STOCHASTIC PROPERTIES OF THE FRIEDMAN DYNAMICAL SYSTEM

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Some mathematical aspects of the stochastic cosmology are discussed in its relationship to the corresponding ordinary Friedman world models. In particular, it is shown that if the strong and Lorentz energy conditions are known, or the potential function is given, or a stochastic measure is suitably defined then the structure of the phase plane of the Friedman dynamical system is determined.

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Introduction

In the series of works [1–3] a stochastic world model has been constructed. Probabilistic elements have been introduced into the ordinary Friedman cosmological model through a certain instability of its matter composition. In technical terms: the equation of state has been perturbed by a "white noise", Einstein's equations, together with the corresponding Fokker-Planck equation, solved and a stochastic description of the cosmic evolution obtained. In our view, the most interesting element of the stochastic world picture is an irrelevance of singularities. The ordinary Friedman models are represented by curves on a phase plane ((ε, H) — plane, for instance), and the singularities are properties of these curves. On the other hand, stochastic evolution takes place on the entire phase plane, and singularities form in it zero-measure sets. This clearly shows that a stochastic measure defined on the Friedman phase plane introduces mathematically new, and some-

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times unexpected, elements. Many problems await their clarification, one of the most important being a non-commutation of the operations of randomization and of imposing symmetries (such as Robertson-Walker symmetries) on the field equation: a randomized Robertson-Walker solution of Einstein's equations need not be the same as Robertson-Walker solution (if it exists) of randomized Einstein's equation [1].

The aim of the present work is to study some mathematical aspects of the stochastic cosmology in relationship to corresponding ordinary "non-perturbed" world models. In particular, we show that the stochastic cosmology correctly determines, through the extrema of the corresponding density function, critical points of the ordinary ("non-perturbed") Friedman phase plane. This demonstrates that even if a randomized Friedman model is not the same as Friedman-like solution of randomized Einstein's equations, the proposed stochastic world models could be treated as a new cosmological system legitimately generalized as compared with the ordinary Friedman cosmology.

The organization of the material is the following. In Sec. 1, we show that the Lorentz and strong energy conditions determine the phase plane structure of the Friedman dynamical system. In Sec. 2, we construct for it the potential function. In Sec. 3, it is demonstrated that a stochastic measure defined on the phase plane of the Friedman dynamical system uniquely determines the set of critical points of this plane.

1. Friedman's dynamical system and the energy conditions

As it is well known, Friedman's cosmological models may be presented in the form of the dynamical system [4–10]

$$\mathscr{F}: \quad \frac{dH}{dt} \equiv \dot{H} = P(H, \varepsilon) \equiv -H^2 - \tfrac{1}{6}(\varepsilon + 3\bar{p} - 2\Lambda), \tag{1.1a}$$

$$\frac{d\varepsilon}{dt} \equiv \dot{\varepsilon} = Q(H, \varepsilon) \equiv -3H(\varepsilon + \bar{p}), \tag{1.1b}$$

where $H = \dot{R}/R$ is the Hubble parameter, ε — energy density, $\bar{p}$ — pressure, and Λ — cosmological constant. In the following, (1.1) will be called the Friedman dynamical system, and denoted by $\mathscr{F}$. The following equation of state has been assumed: $\bar{p} = (\gamma - 1)\varepsilon - 3\zeta(\varepsilon)H$, $1 \leqslant \gamma \leqslant 2$. The dependence $p = p(H, \varepsilon)$ describes the bulk viscosity effects, with the bulk viscosity coefficient $\zeta(\varepsilon) = -\dfrac{1}{3}\dfrac{\partial \bar{p}}{\partial H}$. By assuming $\zeta(\varepsilon) = \dfrac{2}{3}\alpha\varepsilon^m$ (the so-called Belinsky-Khalatnikov parametrization) [4, 5], and by introducing the new variable $E = \varepsilon^{1/2}$, system (1.1) may be expressed in the following form

$$\dot{H} = -H^2 - \frac{3\gamma - 2}{6}E^2 + \alpha H E^{2m} + \frac{\Lambda}{3},$$

$$\dot{E} = -\tfrac{3}{2}(\gamma - 2\alpha H E^{2(m-1)})HE. \tag{1.2}$$

Let us define the following sets on the phase plane (H, ε):

$\lambda = \{\varepsilon+\bar{p}(H,\varepsilon) \geqslant 0\}$, domain of the Lorentz energy condition and its boundary,
$\partial\lambda = \{\varepsilon+\bar{p}(H,\varepsilon) = 0\}$,

$S = \{\varepsilon+3\bar{p}(H,\varepsilon)-2\Lambda \geqslant 0\}$, domain of the strong energy condition and its boundary.
$\partial S = \{\varepsilon+3\bar{p}(H,\varepsilon)-2\Lambda = 0\}$

