Vol. B16 (1985) ACTA PHYSICA POLONICA No 8

x_3-DEPENDENT SU(2) GAUGE POTENTIALS; A NEW EXAMPLE OF CLASSICAL YANG-MILLS MECHANICS*

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(Received December 28, 1984)

Classical Yang-Mills equations for x_3-dependent, static potentials are investigated. Four classes of solutions are found. All solutions are unstable in the Liapunov sense. The solutions do not exhibit chaotic behaviour in the x_3 variable.

PACS numbers: 11.10.Np, 03.50.Kk

1. Introduction

In a number of recent papers it has been observed that purely time-dependent solutions of classical Yang-Mills equations, i.e. $A^a_\mu = A^a_\mu(x_0)$, exhibit a chaotic behaviour, [1]. In particular, the values of the functions $A^a_\mu(x_0)$ change irregularly during successive time intervals, so that one cannot predict the behaviour of $A^a_\mu(x_0)$ for $x_0 \to +\infty$.

In field theory the distinction between the time and the space coordinates is to some extent dissolved. Let us recall, for example, that

— in Euclidean field theory there is no difference between time and space coordinates,

— in relativistic field theory Lorentz transformations, which are the symmetry of the theory, mix time and space coordinates.

Therefore, it is natural to ask whether purely space-dependent classical solutions of Yang-Mills equations, e.g. $A^a_\mu = A^a_\mu(x_3)$, exhibit a chaotic behaviour.

Apart from the problem of chaotic behaviour, we find also another reason why it is interesting to look at solutions of Yang-Mills equations in the simple case of potentials depending only on a single variable. Namely, the equations then still remain to be nontrivial, yet they allow for more or less complete analysis of solutions. The relatively complicated equations we consider below should be compared with trivial equations obtained in the Abelian case, e.g. $d^2A^\mu(x_3)/dx_3^2 = 0$ for x_3-dependent potentials.

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* Paper supported in part by the Polish Ministry of Higher Education, Science and Technology, project MR.I.7.

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We consider the potentials which depend on $\tau = k^2x_2+k^3x_3$; for $k_2 = 0$ the potentials are purely x_3-dependent. For such potentials Yang-Mills equations take the form of Newton equations of classical mechanics. We find four classes of solutions. All solutions are unstable in the Liapunov sense. The answer to the question about the chaotic behaviour is in the negative, in the sense that we observe rather regular behaviour of trajectories for $\tau \to \infty$.

In Section 2 we obtain the Newton equations for the class of Yang-Mills potentials. In Sections 3 and 4 we describe solutions of these equations. In Section 5 we present our point of view on the integrability of the system. Section 6 contains some more general remarks and conclusions.

2. *Equations of motion and their simple implications*

We assume that

$$A^a_\mu = A^a_\mu(k_\varrho x^\varrho), \tag{1}$$

where (k_ϱ) is a constant four-vector. We also assume that A^a_μ obey the algebraic gauge condition

$$k_\varrho A^{a\varrho} = 0. \tag{2}$$

Then, Yang-Mills equations

$$\partial_\mu F^{a\mu\nu} - g f_{abc} A^b_\mu F^{c\mu\nu} = 0, \tag{3}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu,$$

reduce to the following equations

$$k^2 \ddot{A}^{a\nu} + g^2 f_{abc} f_{cde} A^b_\mu A^{d\mu} A^{e\nu} = 0, \tag{4}$$

$$f_{abc} A^b_\mu \dot{A}^{c\mu} = 0. \tag{5}$$

Here dot denotes the derivative with respect to $\tau = k^\varrho x_\varrho$, f_{abc} are the structure constants of the gauge group, i.e., SU(2).

Equation (4) has the form of Newton equation of classical mechanics.

With the help of Jacobi identity

$$f_{abd} f_{dhc} + f_{had} f_{dbc} + f_{bhd} f_{dac} = 0$$

it is easy to show that the τ-derivative of the l.h.s. of (5) is equal to zero. Therefore it is sufficient to impose (5) only at the initial value of τ.

In the following we assume that $A^{a\mu}$ are equal to zero except for

$$A^{10} = x, \quad A^{21} = y. \tag{6}$$

Then, the constraint (5) is obeyed automatically. The gauge condition (2) implies that

$$k_0 = 0, \quad k_1 = 0. \tag{7}$$

Thus, in this case τ does not contain the time variable. For another Ansatz, e.g., $A^{12} \neq 0$, $A^{21} \neq 0$ and all other $A^{a\mu} = 0$, τ can contain the time variable.

