ON THE EQUIVALENCE OF DIFFERENT REGULARIZATION METHODS

By S. Brzezowski

Institute of Physics, Jagellonian University, Cracow*

(Received June 4, 1984)

The $\hat{R}$-operation preceded by the regularization procedure is discussed. Some arguments are given, according to which the results may depend on the method of regularization, introduced in order to avoid divergences in perturbation calculations.

PACS numbers: 03.70.+k, 11.10.Gh

1. Introduction

Divergent integrals, which we meet in perturbation calculations of QFT are the source of technical difficulties, we have to overcome. The regularization is just one of the methods to deal with them.

We often meet a following opinion: the final (and finite) results obtained when different methods of regularization are used do not depend on the method chosen (up to the set of finite renormalization constants). This statement is true if applied to the methods being in actual use: the analytical, dimensional or the Pauli-Villars regularization methods [1–6].

However, examples of other "regularizations" may be presented, giving results which disagree with the standard ones. We shall discuss some examples of such regularizations and then we shall come back to the problem of the universal criterion of the correctness of regularization procedure [7].

2. The regularization and the $\hat{R}$-operation

At first, two definitions.

a. Let us denote the divergent integrals by

$$I(k) = \int d\alpha f(k, \alpha), \tag{1}$$

* Address: Instytut Fizyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland.

where $\boldsymbol{k}$ is the set of external variables (external momenta), $\boldsymbol{\alpha}$ is the set of variables of internal integration, e.g. the internal momenta or the integration variables of the so called α-representation.

b. By $\hat{R}$-operation we mean the product of subtraction operators $\pi(\hat{1}-\hat{M}_\gamma)$ defined e.g. by Bogoliubov [8]. Each operator $(\hat{1}-\hat{M}_\gamma)$ realizes the subtraction connected with subdiagram γ.

There are two methods of dealing with divergences:

1°. The $\hat{R}$-operation without any regularization of the integrand [9]. In this approach the integration over the variables $\boldsymbol{\alpha}$ has to be performed (and is possible) after the $\hat{R}$-operation is realized on the integrand $f(\boldsymbol{k}, \boldsymbol{\alpha})$.

This approach is often criticized, because the left-hand side of the expression

$$\hat{R}[\int d\boldsymbol{\alpha} f(\boldsymbol{k}, \boldsymbol{\alpha})] = \int d\boldsymbol{\alpha} \hat{R} f(\boldsymbol{k}, \boldsymbol{\alpha}) \tag{2}$$

is not well defined (the integral does not exist). In practice, we have to start with the right--hand side and continue the calculations.

2°. In order to avert the formal objections mentioned above, we have to regularize the function $f(\boldsymbol{k}, \boldsymbol{\alpha})$ and in this way to restore the formal meaning to the integral over $\boldsymbol{\alpha}$. Instead of $f(\boldsymbol{k}, \boldsymbol{\alpha})$ we use some function $\phi(\boldsymbol{k}, \boldsymbol{\alpha}, \gamma)$ (γ is some set of variables) satisfying

$$\phi(\boldsymbol{k}, \boldsymbol{\alpha}, \gamma_0) = f(\boldsymbol{k}, \boldsymbol{\alpha}) \tag{3}$$

(or at least ϕ treated as a function of $\boldsymbol{\alpha}$ has to be distributionally convergent to f with $\gamma \to \gamma_0$) and for which the integral $\int \phi(\boldsymbol{k}, \boldsymbol{\alpha}, \gamma) d\boldsymbol{\alpha}$ is well defined if γ is taken from some area containing γ_0 (except γ_0, of course).

Now, the $\hat{R}$-operation is realized on the well defined expression [8, 10]

$$\hat{R} \int \phi(\boldsymbol{k}, \boldsymbol{\alpha}, \gamma) d\boldsymbol{\alpha} = \int \hat{R} \phi(\boldsymbol{k}, \boldsymbol{\alpha}, \gamma) d\boldsymbol{\alpha}. \tag{4}$$

After the right-hand side integration is performed, we are allowed to take $\gamma = \gamma_0$.

Is the method 2° for every regularization equivalent to 1°? If we define regularization as in (3), the answer to this question is, in general, negative.