The role of the energy condition in space-time dynamics has been discussed in Ref. [10] (pp. 88–96). It can be shown that, in the case of system (1.1), resp. (1.2), the energy conditions determine the critical points of the phase plane, namely: (1) static critical points ($H = 0$) are situated in the intersection of ∂S and the ε-axis, (2) non-static critical points ($H \neq 0$) are situated in the intersection of $\partial\lambda$ and the trajectory k_0 of the flat model $k = 0$, which is given by the solution: $\varepsilon = 3H^2-\Lambda$. (1) is directly seen from the form of system (1.1). To prove (2) one should observe that the right-hand side of Eq. (1.1b) identically vanishes on $\partial\lambda$, and that of (1.1a) is $\dot{H} = -H^2+\frac{1}{3}(\varepsilon+\Lambda)$ which after being put equal to zero gives k_0. These results can be summarized in the following

Proposition 1: For dynamical system (1.1), resp. (1.2), {static critical points} $= \partial S \cap$ {ε-axis}, {non-static critical points} $= \partial\lambda \cap k_0$.

It has been also shown by Woszczyna [9] that if energy conditions are given in a "generic way", i.e. if the following conditions are satisfied

$$\operatorname{grad}(\varepsilon+\bar{p})|_{\partial\lambda} \neq 0, \qquad \operatorname{grad}(\varepsilon+3\bar{p}-2\Lambda)|_{\partial S} \neq 0, \tag{1.3}$$

which guarantees that $\partial\lambda$ and ∂S are regular curves (with no self-crossing points), then the energy conditions determine also the character of the critical points unless the eigenvalues of the linearization matrix are purely imaginary.

2. *Potential function for the Friedman dynamical system*

For the dissipationless case ($\alpha = 0$), system (1.2) with the equation of state $\bar{p} = (\gamma-1)\varepsilon$ may be integrated to obtain

$$E^2-3H^2+\Lambda = k\left(\frac{E}{E_0}\right)^{4/3\gamma}, \tag{2.1}$$

where E_0 is a constant. With the help of the transformation

$$\bar{H} = aH, \qquad \bar{E} = bE \tag{2.2}$$

system (1.2) may be reduced to the gradient form:

$$\dot{\bar{H}} = \bar{P}(\bar{H},\bar{E}) \equiv -\frac{\partial V}{\partial \bar{H}},$$

$$\dot{\bar{E}} = \bar{Q}(\bar{H},\bar{E}) \equiv -\frac{\partial V}{\partial \bar{E}} \tag{2.3}$$

with the potential function

$$V(\bar{H}, \bar{E}) = \frac{\bar{H}^3}{3a} + \frac{3\gamma}{4a}\bar{H}\bar{E}^2 - \frac{\Lambda a}{3}\bar{H} + V_0; \quad V = \text{const.} \tag{2.4}$$

The continuity postulate for $V\left(\frac{\partial \bar{P}}{\partial \bar{E}} = \frac{\partial \bar{Q}}{\partial \bar{H}}\right)$ imposes the condition $\frac{b^2}{a^2} = \frac{2(3\gamma-2)}{9\gamma}$ (in (2.4) $b = 1$). Phase trajectories of system (2.3), turn out to be orthogonal to the equipotential surfaces $V(\bar{H}, \bar{E}) = \text{const}$.

By the same procedure one readily finds that the potential function for the Friedman dynamical system with bulk viscosity ($\alpha = 0$) is

$$V(\bar{H}, \bar{E}) = \frac{\bar{H}^3}{3a} + \frac{3\gamma}{4a}\bar{H}\bar{E}^2 - \frac{\alpha\bar{H}^2\bar{E}^{2m}}{2b^{2m}} - \frac{\Lambda a}{3}\bar{H} + V_0 \tag{2.5}$$

with the additional condition $m = \frac{3b^2}{a^2} = \frac{2(3\gamma-2)}{3\gamma}$, which tells us that the constant bulk viscosity coefficient ($m = 0$) implies $\gamma = \frac{2}{3}$ (i.e. this case has no physical signification; $1 \leqslant \gamma \leqslant 2$).

The results of the present section may be collected to the form of the following

Proposition 2: Phase space of the Friedman dynamical system is uniquely determined by the potential function. The potential function for the dissipationless Friedman dynamical system and for that with bulk viscosity is given by (2.4) and (2.5), correspondingly.

