QUARKS IN NUCLEI: su(4) LIE ALGEBRA*

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It is shown that the spectrum generating algebra for the pairing interaction of quarks is the su(4) $\simeq$ so(6) Lie algebra independently of the j-shell.

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1. *Introduction*

The quarkish constitution of the nucleons must manifest itself at short distances, however there is considerable controversy, at the moment, as to how to include the quarkish composition of the nucleon: there has been little success in understanding low-energy nucleon-nucleon forces in the framework of the three constituent quarks, see e.g. the review by Bäckman et al. (1983).

Recently the pairing interaction of quarks within the given j-shell in the nuclear matter has been proposed and discussed by Bleuler et al. (1983). It is extremely important that this simple phenomenological model is reproducing the characteristic properties of the conventional nuclear shell structure. This gives the hope that the more sophisticated phenomenological models of the quark interactions could be helpful in the more deep understanding of the nuclear structure.

The effectiveness of the phenomenological models of this kind comes from the direct relation between these models and the theory of the Lie algebra representations. In fact the total Hamiltonian of the quark pairing forces can be expressed, up to operator of the number of particles, by means of the Casimir operators of the SU($4j+2$) and SO($4j+2$) groups, which has been demonstrated explicitly by Bleuler et al. (1983).

Generally the method of the spectrum generating Lie algebras consists in the identification of the total Hamiltonian operator H with the proper element of the enveloping algebra of some Lie algebra, i.e. one must express H by means of the set of the mutually commuting

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operators in the enveloping algebra. The spectrum generating Lie algebra obviously need not to be the symmetry algebra of the Hamiltonian operator H.

The goal of this note is to point out that the Hamiltonian describing the pairing forces of quarks in nuclei can be directly related to su(4) $\simeq$ so(6) Lie algebra *independently* of the j-shell. In particular this means that the Hamiltonian operator can be expressed up to "trivial" terms by means of the *one* Casimir operator of the SU(4) group and there is *no* need to consider j-dependent groups SU($4j+2$) and SO($4j+2$).

The so(6) Lie algebra as the spectrum generating Lie algebra is the natural next step in the well known family of the so called quasi-spin Lie algebras: so(3) for the pairing correlations of the one kind of nucleons (no isospin) and so(5) for the pairing correlations of the two kinds of fermions (flavour without the colour). We refer to our previous papers (Oziewicz and Gorczyca 1981 and Oziewicz and Ciechanowicz 1984) devoted to the problem of the so(5) Lie algebra as the spectrum generating for the pairing Hamiltonians. Inclusion of the colour results naturally, as we like to show here, in the so(6) Lie algebra. This result opens the possibility of investigation of the phenomenology of the three-and four-quarks correlations in nuclei by means of the exploitation of all, three independent Casimir operators of the so(6) group, along the line indicated in Ref. (Oziewicz and Gorczyca 1981) for the quadrupling forces.

2. *Lie algebras of the tensor operators*

Let $\mathscr{K}$ denote some linear space and $\mathscr{K}^*$ its dual, then the fermion creation ($\equiv$ emission) and annihilation operators e and a respectively can be considered as the linear mappings

$$e : \mathscr{K} \rightarrow \mathrm{End}\,\{\wedge\, \mathscr{K}\}$$
$$a : \mathscr{K}^* \rightarrow \mathrm{End}\,\{\wedge\, \mathscr{K}\},$$

with $\{e_v, a_\alpha\} = \alpha(v)$ for $\forall\, v \in \mathscr{K}$ and $\alpha \in \mathscr{K}^*$. (1)

The above relation (1) tells us nothing but that a is of the degree one and antiderivation of the exterior Grassmann algebra $\wedge\, \mathscr{K}$, (so called Fock space of the quantum states), i.e. that the annihilation operators can be identified with the internal multiplication and creation operators with the external multiplication in the Grassmann algebra.