For the Ansatz (6) equations (4) reduce to

$$k^2\ddot{x} = -g^2xy^2, \quad k^2\ddot{y} = g^2yx^2. \tag{8}$$

From (7) we see that $k^2 < 0$ ($k^2 = k_0^2 - \vec{k}^2$). In the following we use the rescaled variables

$$x \to \sqrt{\frac{|k^2|}{g^2}}\,x, \quad y \to \sqrt{\frac{|k^2|}{g^2}}\,y.$$

For the rescaled variables the Newton equations are

$$\ddot{x} = xy^2, \tag{9a}$$

$$\ddot{y} = -yx^2. \tag{9b}$$

It is easy to check that

$$H = \tfrac{1}{2}(\dot{x}^2 - \dot{y}^2 - x^2y^2) \tag{10}$$

is a constant of the motion. In fact H plays the role of the Hamiltonian for the system (9), therefore we shall call it the energy. The Lagrangian is

$$\mathscr{L} = \tfrac{1}{2}(\dot{x}^2 - \dot{y}^2 + x^2y^2). \tag{11}$$

H is not positive definite. Let us remark that H is not related to the energy of Yang-Mills field

$$\mathscr{E} = \tfrac{1}{2}\int (\vec{E}^a\vec{E}^a + \vec{B}^a\vec{B}^a)d^3\vec{x}.$$

The r.h.s. of (9) we regard as forces. Their vector field

$$\vec{F} = xy\begin{pmatrix} y \\ -x \end{pmatrix}$$

is presented in Fig. 1. $\vec{F}$ is perpendicular to the radius $\vec{r} = (x, y)$, and $|\vec{F}| = \sqrt{x^2y^2(x^2+y^2)}$. It is obvious that there are no closed trajectories in the $x-y$ plane. Hence, there are no periodic motions. Typical trajectories are presented in Figs. 2–5.

Trajectories of the type A extend from $x = -\infty$ to $x = +\infty$, or vice versa. Trajectories of the type B lie entirely in one of the half-planes $x > 0$ or $x < 0$ — they possess the turning point at which $\dot{x} = 0$. Trajectories of the type C start at $x = +\infty$ ($x = -\infty$) for $\tau = -\infty$ and they approach the y-axis, with the point $y = 0$ excluded, from the right (left) for $\tau \to +\infty$. Trajectories of the type D are characterized by $x = 0$ or $y = 0$ — the forces vanish on them.

Trajectories of the type A can have $E = H \leqslant 0$ or $E > 0$. Trajectories of the type B have $E < 0$, because in the turning point $\dot{x} = 0$, $x^2 > 0$. It is easy to see that in the turning point it cannot happen that $y = \dot{y} = 0$. Therefore, the inequality for energy is sharp. Trajectories of the type C end at a point $(0, y)$, $y \neq 0$, for $\tau \to +\infty$. They have $E = 0$. It is easy to prove that there are no trajectories which end on the x-axis, including the point $x = y = 0$, i.e., the trajectory either crosses this line or entirely lies on it. Trajectories of the type D have $E \geqslant 0$ if $y = 0$, or $E < 0$ if $x = 0$.

Solutions of the type D are linear functions of τ because the nonlinear terms vanish for them. They correspond to the Abelian case mentioned in the Introduction. The type D solutions are not stable in the Liapunov sense in the obvious manner. For example, a small departure from the y-axis would result in completely different trajectory such

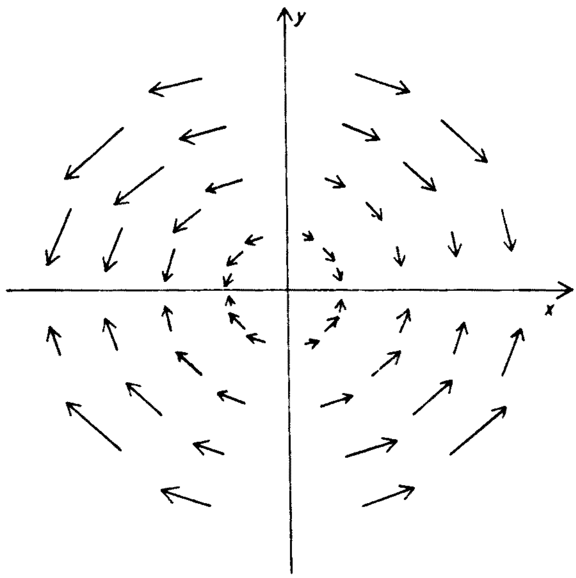

Fig. 1. The force $\vec{F} = xy\begin{pmatrix} y \\ -x \end{pmatrix}$. $\vec{F}$ is tangent to the circles $x^2+y^2 = \text{const}$, $|\vec{F}| = |xy|\,(x^2+y^2)^{1/2}$