As an example let us consider the flat Friedman model with dissipation ($\alpha \neq 0$). For the flat model, $\varepsilon = 3H^2 - \Lambda$ with the equation of state $\bar{p} = (\gamma-1)\varepsilon - 2\alpha\varepsilon^m H$ system (1.1) becomes

$$\dot{H} = -\frac{\gamma}{2}(3H^2-\Lambda) + \alpha H(3H^2-\Lambda)^m, \quad \dot{\varepsilon} = 6H\dot{H}, \tag{2.6}$$

These two equations, when combined, give

$$H = -\tfrac{1}{2}(\varepsilon(H) + \bar{p}(H, \varepsilon(H))) \tag{2.7}$$

which shows that stationary universes ($H = 0$) are situated on $\partial\lambda$. For system (2.6), the potential function is

$$V(H) = \tfrac{1}{2}\gamma(H^3 - \Lambda H) - \frac{\alpha(3H^2-\Lambda)^{m+1}}{6(m+1)} + V_0 \tag{2.8}$$

or, more generally

$$V(H) = \tfrac{1}{2}\int_{H_0}^{H}(\varepsilon+\bar{p})dH' = \tfrac{1}{2}\left(H^3 - \Lambda H + \int_{H_0}^{H}\bar{p}dH'\right) \tag{2.9}$$

For flat cosmological models with $\alpha = 0$, the function $V(H)$, for different values of Λ, is shown in Fig. 1.

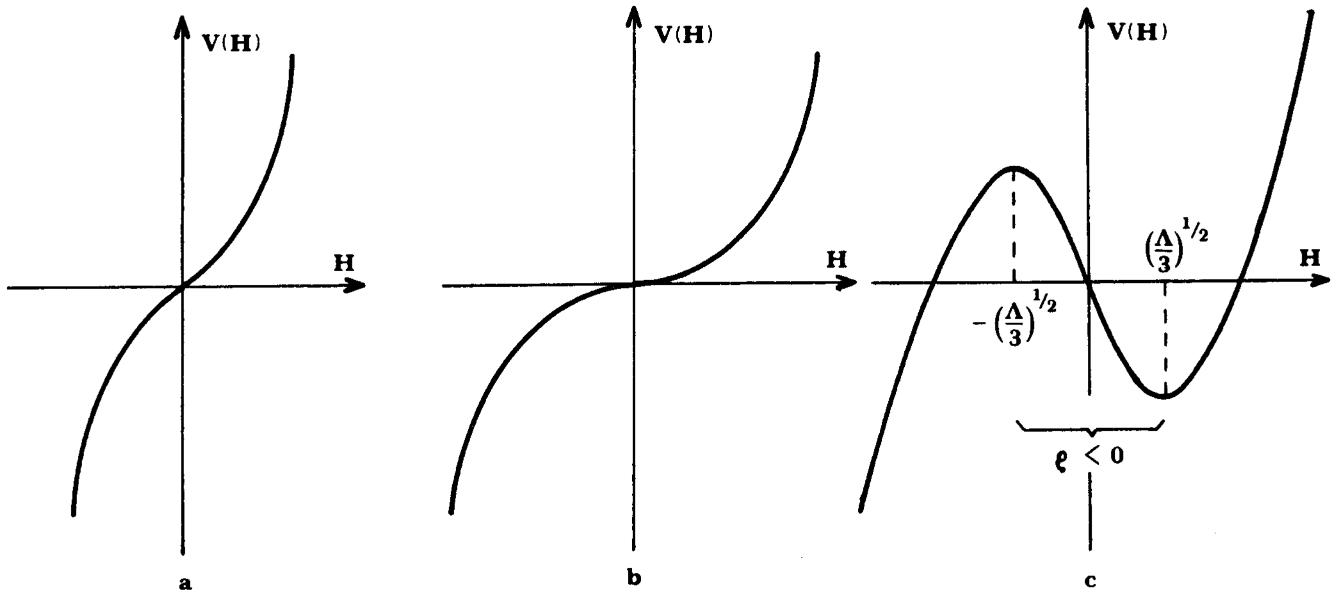

Fig. 1. Potential functions for the flat cosmological models: with (a) $\Lambda < 0$, (b) $\Lambda = 0$, (c) $\Lambda > 0$

