FERMIONS ON A LATTICE*

By E. H. de Groot

Fakultät für Physik, Universität Bielefeld**

(Received November 6, 1984)

Some of the problems which arise in putting fermions on a lattice are discussed.

PACS numbers: 11.15.Ha

In trying to put fermions on a lattice we encounter, as most of you will know, some serious problems which seem to be of a fundamental kind: while it seems to be the natural thing to assign scalars to the sites of the lattice, vectors to the links and tensors to the plaquettes there seems to be no place for the fermions. For want of a better proposal one assigns fermions to the sites all the same. It is then perhaps no surprise that most methods for dealing with fermions have to introduce some kind of "fuzziness" or smearing out, like e.g. the Susskind method where the fermions are smeared out over a hypercube or in the SLAC method where the first-order derivative is of long range leading to non--locality in the interaction. To be a bit more explicit: intimately connected to the notion of (massless) fermions is the appearance of a first order derivative in the Lagrangian and a chiral U(1) invariance:

$$\mathscr{L} = i\bar{\psi}(x)\gamma^{\mu}\partial^{\mu}\psi(x), \tag{1}$$

which is invariant for the global transformation

$$\bar{\psi}(x), \psi(x) \rightarrow \bar{\psi}(x)e^{i\gamma_5\theta}, e^{i\gamma_5\theta}\psi(x).$$

Naive reasoning would then imply that the associated axial current would be conserved. Actually this is not the case in the continuum theory. It suffers from the well known Adler--Bell-Jackiw anomaly [1]. On the other hand, although not conserved in the continuum theory, the axial current will always be conserved on the lattice. So, how is the lattice regularization going to produce the continuum anomaly? There are three ways of doing this:

---

* Presented at the XXIV Cracow School of Theoretical Physics, Zakopane, June 6–19, 1984.

** Address: Fakultät für Physik, Universität Bielefeld, Postfach 8640, 4800 Bielefeld 1, F.R. Germany.

1. By only restoring the $U(1)_{chiral}$ symmetry in the continuum limit; so the lattice formulation does not have the symmetry, like in the Wilson formulation [2, 3] or for the so-called "staggered" fermions of Susskind and Kogut [4, 5].
2. By doubling the number of fermion species so that the anomaly is cancelled, as in the naive fermion formulation [3].
3. The conserved axial current on the lattice diverges in the continuum limit (if one uses a non-local derivative, keeping $U(1)_{chiral}$ invariance) [6].

That one cannot have everything is actually shown by the Nielsen-Ninomiya theorem [7] which states that you cannot avoid species doubling unless you give up on either $U(1)_{chiral}$, locality, hermiticity or translational invariance (over any finite set of lattice sites).

Let us demonstrate the theorem with a fairly general example by writing down the massless Dirac action on an Euclidean d-dimensional (d = even) lattice, where we have assigned independent Grassmann variables $\psi^\dagger$, ψ with each $2^{d/2}$ components to each site.

$$S = i \sum_{x,x',\mu} \psi^\dagger(x)\gamma^\mu\partial^\mu(x, x')\psi(x'), \tag{2}$$

where the γ^μ are hermitian and unitary (Euclidean metric!) obeing

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} \qquad \gamma_5 \equiv -(i)^{d/2}\gamma_1\gamma_2 \cdots \gamma_d = \gamma_5^\dagger. \tag{3}$$

So we clearly have

$$\{\gamma^\mu, \gamma^5\} = 0 \tag{4}$$

and thus S is invariant under the "chiral" U(1) transformation

$$\psi^\dagger(x), \psi(x) \to \psi^\dagger(x)e^{i\gamma_5\theta}, e^{i\gamma_5\theta}\psi(x). \tag{5}$$

This chiral U(1) symmetry appears because we have ψ's which have more than one component. In the continuum theory it is necessary to have $2^{d/2}$ components in order to have Lorentz-invariance: a four dimensional rotation mixes the components of ψ and so does the parity operation:

$$x_\nu \to -x_\nu \qquad P = i\gamma^\nu\gamma_5 \tag{6}$$

which clearly anticommutes with γ_5

$$\{P, \gamma_5\}_+ = 0. \tag{7}$$

The point now is that on a lattice we do not need these $2^{d/2}$ component ψ's: we can define parity and rotations $\left(\text{over } \frac{\pi}{2} \text{ of course}\right)$ in such a way that we only need one-component objects, at least if we define the derivative on a lattice in the most naive way:

$$\frac{d}{dx}f(x) \to \frac{f(x+a)-f(x-a)}{2a} \tag{8}$$

or

$$\frac{d}{dx} \to \sum_{x'} \frac{1}{2a} (\delta_{x',x+a} - \delta_{x,x'+a}) \tag{9}$$

which in p space reads

$$p \to \frac{\sin ap}{a}, \quad p \in \left(-\frac{\pi}{a}, +\frac{\pi}{a} \right]. \tag{10}$$