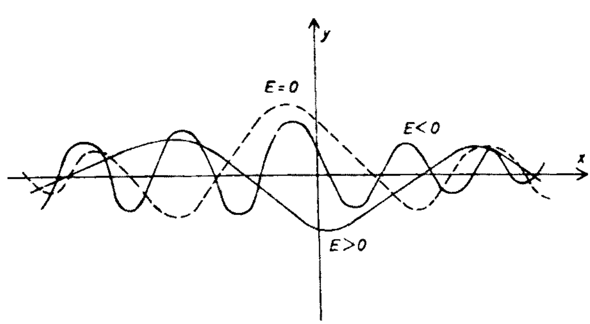

Fig. 2. Examples of trajectories of the type A. The dashed line corresponds to a trajectory with $E = 0$ — it crosses the y-axis and x-axis under the angle $\pi/4$

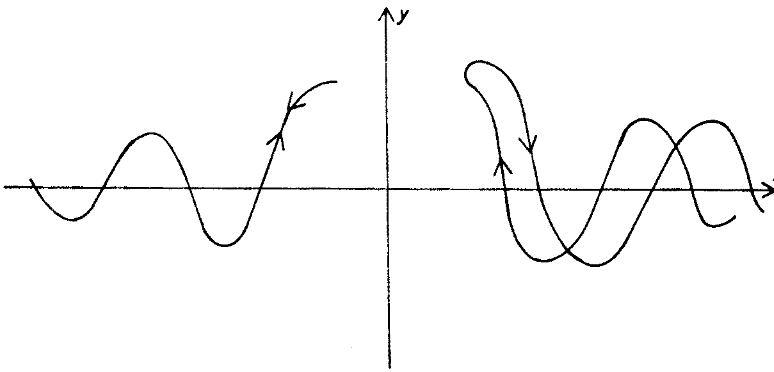

Fig. 3. Examples of trajectories of the type B. The trajectory on the l.h.s. is characterized by $\dot{x} = \dot{y} = 0$ at the turning point

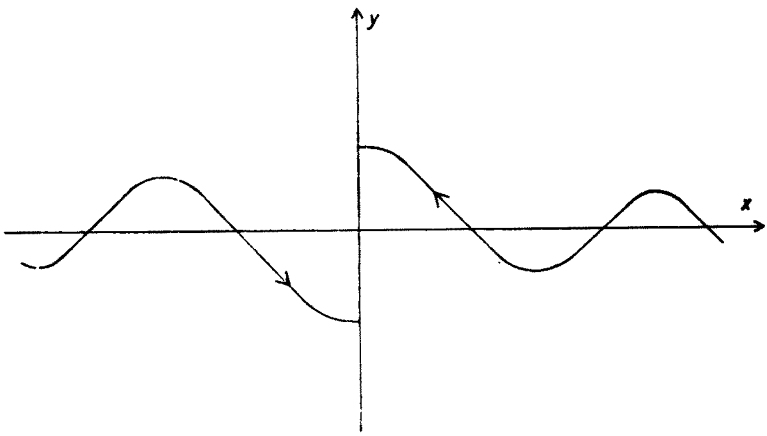

Fig. 4. Example of a trajectory of the type C. Trajectories of this type cross the x-axis under the angle $\pi/4$

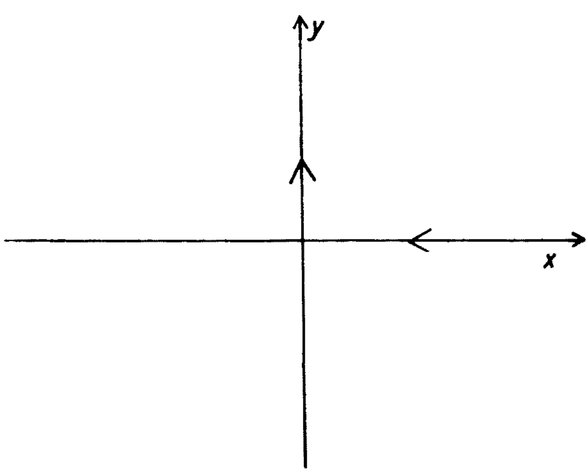

Fig. 5. Examples of trajectories of the type D

that $|x| \to \infty$ for $\tau \to \infty$. Small departure from the x-axis would give a trajectory of the type A or B which have quadratic dependence on τ for large τ, see the next Section. In the following we shall discuss the nontrivial trajectories A, B, C.

3. Solutions of the types A and B

We shall investigate these solutions for $\tau \to \infty$. The behaviour for $\tau \to -\infty$ can be obtained from the analysis presented below by the change $\tau \to -\tau'$. More precisely, the Cauchy problem for $\tau \leqslant 0$ with the data $x(0) = x_0$, $y(0) = y_0$, $dx/d\tau(0) = x_1$, $dy/d\tau(0) = y_1$ is equivalent to the Cauchy problem for $\tau' \geqslant 0$ with the initial data $x(0) = x_0$, $y(0) = y_0$, $dx/d\tau(0) = -x_1$, $dy/d\tau(0) = -y_1$.