Examples:

Let the divergent integral be of the form

$$I(k) = \int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha k} \tag{5}$$

(we assume all variables as dimensionless).

Ad 1°. The $\hat{R}$-operation realized as in the version 1° reduces in this case to the subtraction

$$\hat{R} I(k) = I(k) - I(k_0) = \int_0^\infty \frac{d\alpha}{\alpha} (e^{-\alpha k} - e^{-\alpha k_0})$$

$$= -\ln k - (-\ln k_0) = -\ln k + C. \tag{6}$$

The constant C is added in order to stress the freedom of choice of the subtraction point k_0 (cf. P_γ in [10]). The final result is

$$I_{\rm fin}(k) = -\ln k + C \tag{7}$$

and this will be our standard to compare with other results.

Ad 2°. a. The analytical regularization (a correct one)

$$I_{\rm reg}(k, \gamma) = \int\limits_0^\infty \frac{d\alpha}{\alpha^{1-\gamma}} e^{-\alpha k}, \tag{8}$$

$$\hat{R}I_{\rm reg}(k, \gamma) = \int\limits_0^\infty \frac{d\alpha}{\alpha^{1-\gamma}} (e^{-\alpha k} - e^{-\alpha k_0}) = \Gamma(\gamma)(k^{-\gamma} - k_0^{-\gamma})$$

$$= \left(\frac{1}{\gamma} + \dots\right) \{[(-\ln k) - (-\ln k_0)]\gamma + O(\gamma^2)\}$$

$$\xrightarrow{\gamma \to 0} -\ln k - (-\ln k_0) = -\ln k + C. \tag{9}$$

The result is correct (as compared with [10]).

b. The examples of false regularizations

Let us take

$$I_{\rm reg}(k, \gamma) = \int\limits_0^\infty d\alpha \frac{k^\gamma}{\alpha^{1-\gamma}} e^{-\alpha k}. \tag{10}$$

One can say, the factor k^γ is "strange". However, it does not introduce any new singularity and it tends to 1 (for $k \neq 0$) when γ tends to 0. Now, the subtraction gives

$$RI_{\rm reg}(k, \gamma) = \int\limits_0^\infty \frac{d\alpha}{\alpha^{1-\gamma}} (k^\gamma e^{-\alpha k} - k_0^\gamma e^{-\alpha k_0})$$

$$= \begin{bmatrix}\text{compare with the} \\ \text{calculation of (9)}\end{bmatrix} = \Gamma(\gamma) - \Gamma(\gamma) = 0 \qquad \forall \gamma. \tag{11}$$

So this time, the "amplitude" $I_{\rm fin}(k)$ is a *constant*.

What more, for an arbitrary function $\Psi(k)$ a regularization of the integral (5) can be given, leading to the result $I_{\rm fin}(k) = \Psi(k)$; namely, if

$$I_{\rm reg}(k, \gamma) = \int\limits_0^\infty d\alpha [\phi(k)k]^\gamma \frac{e^{-\alpha k}}{\alpha^{1-\gamma}} \tag{12}$$

the $\hat{R}$-operation gives

$$\hat{R}I_{\text{reg}}(k,\gamma) = \int_0^\infty \frac{d\alpha}{\alpha^{1-\gamma}} \{[\phi(k)k]^\gamma e^{-\alpha k} - [\phi(k_0)k_0]^\gamma e^{-\alpha k_0}\}$$

$$= [\phi(k)^\gamma - \phi(k_0)^\gamma]\Gamma(\gamma) \xrightarrow{\gamma\to 0} \ln\phi(k) - \ln\phi(k_0). \tag{13}$$

Now, it is enough to take

$$\phi(k) = \exp(\Psi(k)). \tag{14}$$

The last two examples may be called the "pathological" ones. However, it becomes clear that the opinion, according to which "the result of calculations does not depend on the method of regularization used", needs some specification of the class of proper regularizations.

In our negative examples the fault was in the presence of the external variable k in the regularizing factor. However, the statement, according to which in proper regularizations the external variables have to be separated from the regularizing factor, would be too strong. For example, if α's are the internal momenta, the analytical regularization

$$\frac{1}{p^2-m^2} \to \frac{1}{p^2-m^2}\left(\frac{1}{p^2-m^2}\right)^\gamma \tag{15}$$

contains external variables in the regularizing factor $(p^2-m^2)^{-\gamma}$. (In this case, however, the separation *is* gained after the α-representation is introduced.)

