MACH'S PRINCIPLE AND THE MISSING MASS IN CLUSTERS OF GALAXIES

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It is shown that Sciama's inertial force law can be used to provide an explanation of why clusters of galaxies appear to be bound together more than gravitational force from their luminous masses would suggest. Such an explanation gives an approximate value of the constant in the inertial force law. Shortcomings when the law is applied cosmologically are noted. The law is shown not to be amenable to solar system tests, and not to be equivalent to Milgrom's force law.

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The mass of galaxies which occur in clusters can be calculated from either their relative dynamics or from their intrinsic luminosity. Dynamical considerations give a much larger mass, see for example Rood [1], several explanations of this have been put forward such as: large amounts of non-luminous gas, heavy neutrinos, black holes, and new force laws. Here Mach's principle in the form of Sciama's inertial force law [2] is put forward as a possible explanation. The physical considerations underlying the inertial force law are sketched, and the implications that this type of law only has effects for enormous masses or accelerations such as those that occur at very short distances in central orbit problems is stressed. This approach is shown to be inequivalent to the modified force law discussed in great detail by Milgrom [3]. Finally the significance of the inertial force law to the dynamical situations on the scale of the Universe, clusters of galaxies, and the solar system is discussed.

Sciama's inertial force law arises from investigating the source of the force exerted on a body in an accelerating frame. In an inertial frame we have Newton's second law $F = ma$; but if we use the body, upon which the force is acting, as the frame of rest we have $F - ma = 0$. The $-ma$ force has no local source and is an inertial force in contrast to F. By Mach's principle we say that this force occurs because the body is accelerating relative to the 'fixed stars'. Assuming that the inertial mass has both an active and a passive role in producing inertial force, gives an inertial force $F_I \propto m_1 m_2$; as noted above such a force occurs when accelerating relative to the "fixed stars" so that

$F_I \propto m_1 m_2 a$; we expect the force to be longer ranged than the gravitational force and assuming for simplicity that it depends on an integer power of distance r gives $F_I \propto m_1 m_2 a/r$, which is the form given by Sciama. Putting k as the constant of proportionality, which has dimension length over mass, gives: $F_I = mk_1 m_2 a/r$. Clearly as such a force has not been detected either the masses or their relative accelerations must be very large. For a particle in orbit around a large mass M_S we have $a = GM_S/r^2$ and for the necessary large accelerations either r has to be unrealistically small or M_S unrealistically large, this is discussed in more detail later.

Comparison with Milgrom's law is made by considering only a small mass m_1 and a large mass M_S then the sum of the inertial force F_I and the gravitational force F_G is

$$F_T = F_I + F_G = \frac{km_1 M_S}{r} \cdot \frac{M_S G}{r^2} + \frac{m_1 M_S G}{r^2} = \frac{m_1 M_S G}{r^2}\left(1 + \frac{kM_S}{r}\right). \tag{1}$$

Milgrom uses a modification of Newton's second law

$$m\mu\left(\frac{a}{a_0}\right) = F, \tag{2}$$

where a_0 is a constant and

$$\mu(x \gg 1) \approx 1, \quad \mu(x \ll 1) \approx x \tag{3}$$

again considering a small mass m_1 and a large mass M_S, and using $a = GM_S/r^2$ we obtain

$$m\mu\left(\frac{GM_S}{a_0 r^2}\right)\frac{GM_S}{r^2} = F. \tag{4}$$

For the two approaches to be equivalent we must have

$$\mu\left(\frac{GM_S}{a_0 r^2}\right) = \left(1 + \frac{kM_S}{r}\right), \tag{5}$$

which is clearly not the case because of the different dependence on the distance r. Essentially Milgrom assumes that when there is a large acceleration Newton's second law holds, whereas Sciama's law gives the opposite effect.

In order to investigate the cosmological significance of Sciama's law again consider a central orbit problem with small mass m_1 and a large mass M_S. Now in m_1's frame of reference m_1 experiences an inertial force due to the "fixed stars", because we only wish a rough order of magnitude for k we can take this force as being due to the mass of the Universe $M_U \approx 10^{52}$ kg at a distance of the radius of the Universe $R \approx 10^{26}$ m, this gives an inertial force

$$F_I = \frac{km_1 M_U a}{R} = \frac{km_1 M_U G M_S}{Rr^2}. \tag{6}$$

In m_1's frame F_I and F_G must balance so that

$$F_G = \frac{GM_S m_1}{r^2} = F_I = \frac{km_1 M_U GM_S}{Rr_2}, \tag{7}$$

which implies that $k = R/M_U \approx 3 \times 10^{-26}$ m · kg^{-1}. Leaving aside central orbit problems we now investigate whether such a value of k is reasonable by looking at the effect it would have on the relative motions of stars within the Galaxy. Considering two stars accelerating with respect to one another, we can ask when does the inertial force become of the same order as the gravitational force, we have

$$F_I = \frac{kM_1 M_2 a}{r} \approx F_G = \frac{GM_1 M_2}{r^2}, \tag{8}$$

which implies a $r \sim G/k \approx 2 \times 10^{15}$ m^{2}s^{-2}. For an acceleration of one ms^{-2} we find that r is about one light year so k is clearly too large. An extension of the inertial law to

$$F_I = \frac{R^x}{M_U} \frac{M_1 m_2 a}{r^x}, \tag{9}$$

does not overcome this problem. We can see what the maximum distance that F will be less than F_G for $x > 0$ in the above modification. We have

$$\frac{R^x}{M_U} \frac{M_1 M_2 a}{r^x} < \frac{GM_1 M_2}{r^2}, \tag{10}$$

so that

$$0 < x < \frac{\ln (GM_U/(ar^2))}{\ln (R/r)} \approx \frac{\ln (6 \times 10^{41}/(ar^2))}{\ln (3 \times 10^{26}/r)}. \tag{11}$$