The behaviour of trajectories for τ close to zero has to be obtained by numerical methods. We adjust the τ-variable in such a way that $\tau = 0$ corresponds to the turning point for trajectories of the type B, or to the point $x = 0$ for trajectories of the type A.

Now, let us fix our attention on the trajectories which for $\tau \to +\infty$ flow to the right in the $x-y$ plane, Fig. 2. That is, for $\tau > 0$

$$\dot{x} = +\sqrt{2E+\dot{y}^2+x^2y^2} > 0, \tag{12}$$

and $x > 0$. We shall consider these trajectories for large τ. The trajectories flowing to the left can be obtained by the replacement $x \to -x$. It follows from (12) that in the considered region $x(\tau)$ is monotonically increasing function of τ. Therefore we can regard y and p as functions of x. Then,

$$\frac{dy}{dx} = p(2E+p^2+x^2y^2)^{-1/2},$$

$$\frac{dp}{dx} = -yx^2(2E+p^2+x^2y^2)^{-1/2}, \tag{13}$$

where $p = \dot{y}$.

From (13) it follows that

$$\frac{d(x^2y^2+p^2)}{dx} = 2xy^2,$$

i.e.,

$$x^2y^2(x)+p^2(x) = x_0^2y^2(x_0)+p^2(x_0)+2\int_{x_0}^{x} sy^2(s)ds. \tag{14}$$

Thus, $x^2y^2+p^2$ increases with x (and with τ, because $\dot{x} > 0$). Comparing (14) with (10) we see that

$$\dot{x}^2-2E = x_0^2y^2(x_0)+p^2(x_0)+2\int_{x_0}^{x} sy^2(s)ds.$$

Formula (14) can be rewritten in the form

$$\dot{y}^2 f^{-2} + y^2 b^{-2} = 1, \tag{15}$$

where

$$f^2 = \dot{x}^2 - 2E > 0, \quad b^2 = x^{-2}(\dot{x}^2 - 2E).$$

Thus, y and $\dot{y}$ lie on a τ-dependent ellipse. It is easy to check that for $\tau \to +\infty$

$$\frac{db^2}{d\tau} \leqslant 0.$$

Thus, b^2 decreases (and f^2 increases).

Let us now investigate the trajectories in more detail. Because of (15) we can write that

$$y = x^{-1} f(x) \cos \varphi(x), \quad p = f(x) \sin \varphi(x). \tag{16}$$

From equations (13) it follows that the functions

$$\chi = 2\varphi, \quad \vartheta = 2 \ln f - \ln x$$

obey the equations

$$\vartheta'(x) = x^{-1} \cos \chi(x) \tag{17a}$$

$$\chi'(x) = -x^{-1} \sin \chi(x) - 2x(2E + x \exp(\vartheta(x)))^{-1/2}, \tag{17b}$$

where coma denotes d/dx.

It is easy to find an approximate solution (for $x \to +\infty$) of this set of equations. The form of (17a) suggests that for $x \to \infty$

$$\vartheta'(x) \approx 0,$$

i.e.,

$$\vartheta(x) \approx c_0. \tag{18}$$

Then, (17b) gives

$$\chi'(x) \approx -2\sqrt{x} \exp(-c_0/2),$$

i.e.,

$$\chi(x) \approx -\tfrac{4}{3} x^{3/2} \exp(-c_0/2) + c_1. \tag{19}$$

Thus,

$$y(x) \approx \exp(c_0/2) x^{-1/2} \cos(c_{1/2} - \tfrac{2}{3} \exp(-c_0/2) x^{3/2}),$$

$$p(x) \approx \exp(c_0/2) x^{1/2} \sin(c_{1/2} - \tfrac{2}{3} x^{3/2} \exp(-c_0/2)). \tag{20}$$

It is easy to check that this solution is consistent with (14).

The dependence of x, y, p on τ is not difficult to obtain. From (12), (20) we obtain the equation

$$\dot{x} \approx \sqrt{2E + x \exp c_0} \tag{21}$$

which has the solution

$$x(\tau) = \tfrac{1}{4}\exp(c_0)(\tau-\tau_0)^2+\dot{x}(\tau_0)(\tau-\tau_0)+x(\tau_0). \tag{22}$$

The τ dependence of y, p follows from (20). On the other hand, (9a) and (20) give the equation

$$\ddot{x} = \exp(c_0)\cos^2(c_{1/2}-\tfrac{2}{3}\exp(-c_0/2)x^{3/2}).$$

The solution (22) is consistent with this equation — it amounts to the replacement of $\cos^2$ by a mean value of it which, as we see from (22), turns out to be 1/2.

From (22) and (20) it follows that the trajectory is not chaotic — one can predict x, y for $\tau \to \infty$.