Suppose that $\mathscr{K}$ is the (irreducible) G-space for some (Lie) group G. Then we have the natural G-invariant isomorphism $\Phi : \mathscr{K} \rightarrow \mathscr{K}^*$ which is defined by means of the Clebsch-Gordan coupling to the one-dimensional (scalar) representation of G. G--invariance means that $g^* \circ \Phi = \Phi \circ g^{-1}$ where $g \in$ G and g^* denotes the pull-back of g. For the SU(2) case Φ is essentially the T-transformation in (Bleuler et al. 1983). However, we are choosing the different convention

$$(\Phi v)w \equiv (-)^j \hat{j} \langle 0 | v \otimes w \rangle \quad \text{for} \quad v \text{ or } w \in [j]. \tag{2}$$

Here $[j]$ is the carrier space of the linear irreducible representation of SU(2) with $\dim [j] = 2j+1$.

Now it is convenient to define the "bosonic" operators in $\mathscr{K} \otimes \mathscr{K}$,

$$E \equiv e \otimes e \quad \text{and} \quad A \equiv (a \otimes a) \circ \Phi. \tag{3}$$

Let

$$\mathscr{K} \otimes \mathscr{K} = \oplus \mathscr{K}^J \quad \text{and} \quad \mathscr{K}^J \otimes \mathscr{K}^K = \oplus \mathscr{K}^L_{JK}. \tag{4}$$

Then we define

$$N^L_{JK} \equiv E \wedge A \circ P^L_{JK} \tag{5}$$

where P^L_{JK} is the projector on $\mathscr{K}^L_{JK}$ subspace and

$$(E \wedge A)(v \otimes w) \overset{\mathrm{df}}{\equiv} [E^v, A^w]. \tag{6}$$

For the particular case the operators (5) are related to the number and isospin operators, cf. with the definitions (2.3) and (2.4) in (Oziewicz and Gorczyca 1981).

Now we like to calculate N^L_{JK} operators for the particular case when $\mathscr{K} = [j]$ and (4) means the SU(2)-irreducible splittings, i.e.

$$[j] \otimes [j] = \bigoplus_{J=0}^{2j} [J], \text{ etc.} \tag{7}$$

We obtain the following result

$$N^0_{JJ} = \hat{J}(1-(-)^{2j+J})((-)^J + 2\hat{j}^{-2} N_j), \tag{8}$$

where $\hat{J} \equiv \sqrt{2J+1}$ and N_j denotes the number operator

$$N_j \equiv N \circ (-)^j \hat{j} P^0_{jj}, \tag{9}$$

where

$$N \equiv e \otimes (a \circ \Phi). \tag{10}$$

In (9) the factor $(-)^j \hat{j}$ is due to our convention (2). In a similar way for $L \neq 0$ we get

$$N^L_{JK} = \hat{J}\hat{K} \begin{Bmatrix} J & K & L \\ j & j & j \end{Bmatrix} ((-)^{2j+J} + (-)^{2j+K} - (-)^{J+K} - 1)(-)^{L+j} N \circ P^L_{jj}. \tag{11}$$

In the last formula, $\{\ldots\}$ denotes the Racah coefficient for the SU(2) group, see e.g. (Varschalovich et al. 1975) or (Jucys and Bandzaitis 1977).

Let us define

$$E^L_{JK} \equiv N \wedge E \circ P^L_{JK},$$

$$A^L_{JK} \equiv N \wedge A \circ P^L_{JK},$$

$$M^L_{JK} \equiv N \wedge N \circ P^L_{JK} \tag{12}$$

in a way similar to Eqs. (5–6). For $\mathscr{K} = [j]$ we get

$$E^L_{JK} = (1-(-)^{2j-K})(-)^{j-K}\hat{K}\hat{J}\begin{Bmatrix} J & K & L \\ j & j & j \end{Bmatrix} E \circ P^L_{jj},$$

$$A^L_{JK} = -(-)^J(1-(-)^{2j-K})(-)^{J-K}\hat{K}\hat{J}\begin{Bmatrix} J & K & L \\ j & j & j \end{Bmatrix} A \circ P^L_{jj},$$

$$M^L_{JK} = (-)^{j+L}(1-(-)^{J+K+L})\hat{J}\hat{K}\begin{Bmatrix} J & K & L \\ j & j & j \end{Bmatrix} N \circ P^L_{jj}. \tag{13}$$

These expressions show that the E, A and N operators (3) and (10) jointly with the corresponding set of the projectors $\{P^L_{jj}\}$ are generating the Lie algebras.

