PHASE SPACE ANALOGY TO HOJMAN AND HARLESTON THEOREM

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A theorem known for one-dimensional q-equivalent Hamiltonians is extended to the multidimensional case. The construction of new integrals of motion is put forward illustrated by the quantity $\sum_{i,j} \frac{\partial \bar{p}_i}{\partial p_j}$.

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1. *Introduction*

It is a well-known fact that the Lagrangians: $L = \frac{1}{2} m\dot{q}^2$ and $\bar{L} = -mc\sqrt{1-\frac{\dot{q}^2}{c^2}}$ give straight line trajectories. An identical remark may refer to the Hamiltonians. Although the Lagrangians describe free particles motion, the physics of the particles is completely different. It is interesting to study the class of the Lagrangians and Hamiltonians respectively, which result in the same trajectories in the corresponding configuration spaces. We do not take into consideration the Lagrangians which differ from a total time derivative and canonical transformation on a phase space.

The relation between these Lagrangians and Hamiltonians respectively in the one-dimensional case was studied by Currie and Saletan [1]. They showed that, if we have two Lagrangians, L and $\bar{L}$, such that the solutions of their Euler-Lagrange equations are the same, the following equality can be written down:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = f(q, \dot{q}, t)\left(\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}} - \frac{\partial \bar{L}}{\partial q}\right), \tag{1}$$

where $f(q, \dot{q}, t)$ is a constant of motion. The Lagrangians subjected to this condition are called the s-equivalent Lagrangians.

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For two Hamiltonians, H and $\bar{H}$, such that

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial \bar{H}}{\partial \bar{p}}, \tag{2}$$

$$-\dot{p} = \frac{\partial H}{\partial q}, \tag{3}$$

$$-\dot{\bar{p}} = \frac{\partial \bar{H}}{\partial q}, \tag{4}$$

the dependence was obtained

$$\frac{\partial^2 H}{\partial p^2} = f(q, p, t)\frac{\partial^2 \bar{H}}{\partial \bar{p}^2}, \tag{5}$$

where $f = \partial \bar{p}/\partial p$ is a constant of motion. The transformation from (q, p) variables to $(q, \bar{p})$ variables is called a "fouling transformation" [2] and the corresponding Hamiltonians are called q-equivalent Hamiltonians.

For the s-equivalent Lagrangians in the multidimensional case, we proposed in our paper [3] that

$$\bar{L}_r = \Lambda_{rs}(q_i, \dot{q}_i, t)L_s, \quad (r, s, i = 1, 2 \ldots N), \tag{6}$$

where

$$L_s \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0, \tag{7}$$

$$\bar{L}_r \equiv \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}_r} - \frac{\partial \bar{L}}{\partial q_r} = 0. \tag{8}$$

Hojman and Harleston [4] showed that the trace of matrix Λ_{rs} and its any positive integer power are constants of motion.

The above theorem was also proved by Henneaux [5] basing on geometrical arguments. Gonzalez-Gascon [6] has shown it directly from Euler's equations; whereas Farias and Negri [7] by using the Helmholtz conditions.

Below, we study the q-equivalent Hamiltonians in the multidimensional case which provides us with the following theorem:

If $H(q_i, p_i)$ and $\bar{H}(q_i, \bar{p}_i)$ are q-equivalent i.e. if the trajectories generated by these Hamiltonians are, in the configuration space, the same, then the following equalities take place

$$\frac{\partial^2 H}{\partial p_j \partial p_i} = \sum_{k=1}^{N} \frac{\partial \bar{p}_k}{\partial p_j}\frac{\partial^2 \bar{H}}{\partial \bar{p}_k \partial \bar{p}_i}, \tag{9}$$

where the sum $\displaystyle\sum_{j,k=1}^{N} \frac{\partial \bar{p}_k}{\partial p_j} \equiv \mathscr{A}$ is a constant of motion.