Notice that we have suddenly 2 zeros on the lattice $p = 0, \frac{\pi}{a}$. So $\partial^\mu(x, x') = \partial^\mu(x-x')$ only connects sites where x_μ and x'_μ have an odd difference (in fact ± 1) and all other $x_\nu = x'_\nu$.

$$\delta^\mu(x-x') = \frac{1}{2a} (\delta_{x'_\mu, x_\mu+a} - \delta_{x_\mu, x'_\mu+a}) \prod_{\nu \neq \mu} \delta_{x_\nu, x'_\nu}. \tag{11}$$

Now define new fields $\chi^\dagger$, χ instead of $\psi^\dagger$, ψ (we take units such that $a = 1$)

$$\chi(x) \equiv \gamma_1^{x_1} \gamma_2^{x_2} \ldots \gamma_d^{x_d} \psi(x), \qquad \chi^\dagger(x) = \psi^\dagger(x) \gamma_d^{x_d} \ldots \gamma_1^{x_1}. \tag{12}$$

Inserting this into S we observe that as $\partial^\mu(x-x')$ only connects x_μ and x'_μ with an odd difference we get an extra γ^μ into S, cancelling the γ^μ already present in S, leading to:

$$S = \sum_{x,x',\mu} \chi^\dagger(x) \eta^\mu(x) \delta^\mu(x-x') \chi(x'), \tag{13}$$

where the $\eta^\mu(x)$ are now only *phase factors*:

$$\eta^\mu(x) = (-)^{\sum_{\nu=1}^{\mu-1} x_\nu} \times I, \qquad \text{(we have taken } a \equiv 1) \tag{14}$$

where I is the unit matrix in "spinor" space! So if we write out the "spinor" product $\chi^\dagger \chi$ in S we simply get $2^{d/2}$ identical expressions. We have the same theory written down $2^{d/2}$ times, and so it is no surprise that we can define rotations and parity on one component only, e.g.

$$x_\nu \to -x_\nu \qquad P_{\text{new}} = (-)^{x_\nu}. \tag{15}$$

Replacing $X^{(\dagger)}(x_1, x_2, \ldots, x_\nu, \ldots, x_d)$ by $(-)^{x_\nu} X^{(\dagger)}(x_1, x_2, \ldots, -x_\nu, \ldots, x_d)$ leaves S invariant. We observe that as a consequence this new parity operator of course commutes with γ_5

$$[P_{\text{new}}, \gamma_5]_- = 0 \tag{16}$$

and thus γ_5 cannot be identified with chiral symmetry. In fact, the original reason why we had to have ψ's with $2^{d/2}$ components (i.e. Lorentz invariance) is non-existent on the lattice and we can just as well take the χ's to be one-component objects, so-called "staggered" fermions, which we will do from now on. Still we have to face a problem: where are the spinor degrees of freedom which should reappear in the continuum limit? The answer is simple: the ψ's which we were working with actually represented not one fermion but 2^d fermions all the time, corresponding to the fact that the Fourier transform of our derivative had two zeros $\left(p = 0, \frac{\pi}{a}\right)$ and not just one ($p = 0$) as in the continuum theory. The consequence is that the inverse free propagator on a lattice which now reads

$$\sum_{\mu} \gamma^{\mu} \frac{\sin ap^{\mu}}{a} \quad \text{with} \quad p^{\mu} \in \left(-\frac{\pi}{a}, +\frac{\pi}{a}\right] \tag{17}$$

has 2^d zeros $\left(\text{each } p^{\mu} \text{ can be } 0 \text{ or } \frac{\pi}{a}\right)$, corresponding to 2^d fermions with $2^{d/2}$ components making a total of $2^{3d/2}$ degrees of freedom. So taking the χ's to be one component object leaves us still with 2^d degrees of freedom which *in the continuum limit* gives us $2^{d/2}$ species of fermions with $2^{d/2}$ spinor degrees of freedom. So, not only have we lost chiral U(1) symmetry, but we have also obtained a $2^{d/2}$ doubling of fermion species. Worse: we can only separate the spinor degrees of freedom from the flavour degrees of freedom in the continuum limit, where we will have the symmetry

$$U(1)_{\text{phase}} \otimes U(1)_{\text{chiral}} \otimes SU(2^{d/2})_L \otimes SU(2^{d/2})_R. \tag{18}$$