Suppose that $G = \overset{n}{\times} SU(2)$ and let G-irreducible space be $\mathscr{K} = [j_1] \otimes [j_2] \otimes \ldots \otimes [j_n]$. Then the natural generalization of the formulas (8) and (11) above is

$$N^L_{JK} = (\prod_i \delta_{J_iK_i}\delta_{L_i0}\hat{K}_i)(-)^{\Sigma K_i}(i-(-)^{\Sigma 2j_i+K_i})$$

$$+\left(\prod_i \hat{J}_i\hat{K}_i\begin{Bmatrix} J_i & K_i & L_i \\ j_i & j_i & j_i \end{Bmatrix}\right)\Bigg\}((-)^{\Sigma 2j_i+J_i}+(-)^{\Sigma 2j_i+K_i}-(-)^{\Sigma J_i+K_i}-1)$$

$$\times(-)^{\Sigma L_i+J_i}N \circ \overset{n}{\otimes} P^{L_i}_{j_ij_i}. \tag{14}$$

In a similar way one can generalize the other commutators (12–13) and the formula like (14) is one of the main results of the present paper. It should be obvious that in (14) L, J and K denote the multi-indices, say $J \equiv \{J_1, J_2, \ldots, J_n\}$, etc. Now the number operator (9) should read

$$N_j = N \circ \overset{n}{\otimes}(-)^{j_i}\hat{j}_i P^0_{j_ij_i}. \tag{15}$$

Therefore

$$N^0_{JJ} = (\prod \hat{J}_i)(1-(-)^{\Sigma 2j_i+J_i})\{(-)^{\Sigma J_i}+2(\prod(2j_i+1)^{-1})N_j\}. \tag{16}$$

Let us consider several illustrative examples of the formula (16). Let $J = \{0, 1\}$ then

$$N^0_{JJ} = \sqrt{3}\,(1+(-)^{\Sigma 2j_i})\left\{-1+\frac{2}{(2j_1+1)(2j_2+1)}N_j\right\}. \tag{17}$$

For $j_1 = j$ and $j_2 = \frac{1}{2}$ both half-integer we have

$$N^0_{JJ} = 2\sqrt{3}\left(-1+\frac{1}{2j+1}N_J\right), \tag{18}$$

which is well known, see e.g. formula (2.5) in (Oziewicz and Gorczyca 1981).

For $J = \{1, 1\}$ the formula (16) gives

$$N^0_{JJ} = 3(1-(-)^{\Sigma 2j_i})\left\{1+\frac{2}{(2j_1+1)(2j_2+1)}N_j\right\}, \tag{19}$$

which for example for $j_1 = 1$ and $j_2 = \frac{1}{2}$ is reduced to the form

$$N^0_{JJ} = 6(1+\tfrac{1}{3}N_j). \tag{20}$$

The case $J = \{0, 0, 1\}$, (spin, flavour and "colour": $\{j, \frac{1}{2}, 1\}$ corresponds to the pairing forces introduced by Bleuler et al. (1983). We made here the mathematically justified identification which we will discuss in the full line in the next communication. Then the tedious calculations, see the last Section, shows that the spectrum generating Lie algebra for the pairing interaction of quarks is the su(4) $\simeq$ so(6) Lie algebra independently of the j-shell.