2. *Demonstration of the theorem*

Let us consider the q-equivalent Hamiltonians $H(q_i, p_i)$ and $\bar{H}(q_i, \bar{p}_i)$ connected by the "fouling transformation" which result in the equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{\partial \bar{H}}{\partial \bar{p}_i}, \tag{10}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{11}$$

$$\dot{\bar{p}}_i = -\frac{\partial \bar{H}}{\partial q_i}. \tag{12}$$

Making use of $\bar{p}_i = \bar{p}_i(q_i, p_i)$ equations (12) can be rewritten in the form

$$\frac{\partial \bar{H}}{\partial q_i} = -\dot{\bar{p}}_i = \sum_{j=1}^{N} \left(\frac{\partial \bar{p}_i}{\partial p_j} \frac{\partial H}{\partial q_j} - \frac{\partial \bar{p}_i}{\partial q_j} \frac{\partial H}{\partial p_j} \right). \tag{13}$$

Let us take the derivative of (13) with respect to p_k and sum over k. Then, the right-hand side becomes equal to

$$-\sum_{k=1} \frac{\partial}{\partial p_k} \dot{\bar{p}}_i = \sum_{k,j=1}^{N} \left[\frac{\partial}{\partial p_k} \left(\frac{\partial \bar{p}_i}{\partial p_j} \right) \frac{\partial H}{\partial q_j} + \frac{\partial \bar{p}_i}{\partial p_j} \frac{\partial^2 H}{\partial p_k \partial q_j} - \frac{\partial}{\partial p_k} \left(\frac{\partial \bar{p}_i}{\partial q_j} \right) \frac{\partial H}{\partial p_j} - \frac{\partial \bar{p}_i}{\partial q_j} \frac{\partial^2 H}{\partial p_k \partial p_j} \right]. \tag{14}$$

In order to differentiate the left-hand side of (13), first we change the variables from (q_i, p_i) into $(q_i, \bar{p}_i)$ and next, after the differentiation, we come back to the old variables (q_i, p_i). The following identities are helpful in the calculation:

$$\frac{\partial^2 H}{\partial q_l \partial p_i} = \frac{\partial^2 \bar{H}}{\partial q_l \partial \bar{p}_i} + \sum_{m=1}^{N} \frac{\partial \bar{p}_m}{\partial q_l} \frac{\partial^2 \bar{H}}{\partial \bar{p}_m \partial \bar{p}_i}, \tag{15}$$

$$\frac{\partial^2 H}{\partial p_v \partial p_i} = \sum_{n=1}^{N} \frac{\partial \bar{p}_n}{\partial p_v} \frac{\partial^2 \bar{H}}{\partial \bar{p}_n \partial \bar{p}_i}. \tag{16}$$

The formulae (15) and (16) are obtained by differentiating (10) with respect to q_l and p_v, respectively. Moreover:

$$\sum_{k=1}^{N} \frac{\partial}{\partial p_k} \left(\frac{\partial \bar{H}}{\partial q_i} \right) = \sum_{k,n=1}^{N} \frac{\partial \bar{p}_n}{\partial p_k} \frac{\partial^2 \bar{H}}{\partial \bar{p}_n \partial q_i}$$

$$= \sum_{k,n=1}^{N} \frac{\partial \bar{p}_n}{\partial p_k} \frac{\partial^2 H}{\partial q_i \partial p_n} - \sum_{k,n,m=1}^{N} \frac{\partial \bar{p}_n}{\partial p_k} \frac{\partial \bar{p}_m}{\partial q_i} \frac{\partial^2 \bar{H}}{\partial \bar{p}_m \partial \bar{p}_n}$$

$$= \sum_{k,n=1}^{N} \frac{\partial \bar{p}_n}{\partial p_k} \frac{\partial^2 H}{\partial q_i \partial p_n} - \sum_{k,m=1}^{N} \frac{\partial \bar{p}_m}{\partial q_i} \left(\sum_{n=1}^{N} \frac{\partial \bar{p}_n}{\partial p_k} \frac{\partial^2 \bar{H}}{\partial \bar{p}_m \partial \bar{p}_n} \right)$$

$$= \sum_{n,k=1}^{N} \frac{\partial \bar{p}_n}{\partial p_k} \frac{\partial^2 H}{\partial q_i \partial p_n} - \sum_{k,m=1}^{N} \frac{\partial \bar{p}_m}{\partial q_i} \frac{\partial^2 H}{\partial p_k \partial p_m} . \tag{17}$$

From (14) and (17), summing up over the index i and comparing we get

$$\sum_{j=1}^{N} \frac{\partial}{\partial p_j} \left(\sum_{k,i=1}^{N} \frac{\partial \bar{p}_i}{\partial p_k} \right) \frac{\partial H}{\partial q_j} - \sum_{j=1}^{N} \frac{\partial}{\partial q_j} \left(\sum_{k,i=1}^{N} \frac{\partial \bar{p}_i}{\partial p_k} \right) \frac{\partial H}{\partial p_j} = 0, \tag{18}$$

thus $\{H, \mathscr{A}\} = 0$ which proves the theorem in question.

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