Let us return to our example. The regularization

$$I_{\text{reg}}(k,\gamma) = \int_0^\infty \frac{1+k\gamma^2}{\alpha^{1-\gamma}} e^{-\alpha k} d\alpha \tag{16}$$

after the subtraction

$$\hat{R}I_{\text{reg}}(k,\gamma) = \int_0^\infty \left(\frac{1+k\gamma^2}{\alpha^{1-\gamma}} e^{-\alpha k} - \frac{1+k_0\gamma^2}{\alpha^{1-\gamma}} e^{-\alpha k_0}\right) d\alpha$$

$$= \left[(1+k\gamma^2)\frac{1}{k^\gamma} - (1+k_0\gamma^2)\frac{1}{k_0^\gamma}\right]\Gamma(\gamma)$$

$$= \Gamma(\gamma)\left[\frac{1}{k^\gamma} - \frac{1}{k_0^\gamma}\right] + \gamma^2\Gamma(\gamma)\left[\frac{1}{k^{\gamma-1}} - \frac{1}{k_0^{\gamma-1}}\right]$$

$$\xrightarrow{\gamma\to 0} -\ln k - (-\ln k_0) = -\ln k + C \tag{17}$$

gives a correct result, though the regularization factor $(1+k\gamma^2)$ depeds on k. Whereas, if we take $(1+k\gamma)$, we obtain

$$\hat{R}I_{reg}(k) = \Gamma(\gamma)\left[\frac{1}{k^\gamma} - \frac{1}{k_0^\gamma}\right] + \gamma\Gamma(\gamma)\left[\frac{1}{k^{\gamma-1}} - \frac{1}{k_0^{\gamma-1}}\right]$$

$$\xrightarrow{\gamma\to 0} -\ln k - (-\ln k_0) + k - k_0 = -\ln k + k + C \tag{18}$$

and the result is false.

So, the crucial point is not the absence of the external variable (variables) in the regularizing factor, but the character of this dependence.

3. *Recapitulation*

Let us limit ourselves to the regularization of the primitively divergent diagram (subdiagram) Γ, realized by a regularization factor $\chi(\boldsymbol{k}, \boldsymbol{a}, \gamma)$

$$\int f(\boldsymbol{k}, \boldsymbol{\alpha})d\boldsymbol{\alpha} \to \int \phi(\boldsymbol{k}, \boldsymbol{\alpha}, \gamma)d\boldsymbol{\alpha} = \int \chi(\boldsymbol{k}, \boldsymbol{\alpha}, \gamma)f(\boldsymbol{k}, \boldsymbol{\alpha}), \tag{19}$$

where $\chi(\boldsymbol{k}, \boldsymbol{a}, \gamma_0) \equiv 1$, (or, it tends distributionally to 1), and of course the integral $\int \phi(\boldsymbol{k}, \boldsymbol{a}, \gamma)d\boldsymbol{a}$ is well defined for some $\gamma \neq \gamma_0$.

The criterion of the correctness of the regularization of primitively divergent diagrams has been already discussed in [7], where, instead of subtractions, the integrand is simply differentiated with respect to external variables. (These two methods of calculation are equivalent.) The criterion is:

$$\lim_{\gamma\to\gamma_0} \int \partial_{\boldsymbol{k}}^t \phi(\boldsymbol{k}, \boldsymbol{\alpha}, \gamma)d\boldsymbol{\alpha} = \int \partial_{\boldsymbol{k}}^t f(\boldsymbol{k}, \boldsymbol{\alpha})d\boldsymbol{\alpha} \tag{20}$$

where

$$\partial_{\boldsymbol{k}}^t \equiv \frac{\partial^{t_1}}{(\partial k_1)^{t_1}} \cdots \frac{\partial^{t_m}}{(\partial k_m)^{t_m}}, \quad \sum_{i=1}^{m} t_i \geqslant \frac{\omega^\Gamma}{2} + 1 \tag{21}$$

and ω_Γ is the index of the diagram Γ.