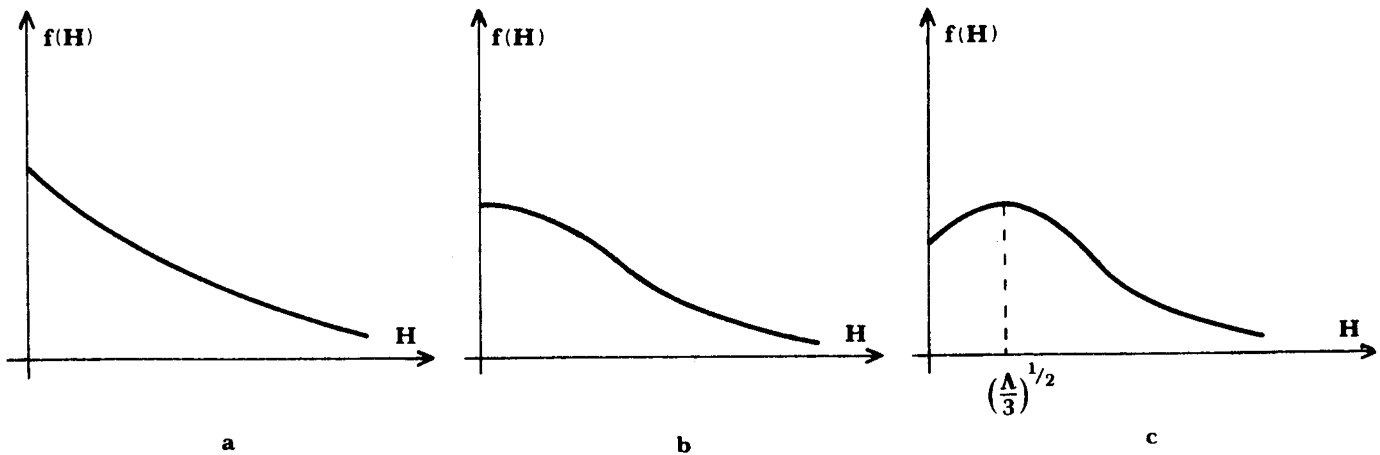

Fig. 2. Density distributions for the flat cosmological models: with (a) $\Lambda < 0$, (b) $\Lambda = 0$, (c) $\Lambda > 0$

3. Stochastic properties of the Friedman dynamical system

Let us consider system (2.3) "perturbed" by the "white noise"

$$\dot{x} = P(x, y)+\eta_1(t) = -\frac{\partial V}{\partial x}+\eta_1(t),$$

$$\dot{y} = Q(x, y)+\eta_2(t) = -\frac{\partial V}{\partial y}+\eta_2(t), \tag{3.1}$$

$$\langle\hat{\eta}_i(t)\rangle = 0, \quad \langle\hat{\eta}_i(t), \hat{\eta}_j(t+\tau)\rangle = 2\theta\delta(\tau), \quad \theta = \text{const}, \quad i,j = 1,2.$$

The wedge over η_i denotes that $\hat{\eta}_i$ is a stochastic process. The corresponding Fokker--Planck equation is

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(K_1 f)-\frac{\partial}{\partial y}(K_2 f)+\theta\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right), \tag{3.2}$$

where $K_1 = P(x, y)$ and $K_2 = Q(x, y)$ are transport coefficients and θ is a diffusion coefficient (see, e.g. [11]).

It can be checked that the expression

$$f(x, y, x_0, y_0) = f_0 \exp\left(-\frac{U(x, y)}{\theta}\right) \tag{3.3}$$

is a stationary solution of Eq. (3.2). One can easily see that the extrema of this density function correspond to the critical points of the non-perturbed system. We have, therefore

Proposition 3: A stochastic measure defined on a phase plane of the Friedman dynamical system determines, through the extrema of the corresponding density function, the set of critical points of the phase plane.

For the flat model ($k = 0$), the potential function is given by (2.9) and from it one can directly see that the extrema of the density functions are situated on $\partial\lambda \cap k_0$ (see, Sec. 1). If $\bar{p} = (\gamma-1)\varepsilon$, the solution of (3.2) is

$$f(H) = N \exp\left(\frac{-\gamma(H^3-\Lambda H)}{2\theta}\right), \tag{3.4}$$

where $N = \frac{1}{3}\left(\frac{\gamma}{2\theta}\right)^{-3}\Gamma(\frac{1}{3})$, for $\Lambda = 0$, and f is to be interpreted as a density distribution function, defined for $H \in \langle 0, +\infty\rangle$ (see, Fig. 2).

The above three propositions may be put together to get the following

Theorem: If (1) boundaries $\partial\lambda$ and ∂S of the Lorentz and strong energy conditions, together with the trajectory of the flat Friedman model k_0, are given,

or (2) the potential function $V(x, y)$ for $\mathscr{F}$ is given,

or (3) the stochastic measure, corresponding to the density function (3.3), is defined on the phase plane of $\mathscr{F}$, then (4) the set of the critical points (their number and positions) are determined on the phase-plane of $\mathcal{F}$.

Moreover, the following implications are also valid: (3) $\Rightarrow$ (2), (4) $\Rightarrow$ (1).

This result shows a consistency of our method to introduce the stochastic measure on the phase plane of the Friedman dynamical system.

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