The system (17) obeys the Lipschitz condition. This is a very nice feature of it, because then one can use rapidly convergent iterative procedure for construction of the solution based on a contractive map, see, e.g., [2]. In order to prove the Lipschitz condition let us write (17) in the form

$$\dot{\vec{\Theta}} = \vec{f}(\vec{\Theta}),$$

where

$$\vec{\Theta} = \begin{pmatrix} \vartheta \\ \chi \end{pmatrix}, \quad \vec{f}(\vec{\Theta}) = x^{-1}\begin{pmatrix} \cos\chi \\ -\sin\chi-2x^2(2E+x\exp\vartheta)^{-1/2} \end{pmatrix}.$$

Using the following inequalities

$$(\cos\chi_1-\cos\chi_2)^2 \leqslant (\chi_1-\chi_2)^2,$$

$$(\sin\chi_1-\sin\chi_2)^2 \leqslant (\chi_1-\chi_2)^2,$$

$$(a+b)^2 \leqslant 2(a^2+b^2),$$

it is easy to check that

$$|\vec{f}(\vec{\Theta}_1)-\vec{f}(\vec{\Theta}_2)|^2 \leqslant \frac{3}{x^2}(\chi_1-\chi_2)^2$$

$$+8x^2[(2E+x\exp\vartheta_1)^{-1/2}-(2E+x\exp\vartheta_2)^{-1/2}]^2. \tag{23}$$

In order to estimate the last bracket on the r.h.s. of (23) we assume for definiteness that $\vartheta_2 \geqslant \vartheta_1$, $\vartheta_2 = \vartheta_1+\delta$, $\delta \geqslant 0$. We have

$$(2E+x\exp\vartheta_1)^{-1/2}-(2E+x\exp\vartheta_2)^{-1/2}$$

$$= -\int_0^\delta \frac{d}{d\sigma}(2E+x\exp(\vartheta_1+\delta))^{-1/2}d\sigma$$

$$= \tfrac{1}{2}x\int_0^\delta \exp(\vartheta_1+\sigma)(2E+x\exp(\vartheta_1+\sigma))^{3/2}d\sigma$$

$$= \tfrac{1}{2}\int_0^\delta h(x\exp(\vartheta_1+\sigma))(2E+x\exp(\vartheta_1+\sigma))^{-1/2}d\sigma,$$

where

$$h(z) = z(2E+z)^{-1}.$$

Now, let us recall that we consider the region of phase space such that $x > 0$, $\dot{x} > 0$. Then,

$$2E+x\exp\vartheta_1 = \dot{x}^2 \geqslant \varepsilon > 0. \tag{24}$$

Therefore, also

$$2E+x\exp(\vartheta_1+\sigma) \geqslant \varepsilon$$

because $\sigma > 0$ and $x\exp\vartheta_1 > 0$. Let us also recall that $f^2 = x\exp\vartheta$ increases with τ. On the other hand, the function $h(z)$ decreases monotonically when $z = x\exp(\vartheta+\sigma)$ increases (for $z > 2|E|$, however this is the case because ot (24)). Putting all this together we obtain

$$\int_0^\delta h(x\exp(\vartheta_1+\sigma))(2E+x\exp(\vartheta_1+\sigma))^{-1/2}d\sigma \leqslant \varepsilon^{-1/2}h(f_0^2)\delta,$$

where

$$f_0^2 = p^2(\tau_0)+x^2(\tau_0)y^2(\tau_0)$$

is fixed by the initial data for the trajectory specified at some $\tau = \tau_0 > 0$.

Therefore, from (23) we obtain the Lipschitz condition

$$\begin{aligned}(\vec{f}(\vec{\Theta}_1)-\vec{f}(\vec{\Theta}_2)|^2 &\leqslant 3x^{-2}(\chi_1-\chi_2)^2+2x^2\varepsilon^{-2}h^2(f_0^2)(\vartheta_1-\vartheta_2)^2\\ &\leqslant m^2(x)(\vec{\Theta}_1-\vec{\Theta}_2)^2,\end{aligned} \tag{25}$$

where

$$m^2(x) = 3x^{-2}+2\varepsilon^{-1}h^2(f_0^2)x^2.$$

Also

$$|f(\vec{\Theta})| \leqslant K(x), \tag{26}$$

where

$$K^2(x) = 3x^{-2}+8\varepsilon^{-1}x^2.$$

Thus, applying the procedure described in [2] one can obtain better and better approximations of the solution of the system (17). The approximate solution (20) can be used as the zeroth order approximation.

The solution (20), (22) is not stable in the Liapunov sense. For example, the difference $x_1(\tau)-x_2(\tau)$ grows quadratically with τ if the two trajectories 1 and 2 have even slightly different c_0. In fact, the instability can already be guessed from the form of the equation (9a). The non-negative force xy^2 enhances differences between initially neighbouring trajectories.