Referring this to (19) we have a condition:

$$\lim_{\gamma\to\gamma_0} \int \partial_{\boldsymbol{k}}[\chi(\boldsymbol{k}, \boldsymbol{\alpha}, \gamma)f(\boldsymbol{k}, \boldsymbol{\alpha})]d\boldsymbol{\alpha} = \int \partial_{\boldsymbol{k}} f(\boldsymbol{k}, \boldsymbol{\alpha})d\boldsymbol{\alpha}. \tag{22}$$

The most simple way to satisfy (22) is to take χ independent on $\boldsymbol{k}$. Here is the place of the analytical, dimensional and Pauli-Villars regularizations (in the α-representation).

However, this is not the only possibility; in order to stay in the framework of (22), it is enough to have $\partial_{\boldsymbol{k}_1}\chi(\boldsymbol{k}, \boldsymbol{a}, \gamma)$ distributionally convergent to zero as $\gamma \to \gamma_0$, in the sense of integration with $\partial_{\boldsymbol{k}_2} f(\boldsymbol{k}, \boldsymbol{a})$, where $\boldsymbol{k}_1 \cup \boldsymbol{k}_2 = \boldsymbol{k}$, $\boldsymbol{k}_1 \neq \emptyset$.

In the few trivial examples we have discussed above, the functions $\chi(k, \alpha, \gamma)$ were: $\alpha^\gamma k^\gamma$, $\alpha^\gamma(1+k\gamma)$, $\alpha^\gamma(1+k\gamma^2)$, whereas, the functions $\partial_k\chi$ respectively: $\alpha^\gamma\gamma k^{\gamma-1}$, $\alpha^\gamma\gamma$, $\alpha^\gamma\gamma^2$.

Only the last one tends to zero (distributionally) when $\gamma \to 0$ (in the sense of integration with $\alpha^{-1} \exp(-\alpha k)$).

Finally, let us compare the usefulness of two methods denoted as 1° and 2°. The $\hat{R}$-operation combined with regularization (2°) is mathematically "clean" — however, in the framework of this method there is no possibility to check the result. For verification we are forced to refer to the universal result calculated by the use of 1°, in other words, to check whether the following four operations:

1. regularization $\hat{\lambda}$: $\hat{\lambda} f = \phi$,
2. subtraction $\hat{R}$,
3. integration $\int \dots d\alpha$,
4. "taking off" the regularization $\hat{\lambda}_0$: $\hat{\lambda}_0 \phi = f$, applied to $f(\boldsymbol{k}, \boldsymbol{\alpha})$ satisfy a commutation relation:

$$2° \to \hat{\lambda}_0 \int \hat{R} \hat{\lambda} f(\boldsymbol{k}, \boldsymbol{\alpha}) d\boldsymbol{\alpha} = \int \hat{R} \hat{\lambda}_0 \hat{\lambda} f(\boldsymbol{k}, \boldsymbol{\alpha}) d\boldsymbol{\alpha} \equiv \int \hat{R} f(\boldsymbol{k}, \boldsymbol{\alpha}) d\alpha \leftarrow 1° \tag{23}$$

being a straightforward generalization of (20) for any complicated diagram.

REFERENCES

[1] W. Pauli, F. Villars, *Rev. Mod. Phys.* **21**, 434 (1949).
[2] J. Rayski, *Acta Phys. Pol.* **9**, 129 (1948).
[3] C. G. Bollini, J. J. Giambiaggi, *Nuovo Cimento* **31**, 550 (1964).
[4] C. G. Bollini, *Nuovo Cimento* **12B**, 20 (1972).
[5] E. Speer, *J. Math. Phys.* **9**, 1404 (1968).
[6] G. t'Hooft, M. Veltman, *Nucl. Phys.* **B44**, 189 (1972).
[7] S. Brzezowski, *Acta Phys. Pol.* **B12**, 295 (1981).
[8] N. N. Bogoliubov, D. V. Shirkov, *Vvedenie v teoriyu kvantovannykh polei,* Moskva 1973.
[9] W. Zimmermann, *Commun. Math. Phys.* **11**, 1 (1968).
[10] D. Zavyalov, *Perenormirovannye diagrammy Feinmana,* Moskva 1979.