c_{n-1})\frac{G}{\bar{m}_2^4}(1-d_n\bar{\varrho}_2)^{2\gamma_1/\beta_1}\right] \tag{25}$$

with $r_n^{(0)}$ given by Eq. (22).

TABLE I

Experimental and theoretical expanded ratios for the $^3S_1$ charmonium channel. The fit parameters are $\Lambda_{\overline{MS}} = 200$ MeV, $\bar{\varrho}_2 = 0.02898$, $\bar{m}_2^2 = 1.62415$ GeV^2 and $G = 0.004963$ GeV^4

n	2	3	4	5	6	7	8	9	10	11	12
$10\times r_n^{(0)}$	0.6597	0.8561	0.9484	1.001	1.034	1.056	1.072	1.083	1.092	1.098	1.103
$10\times r_n$	0.6563	0.8485	0.9357	0.9819	1.008	1.022	1.028	1.029	1.027	1.021	1.013
$10\times r_n^{\exp}$	0.6687	0.8601	0.9422	0.9836	1.006	1.020	1.028	1.033	1.036	1.038	1.0395

TABLE II

Theoretical moments, experimental moments, and unexpanded theoretical ratios for the $^3S_1$ charmonium channel. The fit parameters are $\Lambda_{\overline{MS}} = 200$ MeV, $\bar{\varrho}_1 = 0.02266$, $\bar{m}_1^2 = 1.2847$ GeV^2, $G = 0.003338$ GeV^4

n	1	2	3	4	5	6	7	8	9	10	11	12
$10^{n+2}\times M_n^{(0)}$	3.943	2.612	2.237	2.118	2.113	2.177	2.288	2.434	2.620	2.841	3.098	3.388
$10^{n+2}\times M_n$	3.936	2.597	2.208	2.069	2.033	2.052	2.098	2.157	2.222	2.281	2.323	2.332
$10^{n+2}\times M_n^{\exp}$	3.831	2.562	2.203	2.076	2.042	2.055	2.095	2.153	2.224	2.304	2.391	2.486
$10\times r_n^{(0)}$		0.6623	0.8565	0.9470	0.9977	1.030	1.051	1.064	1.076	1.084	1.091	1.093
$10\times r_n$		0.6597	0.8505	0.9369	0.9826	1.009	1.023	1.028	1.030	1.027	1.019	1.004

Applications: we give here only a summary of the main results, leaving a more detailed account to a future publication. We first performed fits of the $^3S_1$ charmonium channel, both with expanded and unexpanded ratios, trying to determine G. The fits were done by first assuming a given value of $\Lambda_{\overline{MS}}$ between 0 and 300 MeV, then adjusting $\bar{\varrho}_1$ (unexpanded ratios) or $\bar{\varrho}_2$ (expanded ratios) to fit r_3, with the gluon condensate contribution neglected: the knowledge of $\Lambda_{\overline{MS}}$ and $\bar{\varrho}_1(\bar{\varrho}_2)$ determines $\bar{m}_1(\bar{m}_2)$ by Eq. (14b) ((22b)). Finally, we adjust G to reproduce the higher moments (in practice, we used $n = 8$). We took over values of a_n, b_n and c_n from Ref. [6]. The quoted values of $\Lambda_{\overline{MS}}$ refer to 3 flavors and can be simply obtained by using 3 flavors for β_1 and β_2 in Eq. (14b) and (22b) (K_1 and K_2, as well as all other formulas, have to be computed with $f = 4$, however). Fits acceptable within experimental errors[3] were obtained up to $n = 10$ for $0 < \Lambda_{\overline{MS}} < 300$ MeV, G decreasing with $\Lambda_{\overline{MS}}$, $\Lambda_{\overline{MS}} = 0$ (i.e., neglect of all perturbative corrections) being compatible with $G_{SVZ} \simeq 0.012$ GeV^4, and $\Lambda_{\overline{MS}} = 300$ MeV being compatible with $G = 0$. Fitting other channels however restrict $\Lambda_{\overline{MS}}$ to the range 150 MeV $<$ $<$ 250 MeV and require $G \neq 0$ (see below). As an example we give the fit for $\Lambda_{\overline{MS}} = 200$ MeV using expanded ratios (Table I) and unexpanded ratios (Table II). One should note the following features:

(*i*) For 150 MeV $< \Lambda_{\overline{MS}} <$ 250 MeV, G turn out to be in average a factor of 3 smaller

[3] We used data from Ref. [7].

than G_{SVZ}. For unexpanded ratios, this is partly due to the use of a larger value of the effective coupling $\bar{\alpha}_s = 4\pi\bar{\varrho}$ ($\Lambda_{\overline{MS}} = 200$ MeV gives $\bar{\alpha}_s \simeq 0.285$), and partly to the use of formulas different from SVZ. For expanded ratios, the main effect is due to the larger value of $\bar{\alpha}_s$ ($\bar{\alpha}_s \simeq 0.364$ for $\Lambda_{\overline{MS}} = 200$ MeV), since the formula Eq. (22a) turns out to be quite close numerically to the form used by SVZ. Indeed, to first order in $\bar{\varrho}_2$, we get:

$$[1-d_n\bar{\varrho}_2]^{\gamma_1/\beta_1} \simeq 1-[(b_2-b_1)-(b_n-b_{n-1})]\bar{\varrho}_2 \quad \text{with} \quad b_2-b_1 = 10.73,$$

whereas the corresponding correction with the SVZ euclidean mass definition is:

$$1-\left[\frac{4\ln 2}{\pi} \times 4\pi-(b_n-b_{n-1})\right]\varrho \quad \text{with} \quad \frac{4\ln 2}{\pi} \times 4\pi = 11.09.$$

(*ii*) The purely perturbative contribution $r_n^{(0)}$ for unexpanded ratios increases monotonically even up to $n = 12$, contrary to what happens in the usual formulation, where $r_n^{(0)}$ displays an unphysical maximum [7] limiting the validity of perturbation theory to $n \lesssim 8$. There is therefore no sign of perturbation theory breakdown which could forbid the use of unexpanded ratios in the investigated range of n ($n \lesssim 10$).

(*iii*) It is non trivial that *moments* themselves are satisfactorily reproduced, whereas no moment has been used as input in the fit with unexpanded ratios.

(*iv*) The fits with expanded and non expanded ratios are consistent with each other, in the sense that the fit parameters $(\overline{m}_2, \bar{\varrho}_2)$ and $(\overline{m}_1, \bar{\varrho}_1)$ satisfy the relations expected theoretically between them. Indeed, from Eq. (13a) one expects:

$$\frac{\overline{m}_1^2}{\overline{m}_2^2} = \left(\frac{\bar{\varrho}_1}{\bar{\varrho}_2}\right)^{\gamma_1/\beta_1}$$

and from Eq. (13b) (neglecting the 2-loop contribution):

$$\frac{1}{\bar{\varrho}_1} - \frac{1}{\bar{\varrho}_2} = K_2-K_1 = 9.984$$

whereas the fitted parameters are found to satisfy the relations:

$$\left(\frac{\overline{m}_1^2}{\overline{m}_2^2}\right)_{\text{fit}} = 0.7910, \qquad \left(\frac{\bar{\varrho}_1}{\bar{\varrho}_2}\right)_{\text{fit}}^{\gamma_1/\beta_1} = 0.78965$$

and

$$\left(\frac{1}{\bar{\varrho}_1} - \frac{1}{\bar{\varrho}_2}\right)_{\text{fit}} = 9.624.$$

We find this agreement non-trivial. Furthermore, both fits yield similar values for G, differing only by 50%.

Predictions for other channels: with the parameters $\overline{m}_{1,2}$ and $\bar{\varrho}_{1,2}$ for unexpanded and expanded ratios determined in the ${}^3S_1$ channel, we can deduce the corresponding parameters $(\bar{m}_{1,2})_J$ and $(\bar{\varrho}_{1,2})_J$ in the channel J by the one-loop formulas (see Eq. (13)):

$$(\bar{\varrho}_{1,2})_J = \frac{\bar{\varrho}_{1,2}}{1-[(K_{1,2})_J - K_{1,2}]\bar{\varrho}_{1,2}},$$

$$(\bar{m}^2_{1,2})_J = (\bar{m}_{1,2})^2 \left[\frac{(\bar{\varrho}_{1,2})_J}{\bar{\varrho}_{1,2}}\right]^{\gamma_1/\beta_1}, \tag{26}$$