On the lattice, on the other hand we only have $U(1)_{\text{phase}}$ and the U(1) symmetry:

$$\chi^{\dagger}(x), \chi(x) \to \chi^{\dagger}(x) e^{i(-)^{\sum_{\nu} x_{\nu}}\theta}, e^{i(-)^{\sum_{\nu} x_{\nu}}\theta} \chi(x) \tag{19}$$

for one component χ's.

This U(1) symmetry is *not* our original chiral U(1) symmetry. On the spinor degrees of freedom its generator acts indeed like γ_5 but in different ways for different flavours ($+\gamma_5$ for one species, $-\gamma_5$ for another) and this generator is therefore a member of the $SU(2^{d/2})_L \otimes SU(2^{d/2})_R$ and well of the subgroup $SU(2^{d/2})_{L-R}$, i.e. an axial generator of flavour non-singlet type.

This is of great interest as in continuum QCD we have the global symmetry

$$U(1)_{\text{chiral}} \otimes SU(2)_L \otimes SU(2)_R$$

if the u and d quark are massless. The π-meson is then supposed to be the result of the spontaneous breakdown of the $SU_L \otimes SU_R$ to SU_{L+R} giving us three Goldstone pions corresponding to the three broken generators of SU_{L-R}. On the lattice as we have seen we only have the symmetry corresponding to one of the generators of SU_{L-R}, but still we can try to find out if this symmetry breaks down spontaneously or not. If it does, it will of course only give us one Goldstone pion; the other two should only appear in the continuum limit. All the same it is a good test of the basic phenomenon of chiral symmetry breaking. In the strong coupling limit one can indeed prove this breakdown explicitly [8] while the computer calculations for finite coupling also indicate that the breakdown takes place.

So as far as the pion problem is concerned this is very useful, although we get only one pion and not three. It seems a shame that we loose so much of the continuum symmetry by defining our lattice theory in this way. All the trouble of course came from defining our derivative on the lattice as (take $a = 1$ and $d = 1$):

$$\frac{\psi(x+1)-\psi(x-1)}{2}. \tag{20}$$

It gave us the proliferation of fermions ($1 \to 2^d$) which automatically reduced themselves (the one component X's) to $2^{d/2}$ fermions, killing $U(1)_{\text{chiral}}$ symmetry.

So why not try another derivative, like e.g.

$$\frac{\psi(x+\alpha)-\psi(x-\alpha)}{2\alpha} \quad \text{for some } \alpha \neq 1, \quad x = \text{integer}. \tag{21}$$

The $\psi(x)$ are only defined for integer x so we have to define $\psi(x+\alpha)$ in terms of the $\psi(x)$ before we can give a meaning to this derivative. This can easily be done by Fourier transforming the $\psi(x)$ with x integer

$$\psi(x) = \frac{1}{\sqrt{2N+1}} \sum_{k=-N}^{+N} a(k) e^{2\pi ikx/(2N+1)}, \tag{22}$$

where we have taken a finite one-dimensional lattice with $2N+1$ sites. So instead of $2N+1$ variables $\psi(x)$ we now have $2N+1$ variables $a(k)$, defined by

$$a(k) = \frac{1}{\sqrt{2N+1}} \sum_{x=-N}^{+N} \psi(x) e^{-2\pi ikx/(2N+1)}. \tag{23}$$

These we can now use to define ψ for non-integer argument by

$$\psi(x+\alpha) = \frac{1}{\sqrt{2N+1}} \sum_{k=-N}^{+N} a(k) e^{2\pi ik(x+\alpha)/(2N+1)} = \text{insert (23)}$$

$$= \sum_{x'=-N}^{+N} \Delta_N(x+\alpha-x')\psi(x') \tag{24}$$

with

$$\Delta_N(\beta) \equiv \frac{1}{2N+1} \sum_{k=-N}^{+N} e^{2\pi ik\beta/(2N+1)} = \frac{\sin \pi\beta}{(2N+1)\sin(\pi\beta/(2N+1))}, \tag{25}$$

which for an infinite lattice ($N \to \infty$) becomes

$$\Delta_\infty(\beta) = \frac{\sin \pi\beta}{\pi\beta}. \tag{26}$$

Inserting this into our expression for $\psi(x+\alpha)$ we obtain for our derivative (on an infinite lattice):

$$\frac{\psi(x+\alpha)-\psi(x-\alpha)}{2\alpha} = \frac{\sin \pi\alpha}{\pi\alpha} \sum_{n=1}^{\infty} (-)^{n-1} \frac{n}{n^2-\alpha^2} (\psi(x+n)-\psi(x-n)). \tag{27}$$