For $a = 1$ ms^{-2}, x becomes negative for $r > 8 \times 10^{20}$ m. Therefore a modification of this type does not overcome the problem of the range being too short as the radius of the Galaxy $\sim 10^{21}$ m and we would expect gravitational forces to dominate there.

In spite of the shortcomings of the inertial law to explain phenomena on the cosmological scale we may consider if it has any use on the scale of clusters of galaxies and hope that if it has that a suitable modification might describe its cosmical form. Clusters of galaxies appear to be held together more than their luminous masses would suggest, maybe this could be explained by an extra Machian binding force. If we assume an inertial law of the original type and ask what value of k would give the inertial force of the same order of magnitude as the gravitational force we obtain

$$F_I = \frac{kM_{CL} M_G a}{r} \sim F_G = \frac{GM_{CL} M_G}{r^2}, \tag{12}$$

where M_{CL} is the mass of a cluster of galaxies, M_G is the mass of the particular galaxy in question, and a is their relative acceleration. We obtain a value for k of $k \sim G/ar$, for $r = 10^{22}$ m and $a = 1$ ms^{-2} this is $\sim 10^{-34}$ kg^{-1}. A value of this order would meet the requirements of binding the galaxies together more than would otherwise be expected and at the same time not interfere too much with their internal structure.

It is necessary to check that this value of k is consistent with the dynamics of the solar system. We have a central orbit problem in which the inward acceleration is

$$f = -\frac{GM_S}{r^2} - \frac{kM_S a}{r} = -\frac{GM_S}{r}\left(1 + \frac{kM_S}{r}\right) \quad \text{as} \quad a \approx \frac{GM_S}{r^2}. \tag{13}$$

Therefore we have a modified Binet equation

$$\frac{d^2u}{d^2} + u(1-\kappa) = \frac{\mu}{h^2}, \tag{14}$$

where $\mu = GM_S$, $\kappa = \dfrac{GM_S^2 k}{h^2}$, $h = r^2$ and $u = 1/r$, with solution

$$u = \frac{\mu}{h(1-\kappa)} + e\cos\sqrt{1-\kappa}\,\theta \tag{15}$$

so that there is an advance in the perhelion of $\pi\kappa$, compared to the general relativity value of $6\pi G^2M_S^2/(c^2h^2)$. We see that the ratio of the general relativity effect to the inertial law effect is $6G/c^2 \approx 4\times 10^{-27}$ to k, as we are considering values of k of the order of 10^{-34}, we see that the general relativity effect completely masks the inertial induction effect, or alternatively we can say that the dynamics of the solar system put an upper limit on k of the order of 10^{-28} mkg^{-1}. If we treat the extra term in the general relativistic Binet equation as being due to a Newtonian force we have

$$\begin{aligned} F_N &= GM_SM_0/r^2, \\ F_{Rel} &= 3GM_0h^2u^2 = 3GM_0\dot\theta^2 \quad \text{or} \quad 3G^2M_S^2M_0/r^3c^2, \\ F_{II} &= kM_SM_0a/r, \\ U_N &= -GM_SM_0/r, \\ U_{Rel} &= -3G^2M_S^2M_0/2r^2c^2, \\ U_{II} &= -kGM_S^2M_0/2r^2, \end{aligned} \tag{16}$$

and so we see that the general relativistic effect would be expected to dominate in central orbit problems. Treating the general relativistic term in this way has the unusual consequence that the force F_{Rel} can be expressed in a form not explicitly dependent on r only on $\dot\theta$ (although $\dot\theta^2 = GM_S/r^3$), and the ratio of potential of the general relativistic case to the Newtonian case is

$$\frac{U_N}{U_{Rel}} = \frac{2rc^2}{3GM_S} \approx 10^{-15}\times r \quad \text{for} \quad M_S = 10^{42}\ \text{kg}, \tag{17}$$

which would appear to have little effect when added to the viral theorem and applied to galaxies. It is interesting to note that in central orbit problems the inertial force law becomes a r^{-3} short distance law, because we approximate acceleration by GM_S/r^2, but that its effect is so weak that it would not be expected to contribute until the distance r is too small to be of significance in the problem. In conclusion we may say that we would expect the dynamics of galaxies to put upper bounds on the size of constants in conjectured long range force laws and that the inertial force law would appear to be of some use in explaining the problem of missing mass, although it fails when applied cosmologically, however we may hope that a suitable modification would work when applied cosmologically and also explain Mach's principle.

REFERENCES

[1] H. J. Rood, *Rep. Prog. Phys.* **44**, 1077 (1981).
[2] D. W. Sciama, *The Physical Foundations of General Relativity*, Heinemann Educational Books Ltd. 1969.
[3] M. Milgrom, *Astrophys. J.* **270**, 365 (1983).