where symbols without the subscript J refer to the $^3S_1$ channel, and G is taken the same in all channels. The mass m_J of the lowest lying resonance in channel J is estimated as usual from the J channel ratios r_n^J by $m_J \lesssim (r^J_{n,\max})^{-1/2}$. Values of $(a_n)_J$, $(b_n)_J$ and $(c_n)_J$ are taken from Ref. [6], except for the pseudoscalar channel, where we used values from Ref. [8] for reasons explained below. We proceed as in Ref. [9], not including any continuum contribution, but allowing $(r^J_{n,\max})^{-1/2}$ (which is usually reached for $n \simeq 8$–9), to exceed the experimental value of m_J by at most 30 MeV (as suggested by data in the $^3S_1$ channel), to account for the contribution of higher excited states and the continuum. The results for $\Lambda_{\overline{MS}} = 200$ MeV are given in Table III. We note that all states are satisfactorily reproduced *except the scalar and pseudoscalar ones*, where the predictions exceed by about 100 MeV the experimental numbers. This fact may indicate the presence of new non perturbative contributions affecting exclusively the $0^{\pm}$ channels, such as direct instantons [10]. We have not given unexpanded ratios predictions for all currents having non zero anomalous dimensions [11], except the pseudoscalar current. The reason is that their correlation functions (hence the moments) are not RS invariant objects, since they depend on the operator RS convention used for the current. The way out consists in factorizing out the unphysical RS dependent part of these currents in $\overline{\text{MS}}$ scheme using the "RG invariant operators and Green's functions" introduced in Ref. [4], which cannot be done on the basis of the results of Ref. [6] alone, since the currents there are not renormalized by minimum subtraction. Note that the RS dependence is contained in an overall factor which cancell in ratios r_n^J, which are indeed RS invariant. This problem can be circumvented for pseudoscalar and scalar currents, since their $\overline{\text{MS}}$ anomalous dimension is closely related to the running mass anomalous dimension in $\overline{\text{MS}}$ scheme. Indeed, the relation $\partial_\mu J^5_\mu = 2mi\bar{c}\gamma_5 c$, where J^5_μ is the charm axial vector current, shows that $m\bar{c}\gamma_5 c$ is a RS

TABLE III

Experimental and theoretical masses (in GeV) of the low lying charmonium states, for $\Lambda_{\overline{MS}} = 200$ MeV

State	Experiment	Expanded ratios	Unexpanded ratios
$^3P_1$	3.51	3.535	3.545
$^3P_2$	3.56	3.561	
$^1P_1$		3.533	
$^1S_0$	2.98	3.083	3.078
$^3P_0$	3.41	3.500	

TABLE IV

Experimental and theoretical perturbative ratios and moments for the $^3S_1$ bottonium channel, for $\Lambda_{\overline{MS}}^{f=4} = 164.4$ MeV. The fits parameters are $\bar{\varrho}_2 = 0.01756$, $\overline{m}_2^2 = 17.75$ GeV^2 (expanded ratios), and $\bar{\varrho}_1 = 0.01521$, $\overline{m}_1^2 = 15.235$ GeV^2 (unexpanded ratios)

n	1	2	3	4	5	6	7	8	9	10	11
$10^2 \times r_n$ (expanded ratios)		0.6037	0.8037	0.9045	0.9652	1.005	1.034	1.055	1.071	1.084	1.094
$10^2 \times r_n^{\exp}$		0.6089	0.8116	0.9111	0.96895	1.006	1.031	1.049	1.063	1.073	1.080
$10^{2n+1} \times M_n$	0.3325	0.2010	0.1616	0.1460	0.1407	0.1412	0.1456	0.1532	0.1637	0.1768	0.1928
$10^{2n+1} \times M_n^{\exp}$	0.3229	0.1966	0.1596	0.1454	0.1409	0.1417	0.1461	0.1533	0.1629	0.1747	0.1888
$10^2 \times r_n$ (unexpanded ratios)		0.6046	0.8036	0.9037	0.9636	1.003	1.031	1.052	1.069	1.080	1.091