4. *Solutions of the type C*

Solutions of this type can be regarded as a limit of solutions of type B when the turning point approaches the y-axis (with the point $x = y = 0$ excluded). It turns out that the trajectory reaches the would be turning point only for $\tau \to \infty$. The shape of the trajectory for $\tau \to -\infty$ is given by the analysis given in the previous Section, i.e. by (20).

Solutions of the type C are not stable in the Liapunov sense — a small change of the initial data given at $\tau = \tau_0$ can give $E \neq 0$, and therefore the trajectory with the new initial data will be of the type A or B.

Below we present an approximate form of solutions for $\tau \to +\infty$ and we again prove the Lipschitz property. This property has to be proved anew because the considerations of the previous Section relied heavily on the particular features of trajectories A, B.

Trajectories of the type C have $E = 0$, because for $\tau \to +\infty$ $y \to y_0 = \text{const}$, $\dot{y} \to 0$, $x \to 0$, $\dot{x} \to 0$.

We restrict our attention to the trajectories approaching the y-axis from the right, i.e. $x > 0$. The other trajectories can be obtained by the reflection $x \to -x$.

Let us start from the observation that for these trajectories $\dot{x} < 0$ for any finite τ. Otherwise the trajectory would run away from the y-axis. Therefore we can consider y and $p = \dot{y}$ as functions of x instead of τ. Because now

$$\dot{x} = -\sqrt{p^2+x^2y^2} < 0,$$

we obtain that

$$\frac{dy}{dx} = -p(p^2+x^2y^2)^{-1/2},$$

$$\frac{dp}{dx} = yx^2(p^2+x^2y^2)^{-1/2}. \tag{27}$$

Introducing $\xi = x^{-1}$ we have

$$\frac{dy}{d\xi} = \xi^{-1}p(p^2\xi^2+y^2)^{-1/2}, \tag{28}$$

$$\frac{dp}{d\xi} = -\xi^{-3}y(p^2\xi^2+y^2)^{-1/2}.$$

Using (28) it is easy to prove that

$$\frac{d(y^2+\xi^2p^2)}{d\xi} = 2\xi p^2,$$

i.e.,

$$y^2(\xi)+\xi^2p^2(\xi) = y^2(\xi_0)+\xi_0^2p^2(\xi_0)+2\int_{\xi_0}^{\xi} sp^2(s)ds. \tag{29}$$

From (29) it follhws that y, p lie on a ξ-dependent ellipse. Therefore, we write

$$y = g(\xi)\sin\varphi, \tag{30a}$$

$$p = \xi^{-1}g(\xi)\cos\varphi. \tag{30b}$$

From (29) it follows that g^2 is an increasing function of ξ.

The equations (28) give for

$$\chi = 2\varphi, \quad \vartheta = 2\ln g - \ln\xi$$

the following equations

$$\vartheta' = \xi^{-1}\cos\chi(\xi), \tag{31a}$$

$$\chi' = -\xi^{-1}\sin\chi + 2\xi^{-5/2}\exp(-\vartheta/2), \tag{31b}$$

where comma denotes d/dx.

We would like to obtain an approximate solution for the system (31) for $\xi \geqslant \xi_0$, ξ_0 being sufficiently large. To this end one has to proceede in a different manner than in the previous Section (compare with (18)). We expect that $y \to y_0 \neq 0$. Therefore (30a) implies that $\varphi(\xi)$ has a limit $\varphi_0 \neq k\pi$ for $\xi \to \infty$. Therefore, for $\xi > \xi_0$

$$\vartheta' \approx \xi^{-1}\cos 2\varphi_0,$$

i.e.,

$$\vartheta \approx c_0 + \cos(2\varphi_0)\ln\xi. \tag{32}$$

Thus, the assumption $\vartheta = c_0$ made in (18) is not valid here. The reason is that now φ has a limit when $\xi \to \infty$, while in (18) $\cos\chi$ oscillates for $x \to \infty$. Because of (32) the second term on th? r.h.s. of (31b) tends to zero for $\xi \to \infty$. Thus, the approximate form of (31b) is

$$\chi' \approx -\xi^{-1}\sin 2\varphi_0.$$

This gives $\chi \sim \ln\xi$, in contradiction with the assumption that $\lim_{\xi\to\infty}\chi(\xi) = 2\varphi_0$, unless

$$2\varphi_0 = \pi, 3\pi \tag{33}$$

(larger values of $2\varphi_0$ give $\varphi_0 > 2\pi$). Thus,

$$\vartheta = c_0 - \ln\xi. \tag{34}$$

Then, (31b) gives

$$\chi' = -\xi^{-1}\sin\chi + 2\xi^{-2}\exp(-c_0/2). \tag{35}$$

This equation has the approximate solution

$$\chi = \pi - \xi^{-1}\exp(-c_0/2) + 2k\pi. \tag{36}$$

Here $k = 0, 1$, in accordance with (33). Inserting (36) into (31a) we obtain the improved ϑ:

$$\vartheta = c_0 - \ln\xi - \tfrac{1}{4}\exp(-c_0)\xi^{-2}. \tag{37}$$

Formulae (36), (37), (30) give

$$y = \pm|y_0|\,(1-\tfrac{1}{4}|y_0|^{-2}x^2)+O(x^4),$$

$$\dot{y} = \pm\tfrac{1}{2}\,x^2+O(x^4),$$

where

$$|y_0| = \exp(c_0/2), \qquad x = \xi^{-1} \to 0.$$

This approximate solution is consistent with (29).

The τ dependence of x, y can be obtained from the equation

$$\dot{x} = -(p^2+x^2y^2)^{-1/2}.$$

For small x this equation gives

$$\dot{x} = -x|y_0|,$$

i.e.,

$$x(\tau) = x_0 \exp(-|y_0|\tau) \tag{38}$$

for $\tau \to +\infty$. This formula is consistent with (9a).

Finally, let us prove the Lipschitz property for the system (31). This system has the form

$$\dot{\vec{\Theta}} = \vec{f}(\vec{\Theta}),$$

where now

$$\vec{\Theta} = \begin{pmatrix} \vartheta \\ \chi \end{pmatrix}, \quad \vec{f}(\vec{\Theta}) = \xi^{-1}\begin{pmatrix} \cos\chi \\ -\sin\chi+2\xi^{-3/2}\exp(-\vartheta/2) \end{pmatrix}.$$

Calculations analogous to those presented in Section 3 give

$$|\vec{f}(\vec{\Theta}_1)-\vec{f}(\vec{\Theta}_2)|^2 \leqslant 3\xi^{-2}(\chi_1-\chi_2)^2+2\xi^{-4}g^{-2}(\xi_0)(\vartheta_1-\vartheta_2)^2 \leqslant m^2(\xi)\,|\vec{\Theta}_1-\vec{\Theta}_2|^2, \tag{39}$$

where

$$m^2(\xi) = 3\xi^{-2}+2\xi^{-4}g^{-2}(\xi_0).$$

Here $g^2(\xi_0) = g^2(\xi(\tau = \tau_0))$, where τ_0 is sufficiently large. In order to obtain (39) we have made the following estimation ($\vartheta_2 > \vartheta_1$):

$$\xi^{-1/2}\,[\exp(-\vartheta_{1/2})-\exp(-\vartheta_{2/2})] = \xi^{-1/2}\exp(-\vartheta_{1/2})\left(1-\exp\frac{\vartheta_1-\vartheta_2}{2}\right)$$

$$\leqslant g^{-1}(\xi_0)\int_0^{\frac{\vartheta_2-\vartheta_1}{2}} e^{-s}ds \leqslant g^{-1}(\xi_2)\,\frac{\vartheta_2-\vartheta_1}{2}.$$

Here we have used the fact that $g(\xi)$ increases, as it follows from (29).

Also, it is easy to prove that

$$|\vec{f}(\vec{\Theta})| \leqslant K(\xi),$$

where

$$K^2(\xi) = 3\xi^{-2}+8\xi^{-4}g^{-2}(\xi_0).$$

Thus, again we can use the iterative procedure for constructing the solution with any desired accuracy, [2], starting from the approximate solution (36), (37) as the zeroth approximation.

We would like to mention that the existence of solutions of the type C was observed in the paper [5], in slightly different context of Yang-Mills equations in the presence of uniformly color-charged plane. However, that paper contains only a numerical analysis of the solution.

5. Remarks on the problem of integrability of the system

The lack of the chaotic behaviour of the trajectories suggests that the system (9) is integrable. However, we know only one exact constant of the motion, namely the Hamiltonian H.

Equations (9) are related to the equations considered by Savvidy and others, [1], by the analytic continuation $y \rightarrow iy$. (Nevertheless, the trajectories of the system (9) are totally different from the trajectories of that system.) Therefore, the second constant of the motion for the system (9) cannot be an analytic function of $x, y, \dot{x}, \dot{y}$, because the analytic continuation would give the second regular constant of the motion for the Savvidy's system, in contradiction to its rather well-established non-integrability.

In fact, we expect that the system (9) is not integrable. We think that the chaotic behaviour of the trajectories is absent because the trajectories for $\tau \rightarrow \infty$ stay in the region of the phase space lying far from singular points of the constant of the motion. Then, the usual cause of chaotic behaviour in a conservative system, that is multiple scattering on the singularities of constants of the motion, [3], would be absent. Of course, definite statements can be made only after the singularities of constants of the motion are found.