Summarizing we see that the resulting, $\mathscr{K}$-induced, Lie algebras are generated by the set of the G-irreducible tensor operators $\{E^J, A^J$ and $N^J\}$ where we are denoting

$$E^J \equiv E \circ P^J_{jj} \quad \text{and} \quad A^J \equiv A \circ P^J_{jj},$$

however

$$N^J \equiv (-)^{j+J} N \circ P^J_{jj}. \tag{21}$$

It is important to realize that for $G = \overset{n}{\times} SU(2)$ and for $\mathscr{K} = [j_1] \otimes \ldots \otimes [j_n]$,

$$(1+(-)^{\Sigma 2j_i+J_i})(E^J \quad \text{and} \quad A^J) = 0. \tag{22}$$

Because

$$\sum_L E \wedge A \circ P^L_{JK} = E^J \wedge A^K, \tag{23}$$

therefore one can present the Lie algebras of the G-irreducible tensor operators in the basis independent way

$$\begin{aligned} E^J \wedge A^K &= (-)^J 2\hat{J}\delta_{JK} - (3+(-)^{J+K})\hat{J}\hat{K}\sum \begin{Bmatrix} J & K & L \\ j & j & j \end{Bmatrix} N^L, \\ N^J \wedge E^K &= (-)^{J+1} 2\hat{J}\hat{K}\sum \begin{Bmatrix} J & K & L \\ j & j & j \end{Bmatrix} E^L, \\ N^J \wedge A^K &= 2\hat{J}\hat{K}\sum \begin{Bmatrix} J & K & L \\ j & j & j \end{Bmatrix} A^L, \\ N^J \wedge N^K &= (-)^{2j}\hat{J}\hat{K}\sum ((-)^{J+K}-(-)^L)\begin{Bmatrix} J & K & L \\ j & j & j \end{Bmatrix} N^L. \end{aligned} \tag{24}$$

It should be obvious that the tensors E and A of rank $J = \{0, \ldots, 0, J_r, \ldots, J_n\}$ are generating the subalgebras. The above Lie algebra (24) is determined completely by the G-irreducible modul $\mathscr{K}$. It is appropriate therefore to refer to this algebra as the *$\mathscr{K}$-algebra*. Does any relation of the corresponding group, $\mathscr{K}$-group, to the Aut $\mathscr{K}$ exist?

3. *Simplest examples*

The trivial example is the case of $J = \{0, \ldots, 0\}$ when putting $E \equiv E^J$, etc. we get the su(2) quasi-spin algebra

$$E \wedge A = 2+2c_{\mathscr{K}}N, \quad N \wedge E = +c_{\mathscr{K}}E, \quad N \wedge A = -c_{\mathscr{K}}A,$$

where $c_{\mathscr{K}} \equiv 2 \prod \hat{j}_i^{-1}$.

Let us consider the Lie algebra of tensor operators (24) for the case $\mathscr{K} = [j] \otimes [\frac{1}{2}]$ with the half-integer j. If $J_1 = 0$ then (22) $\Rightarrow J_2 = 1$. Putting $E \equiv E^{\{0,1\}}$ and $A \equiv A^{\{0,1\}}$ and using the algebraic formulas for the Racah coefficients presented in the monograph by Varschalovich et al. (1975) we get from (24):

$$E \wedge A = -2\sqrt{3}+2\hat{j}^{-1}(\sqrt{6}\,N^0-2N^1),$$

$$N^J \wedge E = (\sqrt{3}\,j)^{-1}\hat{J}\,\sqrt{(J+3)(2-J)}\,E,$$

$$N^J \wedge A = (-)^{J+1}(\sqrt{3}\,\hat{j})^{-1}\hat{J}\,\sqrt{(J+3)(2-J)}\,A,$$

$$N^J \wedge N^K = -(1+(-)^{J+K})\hat{J}\hat{K}\hat{j}^{-1}\begin{Bmatrix} J & K & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} N^1. \tag{25}$$

In particular $N^1 \wedge N^1 = 2\hat{j}^{-1}N^1$. This is nothing but sp(4) $\simeq$ so(5) Lie algebra, cf. with e.g. formulas (2.10) in (Oziewicz and Gorczyca 1981). This is the spectrum generating algebra for the nuclear pairing and quadrupling forces, see the references cited in (Oziewicz and Gorczyca 1981).