Observe that for $\alpha = m \neq 0$ (m = integer) only the term $n = m$ contributes in the summation as

$$\operatorname*{Lim}_{\alpha \to m} (-)^{n-1} \frac{\sin \pi\alpha}{\pi(n-\alpha)} = \delta_{n,m}.$$

Taking $\alpha = 2, 3, 4, \dots$ would only aggravate our doubling problem, while $\alpha = 1$ was our old derivative; so $\alpha = 0$ or non-integer is the only alternative. But for these values of α the summation in (27) never breaks off and worse: for large n the terms go like $\dfrac{n}{n^2-\alpha^2} \sim \dfrac{1}{n}$, which means that the derivative becomes of infinite range. The SLAC derivative [6] e.g. is just the above one (27) with $\alpha = 0$. That it is indeed impossible to avoid a long range derivative if one wants to avoid the doubling problem you can judge from the following. Imagine you superimpose derivatives with different α in (27) with the aim of improving the large n behaviour in (27) i.e. multiply (27) with a weight-function $\varrho(\alpha) \geqslant 0$ and

$$\int_0^\infty \varrho(\alpha) d\alpha = 1. \tag{28}$$

In order to create a short-range derivative (i.e. terms in (27) going *faster* to zero than n^{-1} for large n) we clearly need:

$$\int_0^\infty \varrho(\alpha) \frac{\sin \pi\alpha}{\alpha} d\alpha = 0. \tag{29}$$

It is easy to find such a $\varrho(\alpha)$ but it does not help us as the inverse propagator associated with (27) was $\dfrac{\sin \alpha p}{\alpha}$ and thus, after our weighted average, will be

$$\int_0^\infty \varrho(\alpha) \frac{\sin \alpha p}{\alpha} d\alpha \tag{30}$$

but this again, by (29), has a superfluous 0 at $p = \pi$, which was the thing we wanted to avoid, as it gave us the doubling problem.

So unless we are willing to accept a derivative of infinite range we are sunk, which is just what the Nielsen-Ninomiya theorem was trying to tell us all the time[1]. Although there seems to be nothing wrong with such a derivative in principle [6], to do practical calculations in such a scheme (certainly on a computer) is a very cumbersome task; many computational advantages of the lattice regularization are lost in this way. We clearly have to wait for some brilliant idea about how to put fermions on a lattice.

I am grateful to the organizers of the Zakopane school. I am also indebted to B. Petersson for many stimulating discussions.

REFERENCES

[1] S. L. Adler, 1970 Brandeis University Summer Inst. in Theoretical Physics, ed. S. Deser, M. Grisaru and H. Pendleton, MIT Press, Cambridge, Mass.

[2] K. G. Wilson, *Phys. Rev.* **D10**, 2445 (1974).

[3] L. H. Karsten, J. Smit, *Nucl. Phys.* **B183**, 103 (1981).

[4] J. Kogut, L. Susskind, *Phys. Rev.* **D11**, 395 (1975); L. Susskind, *Phys. Rev.* **D16**, 3031 (1977).

[5] H. S. Sharatchandra, H. J. Thun, P. Weisz, *Nucl. Phys.* **B192**, 205 (1981).

[6] J. M. Rabin, *Phys. Rev.* **D24**, 3218 (1981).

[7] H. Nielsen, M. Ninomiya, *Nucl. Phys.* **B185**, 20 (1981); *Nucl. Phys.* **B193**, 173 (1981).

[8] J. M. Blairon, R. Brout, F. Englert, J. Greensite, *Nucl. Phys.* **B180** [FS2], 439 (1981); H. Kluberg-Stern, A. Morel, B. Petersson, *Nucl. Phys.* **B215** [FS7], 527 (1983).

[9] L. Jacobs, Undoubling chirally-symmetric lattice fermions, Santa Barbara preprint NSF-ITP-82-106 (1982); G. Parisi, Y. C. Zhang, *Phys. Lett.* **132B**, 130 (1983).

---

[1] For an interesting attempt to dodge the theorem by adding a random non-hermitian part to the derivative, see Ref. [9].