However, we think that such investigation for the system (9) is not interesting. Namely, the question of integrability is of prime importance when one looks after behaviour of the trajectories for $\tau \rightarrow \infty$, because in this limit numerical methods for a direct calculation of trajectories are useless, in general. For the system (9) one can predict the behaviour of trajectories in this limit — therefore the question of integrability is not so important.

6. Ending remarks

(a) It is easy to calculate gauge fields $F^a_{\mu\nu}$ for A^a_μ considered in this paper. We have

$$\sqrt{\frac{g^2}{|k^2|}}\, A^{a\nu} = \delta^{a1}\delta^\nu_0 x(\tau)+\delta^{a2}\delta^\nu_1 y(\tau),$$

where $\tau = k^2x_2+k^3x_3$. This gives

$$\sqrt{\frac{g^2}{|k^2|}}\, F^a_{\mu\nu} = \delta^{a1}(g_{\nu 0}k_\mu - g_{\mu 0}k_\nu)\,\frac{dx}{d\tau} + \delta^{a2}(g_{\nu 1}k_\mu - g_{\mu 1}k_\nu)\,\frac{dy}{d\tau}$$

$$+\delta^{a3}\sqrt{|k^2|}\, x(\tau)y(\tau)\,(g_{\mu 1}g_{\nu 0} - g_{\mu 0}g_{\nu 1}), \tag{40}$$

where $(k_\mu) = (0, 0, k_2, k_3)$, $(g_{\mu\nu}) = \mathrm{diag}\,(1, -1, -1, -1)$. Thus, there are nonvanishing color electric and color magnetic fields.

(b) One could also obtain classical Yang-Mills mechanics such that the Newton equation (9b) has the sign plus on the r.h.s. (while Savvidy's mechanics is characterized by the sign minus on the r.h.s. of both (9a) and (9b)). Namely, for $z = A^{21}$, $w = A^{12}$, after the rescalling as in Section 2, we obtain the following Newton equations

$$\frac{d^2w}{d\varrho^2} = wz^2, \qquad \frac{d^2z}{d\varrho^2} = zw^2, \tag{41}$$

where $\varrho = k_0x_0 - k_3x_3$, $k^2 = k_0^2 - k_3^2 < 0$. However, these equations do not seem to be so interesting as those considered in [1], or equations (9). The reason is that here the trajectories can reach the infinity in the $w-z$ plane during finite time ϱ, i.e., the gauge potentials A^{21}, A^{12} have singularities for finite x_0, x_3. For example, let us assume that $R = z$. Then (41) gives

$$\ddot{w} = w^3.$$

This equation has the singular solution

$$w(\varrho) = \sqrt{2}(\varrho_0 - \varrho)^{-1}$$

It seems that all solutions of (41), except the trivial ones ($z = 0$ or $w = 0$), are singular for some finite ϱ.

Let us notice that chaotic behaviour of trajectories seems to be lacking also in the case of equations (41). Equations (9) give regular A^a_μ for all $\tau = k^2x_2+k^3x_3$.

(c) We have observed the lack of chaotic behaviour of gauge potentials when they depend only on τ. This suggests that one should be rather cautious in generalizing to a wider class of Yang-Mills potentials the conclusion of papers [1] about the presence of chaotic behaviour when A^a_μ depend on $\sigma = k_\varrho x^\varrho$, with time-like (k_ϱ). It might well happen that even the gauge potentials initially close in some sense to those considered in [1] would evolve in completely different direction in the space of all gauge potentials, leaving far away the region of chaotic A^a_μ. In order to resolve this question — are the chaotic trajectories typical for full Yang-Mills theory — one should investigate more general A^a_μ, e.g., $A^a_\mu(x_0, r)$, where $r = |\vec{x}|$. However, then one encounters many difficulties. For example, to our best knowledge, there exists no definition of chaotic behaviour for systems conservative with infinite dimensional phase space.

One could reformulate the problem of existence of chaotic behaviour as the problem of integrability of the classical system. If the system is integrable then it should not be regarded as chaotic, because one can predict the behaviour of classical trajectory for large time. Again, we think that it is rather risky to conclude that the full classical Yang--Mills theory is non-integrable because the equations considered in [1] give non-integrable classical Yang-Mills mechanics. For example, it might happen that the constants of the motion are regular (regular here means finite; a more refined definition of regularity remains to be adopted) for all classical Yang-Mills fields which vanish at spatial infinity sufficiently fast — the constants of the motion might involve $\int d^3\vec{x}$. The gauge potentials considered in [1] are outside of this class of potentials. The potentials considered in our paper also do not vanish at spatial infinity. Therefore, we think that the plausible non-integrability of the system (9) would not imply non-integrability of the proper classical Yang-Mills theory which deals with fields vanishing at spatial infinity.

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