4. $\mathscr{K} = [j] \otimes [\frac{1}{2}] \otimes [1]$: *spin* $\otimes$ *flavour* $\otimes$ *colour*

Bleuler et al. (1983) introduced SU(3)-invariant generalized pairing force for quarks in nuclei. This case corresponds to SU(2) × SU(2) × SU(3) group of the spin, flavour and colour, with the irreducible representation $[j] \otimes [\frac{1}{2}] \otimes$ [three dimensional fundamental representation of SU(3)].

Let us consider the simplified mathematical model of the $\overset{3}{\times}$SU(2) group with the space of the irreducible representation $\mathscr{K} = [j] \otimes [\frac{1}{2}] \otimes [1]$. For $J = \{0, 0, J_3\}$ and half-integer j, (22) $\Rightarrow J_3 = 1$. Putting as previously $E \equiv E^J$ and $A \equiv A^J$ we get from (24),

$$E \wedge A = -2\sqrt{3}+\sqrt{2}\,\hat{j}^{-1}(2N^0-N^1-N^2),$$

$$N^L \wedge E = -(2\sqrt{6}\,\hat{j})^{-1}\hat{L}\,[L(L+1)-4]E,$$

$$N^L \wedge A = (-)^L(2\sqrt{6}\,\hat{j})^{-1}\hat{L}\,[L(L+1)-4]A,$$

$$N^J \wedge N^K = \hat{J}\hat{K}(\hat{j}\,\sqrt{2})^{-1}\sum((-)^{J+K}-(-)^L)\begin{Bmatrix} J & K & L \\ 1 & 1 & 1 \end{Bmatrix} N^L. \tag{26}$$

The tedious identification is showing that this algebra (26) is isomorphic to su(4) $\simeq$ so(6) Lie algebra.

Let us indicate that for the SU(2)×SU(2)×SU(3) group with $\mathscr{K} = [j] \otimes [\frac{1}{2}] \otimes 3$ and for $E \equiv E^{\{0,0,3\}}$ and $A \equiv A^{\{0,0,3^*\}}$ we have

$$E \wedge A \simeq \text{const} + N^1 + N^8, \tag{27}$$

what follows from the Clebsch-Gordan splittings

$$3 \otimes 3 = 3^* \oplus 6, \quad 3^* \otimes 3^* = 3 \oplus 6^*, \quad 3 \otimes 3^* = 1 \oplus 8.$$

We ar eomitting here the cumbersome expressions and the proof that the 15-dimensional Lie algebra (27) is isomorphic to su(4) with N^8 operator generating the su(3) Lie subalgebra. This exact identification needs the Racah coefficients for SU(3). We can conclude that the pairing force of Bleuler et al. (1983) $E \cdot A \equiv E \otimes A \circ \delta^1_{33^*}$ can be expressed through the second degree Casimir operator of su(4) algebra.

It is illustrative to see what gives the neglection of the spin in the previous example, i.e. when we consider the flavour $\otimes$ "colour" alone. Then from (22) we see that the full algebra for $\mathscr{K} = [\frac{1}{2}] \otimes [1]$ is generated by $J = \{0, 0\}, \{0, 2\}$ and $\{1, 1\}$. It is interesting that the $\{1, 1\}$ is generating the subalgebra of (24). Let us put $E \equiv E^{\{1,1\}}$ and $A \equiv A^{\{1,1\}}$ then we get from (24):

$$E \wedge A = 6 + 2\sqrt{6}\, N^{\{0,0\}} - 4N^{\{1,0\}} - \sqrt{6}\, N^{\{0,1\}} + 2N^{\{1,1\}}$$
$$-\sqrt{6}\, N^{\{0,2\}} + 2N^{\{1,2\}}, \text{ etc.}$$

We see therefore that this is 56-dimensional Lie algebra. However if we put

$$N^L \propto \sqrt{6}\, N^{\{0,L\}} - 2N^{\{1,L\}},$$

and if we will *violate* the isospin invariance by considering only zero isospin components of the E and A tensor operators then we get the 15-dimensional Lie subalgebra isomorphic to (26).

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