invariant quantity with no anomalous dimension. For the pseudoscalar state, we worked with moments of $\partial_\mu J_\mu^5$, following Ref. [8]: the corresponding values of b_n differ by an n-independent constant from those of Ref. [6]. We also note that anomalous dimensions give a different asymptotic continuum behavior to the various currents: as $Q^2 \to \infty$, the renormalization group tells us that $\mathrm{Im}\, \pi_J(Q^2) \sim [\alpha_s(Q^2)]^{-\gamma_J/\beta_1}$ where γ_J is the J current one-loop anomalous dimension. Hence, as $Q^2 \to \infty$, $\mathrm{Im}\, \pi_J(Q^2) \to 0$ for $^1S_0$ and $^3P_0$ currents (which have $\gamma_J < 0$), while $\mathrm{Im}\, \pi_J(Q^2) \to \infty$ for $^3P_2$ and $^1P_1$ currents (which have $\gamma_J > 0$), and $\mathrm{Im}\, \pi_J(Q^2) \to$ const for $^3S_1$ and $^3P_1$ currents (which have $\gamma_J = 0$). These facts should be taken into account in a careful treatment of the continuum contribution to the moments.

Application to bottonium: we expect the gluon condensate contribution to be completely negligible for bottonium. It is therefore important to check whether a reasonable fit can be obtained using only the perturbative contribution. With the 3 flavors $\Lambda_{\overline{MS}}$ assumed to be 200 MeV, one easily determined the corresponding value for 4 flavors to be: $\Lambda_{\overline{MS}}^{f=4} = 164.4$ MeV, just using 4 flavors for β_1 and β_2 in Eq. (22b) (a similar value is obtained from Eq. (14b)). With this value of $\Lambda_{\overline{MS}}$ we fitted data as quoted in Ref. [12] using Eqs (14) and (22), but adjusted the continuum threshold $E_T = 10.80$ GeV to get agreement with the sum rules (a similar idea has been used in Ref. [13]), since the moments are rather sensitive [14] to the value of E_T, which is poorly known experimentally. In effect, we therefore performed a 2 parameters fit ($\bar{\varrho}_{1,2}$ and E_T), but we did not try to get the best fit. The results are given in Table IV, and show a satisfactory fit can be obtained *both with expanded and non expanded ratios* (the main reason why our result for expanded ratios differs from that of Ref. [13] is that we use a larger value for the effective $\bar{\alpha}_s$, $\bar{\alpha}_s \simeq 0.2207$, which follows naturally from the data and the value of $\Lambda_{\overline{MS}}^{f=4}$). We can check again the consistency between expanded and non-expanded ratios, since we get $\left(\frac{\overline{m}_1^2}{\overline{m}_2^2}\right)_{\text{fit}} = 0.8583$, to be compared to $\left(\frac{\bar{\varrho}_1}{\bar{\varrho}_2}\right)_{\text{fit}}^{\gamma_1/\beta_1} = 0.8608$, and $\left(\frac{1}{\bar{\varrho}_1} - \frac{1}{\bar{\varrho}_2}\right)_{\text{fit}} = 8.7986$, to be compared to $K_2 - K_1 = 9.186$ (for $f = 5$). Since $\hat{\beta}_2 < 0$ for $f = 4, 5$, inclusion of 2-loop term of Eq. (13b) makes agreement even better both for bottonium and charmonium, since one expects $\left(\frac{1}{\bar{\varrho}_1} - \frac{1}{\bar{\varrho}_2}\right) < K_2 - K_1$ for $\hat{\beta}_2 < 0$ and $\bar{\varrho}_1 < \bar{\varrho}_2$.

As a last remark, we note that the present formalism can be easily extended to the Q^2 dependent sum rules of Ref. [11]: $M_n(\xi) = \int \frac{ds}{(s+4\bar{m}^2\xi)^n} \operatorname{Im} \pi(s)$, $\xi \equiv \frac{Q^2}{4\bar{m}^2}$, or to the exponential moments of Ref. [15]: $F(\omega) = \int e^{-\omega s/\bar{m}^2} \operatorname{Im} \pi(s) ds$, $\omega \equiv \sigma \bar{m}^2$, where $\bar{m}$ = any fixed mass (for instance : pole mass). Note however the definition of the variables ξ and ω, hence of the moments $M_n(\xi)$ and $F(\omega)$, depend on the definition of $\bar{m}$. As stressed in Ref. [11], it is interesting to test the ξ-stability of the $M_n(\xi)$ sum rules predictions away from $\xi = 0$, which could also help solving [16] the problems caused by the large power corrections found in Ref. [17].

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